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A Common Framework for Curve Evolution, Segmentation and Anisotropic Diffusion

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Abstract[1]

In recent years, curve evolution has developed into an important tool in Computer Vision and has been applied to a wide variety of problems such as smoothing of shapes, shape analysis and shape recovery. The underlying principle is the evolution of a simple closed curve whose points move in the direction of the normal with prescribed velocity. A fundamental limitation of the method as it stands is that it cannot deal with important image features such as triple points. The method also requires a choice of an "edge-strength" function, defined over the image domain, indicating the likelihood of an object boundary being present at any point in the image domain. This implies a separate preprocessing step, in essence precomputing approximate boundaries in the presence of noise. One also has to choose the initial curve. It is shown here that the different versions of curve evolution used in Computer Vision together with the preprocessing step can be integrated in the form of a new segmentation functional which overcomes these limitations and extends curve evolution models. Moreover, the numerical solutions obtained retain sharp discontinuities or "shocks", thus providing sharp demarcation of object boundaries.

1. Introduction

In recent years, curve evolution has been applied to a wide variety of problems such as smoothing of shapes [6,14], shape analysis [6,7] and shape recovery [3,4,9,10,19,20,24]. The underlying principle is the evolution of a simple closed curve whose points move in the direction of the normal with a prescribed velocity. Kimia, Tannenbaum and Zucker [6] proposed evolution of the curve by letting its points move with velocity consisting of two components: a component proportional to curvature and a constant component corresponding to morphology. Depending on the sign of the constant component of the velocity, the curve can expand (thus joining two disjoint nearby shapes) or contract (thus separating a dumbbell shape into two separate blobs). The formulation involves one parameter which together with time provides a two-dimensional scale space, called "entropy" scale space of the shape. Keeping track of how singularities develop and disappear as the curve evolves provides information regarding the geometry of the shape in terms of its parts and its skeleton [7,8,22].

The above technique assumes that the object boundary (in the form of a simple closed curve) has already been extracted in a *robust* way from an image. This is certainly not the case when the image is very noisy. As a result, attempts have been made in the last couple of years to extend the technique to recover shapes from noisy images in two distinctly different ways. In both cases, a continuous edge strength function v, varying between 0 and 1, is defined over the entire image domain. It equals or approaches the value one at the object boundaries and approaches the value zero where the image gradient is small. One of the methods is a simple extension of the formulation of Kimia et al [3,10,24] in which the velocity of the curve is multiplied by a "stopping term", $(1 - v)$, so that the evolution is slowed down near the object boundaries (in fact stopped where $v = 1$). In an alternate approach [4,9,19,20], the idea is to let the curve evolve towards a geodesic in the metric defined by $(1 - v)^q$ where q is a constant, usually equal to 1 or 2. The corresponding point velocity of the curve along its normal consists of the curvature term as before and an advection term given by the derivative of v in the direction of the normal. There is no constant component corresponding to morphology. There is no stopping term either. The advantage of the first approach is that the morphology term prevents the evolving curve from seeping through small gaps represented by small values of v. Another advantage is that since the constant velocity term is independent of v, it is possible to prevent the curve from collapsing under the effect of curvature where the edge-strength v is negligible. The disadvantage is that the stopping term is essential. Even with the stopping term, if v is less than 1 everywhere, then eventually the evolving curve disappears. The second approach has the advantage of possessing steady state solutions in the form of geodesics without using a stop-

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ping term. The main drawback is that if the initial curve is too far away from the desired geodesic, it will converge to a wrong solution. Of course, it is possible to combine the two approaches and let the evolution be governed by all three types of velocity components. However, analysis of the possible steady state solutions and their dependence on the three parameters becomes much more difficult.

The easiest way to implement the curve evolution is by embedding the intial curve as a level curve in a surface and let all the level curves of the surface evolve simultaneously. The advantage is that changes in the topology of the curve are handled automatically, simplifying the data structure. Numerical scheme of Osher and Sethian [13] may then be used to implement the evolution. This is how the method is normally used.

There are several points to notice about this approach.

1. **Singular points:** Since the shape is represented by a simple closed curve, the method as it stands cannot be used for more complex shapes involving singularities, such as the line diagram of a three-dimensional cube. Important image features such as Y-junctions and T-junctions are outside the scope of this scheme.

2. **Choice of the initial curve:** The method requires specification of an initial curve. In [3,4,9,10], this curve is specified by the user. In [24], lots of small circles, called "balloons" (first used in [5]) are randomly distributed over the image and allowed to evolve. They merge together as they collide. Near the object boundary where v is large, they are slowed down by the stopping term and thus prevented from merging across the boundary, resulting in a double-curve representation of the object boundary. In [19,20], the locus of the zero-crossings of the laplacian of a smoothed version of the image is used as the initial curve.

3. **Edge-strength function:** The image properties are incorporated in the edge strength function v. Virtually without exception, the form selected for v has been

$$v = \frac{||\nabla G_\sigma * g||^k}{1 + ||\nabla G_\sigma * g||^k} \tag{1}$$

where $k = 1$ or 2, G_σ is the Gaussian with standard deviation σ and g is the image intensity. Thus the concerns regarding inadequacy of linear filters and the development of non-linear methods for filtering noise are ignored. As a result, the boundaries may be displaced and the curve might converge to a spurious boundary. For this reason, Ambrosio-Tortorelli edge-strength function is used in [19,20].

4. **Mathematical analysis:** The mathematical technique used for analyzing curve evolution is either the method of viscosity solutions [3,4] or differential geometric methods [6,14]. At present, there is no known way to extend these methods if one were to integrate curve evolution with other computational modules of Computer Vision. This is in contrast to the "direct method" of variational calculus or the stochastic methods which are more flexible for incorporating several pertinent variables in a single formulation.

It turns out that the forms of curve evolution described above are special cases of a new segmentation functional. The new functional provides a common framework for curve evolution and segmentation, resolving the issues raised above. The focus of this paper is on recovery of shapes, but an important observation is that the level curves of the edge-strength function v which appears naturally in the approximation of segmentation functionals constitute a scale space for shapes analogous to the entropy scale space proposed in [8]. In particular, various kinds of shocks and resulting shape skeleton may be computed quite easily from v [23]. Generalization of the new segmentation functional to higher dimension and to vector-valued feature functions g, such as color images is straight-forward [21]. The equations governing curve evolution in the vector-valued case are quite different from the ones proposed in [16].

This paper is organized as follows. In §2, the Ambrosio-Tortorelli edge-strength function is reviewed. A review of recovery of shapes as geodesics by curve evolution is contained in §3. The new segmentation functional is introduced in §4. The "anisotropic" diffusion is discussed in §5. Illustrative examples are described in §6.

2. Edge-Strength Function

The Ambrosio-Tortorelli edge-strength function is a byproduct of their formulation of an elliptic approximation of the following segmentation functional introduced by Mumford and Shah in [12]:

$$\begin{aligned} E_{MS}(u,B) = \alpha \int\int_{R\setminus B} ||\nabla u||^2 dx dy \\ + \beta \int\int_R (u-g)^2 dx dy + |B| \end{aligned} \tag{2}$$

where R is a connected, bounded, open subset of $\mathbf{R}^2$, g is the feature intensity, B is a curve segmenting R, u is the smoothed image $\subset \mathbf{R}^2 \setminus B$, $|B|$ is the length of B and α, β are the weights. Since it is difficult to apply gradient descent with respect to B, Ambrosio and Tortorelli [2] replace B by a continuous variable v and obtain

$$\begin{aligned} E_{AT}(u,v) = \int\int_R \{\alpha(1-v)^2 ||\nabla u||^2 \\ + \beta(u-g)^2 + \frac{\rho}{2}||\nabla v||^2 + \frac{v^2}{2\rho}\} dx dy \end{aligned} \tag{3}$$

The corresponding gradient descent equations are:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = -2\nabla v \cdot \nabla u + (1-v)\nabla^2 u \\
&\qquad - \frac{\beta}{\alpha(1-v)}(u-g) \\
(4)\quad &\frac{\partial v}{\partial t} = \nabla^2 v - \frac{v}{\rho^2} + \frac{2\alpha}{\rho}(1-v)||\nabla u||^2 \\
&\frac{\partial u}{\partial n}|_{\partial R} = 0; \ \frac{\partial v}{\partial n}|_{\partial R} = 0
\end{aligned}
$$

where ∂R denotes the boundary of R and n denotes the direction normal to ∂R. Notice that equation for each variable is a diffusion equation which minimizes a convex quadratic functional in which the other variable is kept fixed. Keeping v fixed, the first equation minimizes

$$(5) \quad \int\int_R \left\{\alpha(1-v)^2||\nabla u||^2 + \beta(u-g)^2\right\}dxdy$$

Keeping u fixed, the second equation minimizes

$$(6) \quad \int\int_R \{||\nabla v||^2 + \frac{1+2\alpha\rho||\nabla u||^2}{\rho^2}\left(v - \frac{2\alpha\rho||\nabla u||^2}{1+2\alpha\rho||\nabla u||^2}\right)^2\}$$

Thus the Ambrosio-Tortorelli edge strength function v is nothing but a nonlinear smoothing of

$$(7) \quad \frac{2\alpha\rho||\nabla u||^2}{1+2\alpha\rho||\nabla u||^2}$$

where u is a simultaneous nonlinear smoothing of g. Note the similarity between (1) and (7).

3. Geodesics and Shape Recovery

Let Γ be a simple closed curve in R. In order to move Γ to where the image intensity gradient and hence v are high, we look for the stationary points of the functional

$$(8) \quad \int_\Gamma (1-v)^q ds$$

where s denotes the arc-length along Γ and q is a fixed constant. Let $C(p,t) : I \times [0,\infty) \rightarrow R$ be the evolving family of curves where I is the unit interval and t denotes time. We require that $C(0,t) = C(1,t)$ for all values of t and the image of $C(p,0)$ in R coincides with Γ. Then the evolution of the curve is governed by the equation

$$(9) \quad \frac{\partial C}{\partial t} = [q\nabla v \cdot N - (1-v)\kappa]N$$

where N is the outward normal and κ is the curvature which is defined such that it is positive when Γ is a circle. In order to implement the evolution of Γ, assume that Γ is embedded in a surface $f_0 : R \rightarrow \mathbf{R}$ as a level curve. Let $f(t,x,y)$ denote the evolving surface such that $f(0,x,y) = f_0(x,y)$. Then, in order to let all the level curves of f_0 evolve simultaneously, consider the functional [19,20]:

$$
\begin{aligned}
(10)\quad &\int_{-\infty}^{\infty}\int_{\Gamma_c} (1-v)^q ds_c dc \\
&= \int\int_R (1-v)^q ||\nabla f|| dxdy
\end{aligned}
$$

where $\Gamma_c = \{(x,y)|f(t,x,y) = c\}$. By calculating the first variation of the last functional, we get the gradient descent equation as

$$
\begin{aligned}
(11)\quad &\frac{\partial f}{\partial t} = -q\nabla v \cdot \nabla f + (1-v)||\nabla f|| \ curv(f) \\
&\frac{\partial f}{\partial n}|_{\partial R} = 0, \quad f|_{t=0} = f_0
\end{aligned}
$$

where

$$(12) \quad curv(f) = \frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{\left(f_x^2 + f_y^2\right)^{3/2}}$$

$curv(f)$ is the curvature of the level curves of f.

4. A New Segmentation Functional

The question now is how to combine the alternative forms of curve evolution into a single framework and resolve the issues raised above. A hint is provided by the similarity between the second functional in (10) and the first term of the Ambrosio-Tortorelli functional (3), suggesting the following functional for consideration:

$$
\begin{aligned}
(13)\quad E_\rho(u,v) = &\int\int_R \{\alpha(1-v)^2||\nabla u|| \\
&+ \beta|u-g| + \frac{\rho}{2}||\nabla v||^2 + \frac{v^2}{2\rho}\}dxdy
\end{aligned}
$$

Analysis of simple one-dimensional cases suggests that as $\rho \rightarrow 0$, $E_\rho(u,v)$ converges to the following functional:

$$
\begin{aligned}
(14)\quad E(u,B) = &\int\int_{R\setminus B} ||\nabla u|| dxdy \\
&+ \frac{\beta}{\alpha}\int\int_R |u-g|dxdy + \int_B \frac{J_u}{1+\alpha J_u}ds
\end{aligned}
$$

where J_u is the jump in u across B, that is, $J_u = |u^+ - u^-|$ where the superscripts + and – refer to the values on two sides of B. Thus each boundary point is weighed according to its level of contrast instead of being assigned a fixed weight as in E_{MS}. The gradient descent equations for $E_\rho(u,v)$ are:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -2\nabla v \cdot \nabla u + (1-v)\|\nabla u\|\; curv(u) \\
&\quad - \frac{\beta}{\alpha(1-v)}\|\nabla u\|\frac{(u-g)}{|u-g|} \\
\frac{\partial v}{\partial t} &= \nabla^2 v - \frac{v}{\rho^2} + \frac{2\alpha}{\rho}(1-v)\|\nabla u\| \\
\frac{\partial u}{\partial n}|_{\partial R} &= 0;\;\; \frac{\partial v}{\partial n}|_{\partial R} = 0
\end{aligned} \tag{15}$$

The second equation is very similar to the equation in Ambrosio-Tortorelli system (4) and was already proposed in [17,18]. The first equation is quite different. It in fact prescribes the three components of the velocity with which the level curves of u move. The first two terms are the same as before in (11). The last term moves the level curve with a point velocity of $\pm\beta/\alpha(1-v)$. The sign is automatically chosen such that this component of velocity pushes the level curve towards the corresponding level curve of g. If v is approximately constant along each level curve, then the last term may be seen to correspond to the constant velocity component used by Kimia et al. Thus the equation may be seen as combining all three types of velocity components described in the introduction.

The different curve evolution schemes discussed in the introduction may now be viewed as special cases of the gradient descent equations (15), obtained by introducing appropriate simplifications. Recall that common to all such schemes is the assumption that the edge-strength function v can be determined *externally*, thus breaking the coupling between u and v. Hence, given v, the only equation to consider is the evolution equation for u. Let $\bar{\kappa}$ denote the parameter β/α. We have the following cases.

1. **Noise filtering by smoothing of level curves [1]:** Set v identically equal to zero. Setting the initial $u = g$, the level curves of u evolve as follows. The level curves of u start moving away from the level curves of g at points where $curv(g) > \bar{\kappa}$. In a steady state, at each point, $curv(u) \leq \bar{\kappa}$ everywhere and where $curv(u) < \bar{\kappa}$, $u = g$. Thus, the evolution wipes out all features with curvature greater than $\bar{\kappa}$.

2. **Entropy scale space [8]:** Again set $v = 0$. Let the initial u correspond to the surface in which the given shape is embedded as a level curve. Replace g by a constant value equal to *max*(initial u) or *min*(initial u) according to the desired sign of the constant velocity component.

3. **Recovery of shapes from noisy images:** Choose a fixed edge strength function v_o and let the initial u correspond to the surface in which the given initial curve is embedded as a level curve. To obtain the evolution described in the first approach [3,10,24], set v and g as in the case of the entropy scale space above, but multiply the entire right hand side by $(1 - v_o)$. Note that the equation is still a gradient descent of $E_\rho(u, 0)$ with respect to u. For the second approach [4,9,19,20], just set $\beta = 0$ and $v = v_o$.

Although $E(u,B)$ appears to be only a minor modification of $E_{MS}(u,B)$, it behaves in a fundamentally different way:

Shocks: The most important property of $E(u,B)$ is that in its numerical implementation via its elliptic approximation $E_\rho(u,v)$, u develops shocks and thus object boundaries are recovered *as actual discontinuities*. This is not true of $E_{AT}(u,v)$. The reason is that the evolution equation for u in [15] is parabolic only along the level curves of u; it is *hyperbolic* in the direction normal to the level curves. Note however that the edge-strength function, v, is still a continuous function and hence, the actual boundaries are to be recovered from the discontinuities of u.

Deblurring: To illustrate the deblurring capability of the new functional, consider the case of a 1–dimensional image domain R. Note that, in contrast to other formulations involving curve evolution, the new functional can be applied even to the one dimensional case. When R is one dimensional, the $curv(u)$ term drops out and the evolution equation for u becomes purely hyperbolic, governed by advection due to v and the "constant" velocity component. Assume that g is zero in the first third of R, increases linearly to 1 in the second and stays equal to 1 in the third. When the diffusion system (15) is applied to this example, v develops a unique maximum at the center of R, inducing motion of points on the graph of u towards the center of R, eventually producing a shock there. The steady state solution is a piecewise constant function with a unique discontinuity at the center of R. Since the penalty is higher for two breaks than a single large break combining the two, *we do not get multiple breaks* along the ramp as is the case with E_{MS}.

Singular Points: Singular points such as triple points are formed just as in the case of E_{MS}.

5. Anisotropic Diffusion

The purpose of this section is illustrate that just as the functional E_{MS} may be viewed as the underlying basis of many segmentation methods as shown in [11], the new segmentation functional may be used as a basis for deriving new approximate methods for speeding up the computation. As an example, a formulation analogous to the "anisotropic diffusion" of Perona and Malik [15] is derived below. The key variable is the edge-strength function v. Depending on how it is chosen, one can derive a variety of formulations. Let us begin by first deriving a typical Perona-Malik diffusion equation from the Ambrosio-Tortorelli system (3).

Set $\alpha = \alpha_0/\rho$ in (3) and consider the limit as $\rho \to 0$, keeping α_0 constant. A "naive" limit is obtained by multiplying the functional by ρ and simply setting $\rho = 0$ in it. The result is the following functional:

$$E_{PM}(u,v) = \int\int_R \left\{ \alpha_0 (1-v)^2 ||\nabla u||^2 + \frac{v^2}{2} \right\} \tag{16}$$

The corresponding gradient descent equations are:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla \cdot (1-v)^2 \nabla u \\ \frac{\partial v}{\partial t} &= 2\alpha_0 (1-v) ||\nabla u||^2 - v \end{aligned} \tag{17}$$

To specialize this system to the Perona-Malik's diffusion, choose an appropriate v as a function of $||\nabla u||$ and let u evolve. An efficient choice for v is the one which minimizes E_{PM} for a fixed u. This is easily obtained by setting the right-hand side in the second equation equal to zero:

$$v = \frac{2\alpha_0 ||\nabla u||^2}{1 + 2\alpha_0 ||\nabla u||^2} \tag{18}$$

Substituting this choice in the first equation, we get

$$\frac{\partial u}{\partial t} = \nabla \cdot \frac{1}{\left(1 + 2\alpha_0 ||\nabla u||^2\right)^2} \nabla u \tag{19}$$

which is a Perona-Malik type of equation. As is well-known, the equation is not well-posed. This is also reflected by the fact that the infimum of E_{PM} for *any* piecewise constant function u is zero. Ambrosio and Tortorelli prove that the correct limit of E_{AT} as $\rho \to 0$ with $\alpha = \alpha_0/\rho$ is the functional

$$\beta \int\int_{R \backslash B} (u-g)^2 dx dy + |B| \tag{20}$$

in which u is restricted to being constant on each connected component of $R \backslash B$. In this sense, Equations (4), with $\alpha = \alpha_0/\rho$ may be regarded as a regularization of the Perona-Malik equation.

The trouble with regularization of the Perona-Malik equation is that it no longer can produce shocks, that is, sharp discontinuities in u. The new segmentation functional offers a way to circumvent this difficulty. To derive a Perona-Malik type diffusion from (13), set $\alpha = \alpha_0/\rho$ in (13). The corresponding evolution equation for u obtained by determining the "naive" limit as above as $\rho \to 0$ is:

$$\frac{\partial u}{\partial t} = ||\nabla u|| \nabla \cdot \frac{1}{(1 + 2\alpha_0 ||\nabla u||)} \frac{\nabla u}{||\nabla u||} \tag{21}$$

which is again ill-posed as expected and Equations (13) provide its regularization.

An interesting alternative to (21) is to set $\beta = 0$ in (15). Experimental results show that solution is a piecewise constant u whose discontinuity locus is *piecewise linear*.

6. Illustrative Examples

The first example is a synthetic image which is chosen to test specific boundary detection issues. The stochastic interpretation of E_{MS} makes it clear that although it is applied to general images in practice, the underlying assumption is that the true image is piecewise constant and the actual image is its corruption by Gaussian noise.Thus to test the correctness of the model, the chosen image should have a piecewise constant image as the underlying true image. (The new functional proposed here replaces the Gaussian model of noise with the Laplacian noise.) The issue is the effect of noise on the placement of the boundaries. The reason for inventing nonlinear filtering in the first place is that nearby edges interact when a linear filter is applied, resulting in their displacement or even merging. The higher the level of noise, the higher the degree of inaccuarcy in boundary placement. Therefore, in the example below, an unrealistically low signal-to-noise ratio is used and a thin object is included to test the interaction between boundaries.

Figure 1a shows the ideal, noiseless image. It consists of 4 squares in the four corners of R, a large ellipse in the middle and a long, thin ellipse nearby. The image is represented on a 256×256 square lattice. The thin ellipse has the maximum width of 9 pixels and the distance between the two ellipses is 18 pixels. Figure 1b is a dithered rendition of the noisy version obtained from the image in Figure 1a by adding Gaussian noise with pixel values ranging from 0 to 255. The signal-to-noise ratio (i.e. the ratio between the standard deviation of image with noise removed and the standard deviation of noise) is 1:4.

Let $\sigma = \sqrt{\alpha/\beta}$ in the original functional $E_{MS}(u, B)$ and $\sigma = \alpha/\beta$ in the new functional $E(u, B)$. Then, σ may be interpreted as the nominal smoothing radius in both cases. Figure 2 depicts the result of implemetation of $E_{MS}(u, B)$ by equations (4) with $\sigma = 8$ and $\rho = 4$ pixels. Figure 2a shows the smoothed image u and Figure

2b depicts the edge strength v. The lighter the area, the higher the value of v. The difficulty of recovering the actual shape boundaries from v is evident.

Figure 3 depicts the result of implementation of $E(u, B)$ by equations (15) with $\sigma = 8$ and $\rho = 4$ pixels. The numerical scheme of Osher and Sethian was used. Unlike the Ambrosio-Tortorelli system (4) which is quite robust with respect to the initial values of u and v and the relative speed with which the two diffusion processes are allowed to proceed, the diffusion system (15) crucially depends on these choices. The reason is that there is a maximum principle which says that the maximum of u cannot exceed the maximum of initial u. In the extreme case, if we choose initial u equal to constant, the first equation in (15) is always satisfied and the second equation drives v to zero. More generally, if we let the initial image diffuse freely for a while by setting $v = 0$, and then activate the second equation, whatever level curves are removed by initial diffusion can never be recovered. In the experiments described below, the initial u was set equal to g and initial v was estimated by applying (7) to a smoothed version of the image. The two diffusion processes were run alternately, each for 8 time steps at a time before switching to the other.

Figure 3a shows the smoothed image u and Figure 3b depicts the edge strength v. Formation of junctions (introduced by accidental noise features) are clearly visible in Figure 3b, especially in the upper left square. Figure 4 shows cross-sections of u. The top graph is a horizontal cross-section though the middle of the top two squares while the bottom graph is a horizontal cross-section though the middle of the whole figure. Figure 4a shows the case of diffusion by system (4) while Figure 4b shows the case of diffusion by system (15). Note the sharp discontinuities in the second case. Figure 5 shows the boundaries obtained by global thresholding of the smoothed image u. The boundary deviates in places from the ideal boundary when it follows some accidental feature introduced by the noise. The worst deviation occurs towards the ends of the thin ellipse. The trouble is that end portions of the ellipse have become almost disconnected from the main body by the noise. (It is possible to discern this by blocking out the main body of the ellipse and looking only at its end portions.)

The second example is that of an MRI scan of the brain, shown in Figure 6. The corresponding u, obtained by applying the diffusion system (15) with $\sigma = 8$ and $\rho = 4$ is shown in Figure 7. Boundaries obtained as level curves corresponding to two separate values of u are shown in Figure 8.

FIGURE 1a **FIGURE 1b**

FIGURE 2a **FIGURE 2b**

FIGURE 3a **FIGURE 3b**

FIGURE 4a **FIGURE 4b**

FIGURE 5 **FIGURE 6**

FIGURE 7 **FIGURE 8**

7. References

[1] L. Alvarez, F. Guichard, P.L. Lions and J.M. Morel: "Axioms and Fundamental Equations of Image Processing", Arch.for Rat.Mech. (1993).

[2] L. Ambrosio and V.M. Tortorelli: "On the Approximation of Functionals depending on Jumps by Quadratic, Elliptic Functionals", Boll. Un. Mat. Ital. (1992).

[3] V. Caselles, F. Catte, T. Coll and F. Dibos: "A Geometric Model for Active Contours", Numerische Mathematik 66, (1993).

[4] V. Caselles, R. Kimmel and G. Sapiro: "Geodesic Active Contours", Fifth International Conference on Computer Vision, (1995).

[5] L. Cohen: "On Active Contour Models and Balloons", CVGIP: Image Understanding, **53(2)**, (Mar, 1991).

[6] B.B. Kimia, A. Tannenbaum and S.W. Zucker: "On the Evolution of Curves via a Function of Curvature. I. The Classical Case", J. Math. Anal. and Appl. v.163, no.2, (January, 1992).

[7] B.B. Kimia, A. Tannenbaum and S.W. Zucker: "Shapes, Shocks and Deformations I: The Components of Shape and the Reaction-Diffusion Space", International Journal of Computer Vision, (1992).

[8] B.B. Kimia, A. Tannenbaum and S.W. Zucker: "Entropy Scale-Space", in *Visual Form*, Ed. C. Arcelli et al, Plenum.

[9] S. Kichenassamy, A. Kumar, P. Olver, A. Tannenbaum and A. Yezzi: "Gradient Flows and Geometric Active Contour Models", Fifth International Conference on Computer Vision, (1995).

[10] R. Malladi, J.A. Sethian and B.C. Vemuri: "Shape Modeling with Front Propagation: A Level Set Approach", IEEE-PAMI 17, (1995).

[11] J.-M. Morel and S. Solimini: Variational Methods in Image Processing, Birkhäuser, (1995).

[12] D. Mumford and J. Shah: "Boundary detection by minimizing functionals, I", Proc. IEEE Conf. on Computer Vision and Pattern Recognition, San Francisco (1985)

[13] S. Osher and J. Sethian: "Fronts Propagating with Curvature Dependent Speed: Algorithms based on the Hamilton-Jacobi Formulation", J. Comp. Physics, 79, (1988).

[14] P.J. Olver, G. Sapiro and A. Tannenbaum: "Differential Invariant Signatures and Flows in Computer Vision, A Symmetry Group Approach", Geometry Driven Diffusion in Computer Vision, Kluwer, (1994).

[15] P. Perona and J. Malik: "Scale-Space and Edge Detection Using Anisotropic Diffusion", Preprint, Dept. of Electrical Engineering and Computer Science, Univ. of Calif, Berkeley, (December,1988)

[16] G. Sapiro: "Color Snakes", IEEE Conf. on Vision and Pattern Recognition, (1996)

[17] J. Shah: "Segmentation by Nonlinear Diffusion", IEEE Conf. on Vision and Pattern Recognition, (1991)

[18] J. Shah: "Segmentation by Nonlinear Diffusion, II", IEEE Conf. on Vision and Pattern Recognition, (1992)

[19] J. Shah: "Uses of Elliptic Approximations in Computer Vision", to appear in *Variational Methods for Discontinuous Structures*, Ed: R. Serapioni and F. Tomarelli, Birkhäuser.

[20] J. Shah: "Shape Recovery from Noisy Images by Curve Evolution" IASTED International Conf. on Signal and Image Processing, (Nov. 1995).

[21] J. Shah: "Curve Evolution and Segmentation Functionals: Application to Color Images", IEEE International Conference on Image Processing, (Sept. 1996)

[22] K. Siddiqi and B.B. Kimia: "Parts of Visual Form: Computational Aspects", IEEE Conf. on Vision and Pattern Recognition, (1993).

[23] S. Tari, J. Shah and H. Pien: "A Computationally Efficient Shape Analysis via Level Sets", IEEE Workshop on Mathematical Methods in Biomedical Image Analysis, (June, 1996)

[24] H. Tek and B.B. Kimia: "Image Segmentation by Reaction-Diffusion Bubbles", Fifth International Conference on Computer Vision, (1995).