ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

PHILIPPE BENILAN
HAIM BREZIS
MICHAEL G. CRANDALL

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 2, nº 4 (1975), p. 523-555

<http://www.numdam.org/item?id=ASNSP_1975_4_2_4_523_0>

A Semilinear Equation in $L^1(\boldsymbol{R}^N)$.

PHILIPPE BENILAN (*) - HAIM BREZIS (**)
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Summary. – *The problem $\beta(u) - \Delta u \ni f$ is studied where $f \in L^1(\boldsymbol{R}^N)$ and β is a maximal monotone graph in $\boldsymbol{R}$ with $0 \in \beta(0)$. If $N \geqslant 3$ the problem is shown to have a unique solution in some Marcinkiewicz space. If $0 \in \operatorname{int} \beta(\boldsymbol{R})$ and $N = 1, 2$ solutions unique up to a constant are obtained; in case $0 \notin \operatorname{int} \beta(\boldsymbol{R})$, it may happen that no solution exists. Finally it is proved that, under some assumptions the solution has a compact support.*

Introduction.

Let β be a maximal monotone graph in $\boldsymbol{R}$ with $0 \in \beta(0)$. In particular, β could be any continuous nondecreasing function on $\boldsymbol{R}$ vanishing at 0. This paper treats the problem

$$(\mathrm{P}) \qquad -\Delta u + \beta(u) \ni f \quad \text{ on } \boldsymbol{R}^N$$

for given $f \in L^1(\boldsymbol{R}^N)$. The problem (P) is considerably more delicate than the regularized version

$$(\mathrm{P}_\varepsilon) \qquad \varepsilon u_\varepsilon - \Delta u_\varepsilon + \beta(u_\varepsilon) \ni f \quad \text{ on } \boldsymbol{R}^N \ (\varepsilon > 0),$$

which falls within the scope of [2]. The estimates $\varepsilon\|u_\varepsilon\|_{L^1} \leqslant \|f\|_{L^1}$ and $\|\Delta u_\varepsilon\|_{L^1} \leqslant 2\|f\|_{L^1}$ are easy to obtain for (P_ε) and they are crucial in the existence and uniqueness proofs. The solutions u of (P) to be obtained here

(*) Université de Paris VI. The work of Benilan was supported in part by NSF Grant GP-38519.
(**) Université de Paris VI.
(***) M.R.C. University of Wisconsin. The work of Crandall was supported in part by NSF Grant GP-38519 and US Army Contract DA-31-124-ARO-D-462.
Pervenuto alla Redazione l'11 Marzo 1975.

will not lie in $L^1(\boldsymbol{R}^N)$ in general, and we will need to use the properties of Δ^{-1} considered as an operator on $L^1(\boldsymbol{R}^N)$ in a very precise way to find suitable estimates on u. Therefore it is not surprising that the fundamental solution of the Laplacian will play a prominent role. In particular, it will be necessary to handle the cases $N=1$, $N=2$ and $N\geqslant 3$ separately. When $N=1$ or $N=2$ we will require some coerciveness from the nonlinear term (namely, $0\in\operatorname{int}\beta(\boldsymbol{R})$).

The main results are summarized below ($M^p(\boldsymbol{R}^N)$ denotes the Marcinkiewicz (or weak-L^p) space (see the Appendix)).

$N\geqslant 3$. For every $f\in L^1(\boldsymbol{R}^N)$ there exists a unique $u\in M^{N/(N-2)}(\boldsymbol{R}^N)$ with $\Delta u\in L^1(\boldsymbol{R}^N)$ satisfying (P).

$N=2$. Let $0\in\operatorname{int}\beta(\boldsymbol{R})$. Then for every $f\in L^1(\boldsymbol{R}^2)$ there is a $u\in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2)$ with $|\operatorname{grad} u|\in M^2(\boldsymbol{R}^2)$ and $\Delta u\in L^1(\boldsymbol{R}^2)$ satisfying (P). In addition, two solutions in this class differ by a constant.

$N=1$. Let $0\in\operatorname{int}\beta(\boldsymbol{R})$. Then for every $f\in L^1(\boldsymbol{R})$ there exists a $u\in W^{1,\infty}(\boldsymbol{R})$ with $d^2u/dx^2\in L^1(\boldsymbol{R})$ satisfying (P). In addition, two solutions in this class differ by a constant.

The plan of the paper is as follows: Some preliminary results and notations are collected in Section 1. The second section develops the general results for $N\geqslant 3$. The third and fourth sections deal with the cases $N=2$ and $N=1$. Section 5 discusses conditions on β under which (P) has a solution $u\in L^p(\boldsymbol{R}^N)$ (for all $N\geqslant 1$). Section 6 considers conditions on f and β under which (P) has a solution with compact support; in this section f need not be in $L^1(\boldsymbol{R}^N)$. We conclude with an appendix describing some properties of the Marcinkiewicz spaces and the Laplacian.

1. – Preliminaries.

We begin this section with some of the notation and definitions used later. If $\Omega\subset\boldsymbol{R}^N$ is Lebesgue measurable, meas Ω denotes its measure. If $f\in L^1(\Omega)$, $\int_\Omega f$ denotes the integral of f over Ω with respect to Lebesgue measure and this is shortened to $\int f$ if $\Omega=\boldsymbol{R}^N$. When it is necessary to indicate the variable of integration we sometimes write $\int_\Omega f(x)\,dx$, etc. The norm in $L^p(\boldsymbol{R}^N)$ is denoted by $\|\ \|_{L^p}$, $1\leqslant p\leqslant\infty$; $M^p(\boldsymbol{R}^N)$, $1<p<\infty$, denotes the Marcinkiewicz space and $\|\ \|_{M^p}$ is its norm (see the Appendix). If u is a function on $\boldsymbol{R}^N$, $[|u|>\lambda]$ denotes $\{x\in\boldsymbol{R}^N\colon |u(x)|>\lambda\}$, etc.

If $k\geqslant 0$ is an integer and $1\leqslant p\leqslant\infty$, $W^{k,p}(\Omega)$ is the Sobolev space of functions u on the open set $\Omega\subseteq\boldsymbol{R}^N$ for which $D^l u\in L^p(\Omega)$ when $|l|\leqslant k$ with its usual norm. $W^{k,p}_0(\Omega)$ is the closure of $\mathcal{D}(\Omega)=C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$. Also, if

$p = 2$ we write H^k for $W^{k,2}$. A function u lies in $W^{k,p}_{\mathrm{loc}}(\Omega)$ if $\zeta u \in W^{k,p}(\Omega)$ for all $\zeta \in \mathfrak{D}(\Omega)$.

Some special classes of functions on $\boldsymbol{R}$ we will use are the cones: $\mathfrak{J}_0 = \{j : \boldsymbol{R} \to [0, \infty] : j$ is convex, lower semi-continuous and $j(0) = 0\}$,

$$\mathfrak{I} = \{p \in C^1(\boldsymbol{R}) \cap L^\infty(\boldsymbol{R}) : p \text{ is nondecreasing}\},$$

and

$$\mathfrak{I}_0 = \{p \in \mathfrak{I} : p(0) = 0\}.$$

Finally ζ_0 will be a fixed function in $\mathfrak{D}(\boldsymbol{R}^N)$ such that $0 \leqslant \zeta_0 \leqslant 1$, $\zeta_0(x) = 1$ if $|x| \leqslant 1$ and $\zeta_0(x) = 0$ if $|x| \geqslant 2$. For $n \geqslant 1$, $\zeta_n(x) = \zeta_0(n^{-1}x)$.

Given $f \in L^1(\boldsymbol{R}^N)$ we say that u in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ is a solution of (P) provided that $\Delta u \in L^1(\boldsymbol{R}^N)$ (in the sense of distributions) and $f(x) + \Delta u(x) \in \beta(u(x))$ a.e. on $\boldsymbol{R}^N$. If $\mathfrak{L}$ is a subset of $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ then (P) is said to be *well-posed in* $\mathfrak{L}$ if the following conditions hold:

(I) If $f \in L^1(\boldsymbol{R}^N)$, then (P) has at least one solution $u \in \mathfrak{L}$. We set $G_\beta f = \{u \in \mathfrak{L} : u$ is a solution of (P)$\}$.

(II) $T_\beta f = \{f + \Delta u : u \in G_\beta f\}$ has exactly one element for $f \in L^1(\boldsymbol{R}^N)$.

(III) $\int j(T_\beta f) \leqslant \int j(f)$ for every $f \in L^1(\boldsymbol{R}^N)$ and $j \in \mathfrak{J}_0$.

(IV) $\int (T_\beta f - T_\beta \hat{f})^+ \leqslant \int (f - \hat{f})^+$, for $f, \hat{f} \in L^1(\boldsymbol{R}^N)$ where $r^+ = \max(r, 0)$.

REMARKS. The definitions of G_β and T_β formally depend on $\mathfrak{L}$, but we will not indicate this dependence explicitly. (III) implies that $T_\beta f \in L^1(\boldsymbol{R}^N)$ if $f \in L^1(\boldsymbol{R}^N)$ by choosing $j(r) = |r|$, while (IV) implies that $T_\beta f \geqslant T_\beta \hat{f}$ if $f \geqslant \hat{f}$ and (interchanging f and $\hat{f}$) $\int |T_\beta f - T_\beta \hat{f}| \leqslant \int |f - \hat{f}|$. Thus T_β is an order-preserving contraction on $L^1(\boldsymbol{R}^N)$ if (P) is well-posed in $\mathfrak{L}$. The requirements (III) and (IV) are natural in this problem and are motivated by the results of Brezis and Strauss [2] to which we refer for references to previous related works. It will be shown that (P) is well-posed in $M^{N/(N-2)}(\boldsymbol{R}^N)$ if $N \geqslant 3$, in $\{u \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2) : |\operatorname{grad} u| \in M^2(\boldsymbol{R}^2)\}$ if $N = 2$ and in $L^1_{\mathrm{loc}}(\boldsymbol{R})$ if $N = 1$.

We begin with a well-known linear result.

LEMMA 1.1. *For every $f \in L^1(\boldsymbol{R}^N)$ and every $\lambda > 0$ there is a unique $u \in L^1(\boldsymbol{R}^N)$ satisfying $u - \lambda \Delta u = f$ in $\mathfrak{D}'(\boldsymbol{R}^N)$. Moreover, $\|u\|_{L^1} \leqslant \|f\|_{L^1}$ and also*

$$\operatorname*{ess\,sup}_{\boldsymbol{R}^N} u \leqslant \max\{0, \operatorname*{ess\,sup}_{\boldsymbol{R}^N} f\}.$$

PROOF OF LEMMA 1.1. We give only an outline (employing elementary functional analysis rather than Fourier analysis). Suppose first that $f \in L^2(\boldsymbol{R}^N)$. Then the standard variational argument shows there is a unique $u \in H^1(\boldsymbol{R}^N)$ such that $u - \lambda \Delta u = f$. For any $p \in \mathcal{T}_0$ such that $p' \in L^\infty(\boldsymbol{R})$ one has $p(u) \in H^1(\boldsymbol{R}^N)$ and

$$\int u p(u) = \int f p(u) + \lambda \int (\Delta u) p(u) = \int f p(u) - \lambda \int p'(u) |\nabla u|^2 \leqslant \int f p(u) .$$

Choosing appropriate p's we easily deduce that

$$\operatorname*{ess\,sup}_{\boldsymbol{R}^N} u \leqslant \max \{0, \operatorname*{ess\,sup}_{\boldsymbol{R}^N} f\}$$

and $\|u\|_{L^1} \leqslant \|f\|_{L^1}$ for $f \in L^2(\boldsymbol{R}^N) \cap L^1(\boldsymbol{R}^N)$. For general $f \in L^1(\boldsymbol{R}^N)$ choose $f_n \in L^2(\boldsymbol{R}^N) \cap L^1(\boldsymbol{R}^N)$ so that $f_n \to f$ in $L^1(\boldsymbol{R}^N)$ (for example, $f_n = \min(n, \max(f, -n))$). The corresponding solutions u_n form a Cauchy sequence in $L^1(\boldsymbol{R}^N)$ (since $f \in L^1(\boldsymbol{R}^N) \cap L^2(\boldsymbol{R}^N) \mapsto u$ is a contraction in $L^1(\boldsymbol{R}^N)$). Therefore $u_n \to u \in L^1(\boldsymbol{R}^N)$ and u satisfies the conditions of Lemma 1.1. Finally we prove uniqueness. Suppose $u \in L^1(\boldsymbol{R}^N)$ satisfies $u - \lambda \Delta u = 0$. Let $\varrho \in \mathcal{D}(\boldsymbol{R}^N)$ and $\tilde{u} = \varrho * u$. Then $\tilde{u} \in C^\infty(\boldsymbol{R}^N) \cap H^1(\boldsymbol{R}^N)$ (since $\|\tilde{u}\|_{L^2} \leqslant \|\varrho\|_{L^2} \|u\|_{L^1}$ and $\|\operatorname{grad} \tilde{u}\|_{L^2} \leqslant \|\operatorname{grad} \varrho\|_{L^2} \|u\|_{L^1}$). Also $\tilde{u} - \lambda \Delta \tilde{u} = 0$. Consequently $\tilde{u} = \varrho * u = 0$ for all $\varrho \in \mathcal{D}(\boldsymbol{R}^N)$ and hence $u = 0$.

It follows from Lemma 1.1 that we can apply [2, Theorem 1] (see also Konishi [4]) with $Au = -\Delta u + \varepsilon u$, $D(A) = \{u \in L^1(\boldsymbol{R}^N) : \Delta u \in L^1(\boldsymbol{R}^N)\}$ to obtain the next lemma which is crucial for the existence proofs.

LEMMA 1.2. *Let $N \geqslant 1$ and $\varepsilon > 0$. For every $f \in L^1(\boldsymbol{R}^N)$ there is a unique $u_\varepsilon \in L^1(\boldsymbol{R}^N)$ with $\Delta u_\varepsilon \in L^1(\boldsymbol{R}^N)$ satisfying* (P_ε). *In addition,* (III) *and* (IV) *hold with β replaced by $\beta + \varepsilon I$.*

In other words, (P_ε), which is (P) with β replaced by $\beta + \varepsilon I$, is well-posed in $L^1(\boldsymbol{R}^N)$. In order to show convergence of the u_ε as $\varepsilon \to 0+$ we will use the following lemma.

LEMMA 1.3. *Let $N \geqslant 1$ and $f \in L^1(\boldsymbol{R}^N)$. Let u_ε be the solution of* (P_ε) *and $w_\varepsilon = T_{\beta+\varepsilon I} f = f + \Delta u_\varepsilon$. In addition, if $N = 1$ or 2, suppose that u_ε is bounded in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$. Then $\{[u_\varepsilon, w_\varepsilon] : \varepsilon > 0\}$ is precompact in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)^2$. Moreover, if $\varepsilon_n \to 0+$ and $[u_{\varepsilon_n}, w_{\varepsilon_n}] \to [u, w]$ in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)^2$, then $w = f + \Delta u \in L^1(\boldsymbol{R}^N)$, u is a solution of* (P), *and $\int j(w) \leqslant \int j(f)$ for every $j \in \mathcal{J}_0$. In addition:*

(1.4) *If $N \geqslant 3$, then $u \in M^{N/(N-2)}(\boldsymbol{R}^N)$.*

(1.5) *If $N \geqslant 2$, $u \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^N)$ and $|\operatorname{grad} u| \in M^{N/(N-1)}(\boldsymbol{R}^N)$.*

(1.6) If $N = 1$, $du/dx \in L^\infty(\boldsymbol{R})$.

PROOF OF LEMMA 1.3. By Lemma 1.2, $T_{\varepsilon I+\beta}$ is a contraction on $L^1(\boldsymbol{R}^N)$. Moreover, $T_{\varepsilon I+\beta}$ is clearly translation invariant and $0 = T_{\varepsilon I+\beta}0$. Thus $w_\varepsilon = T_{\varepsilon I+\beta}f$ satisfies $\|w_\varepsilon\|_{L^1} \leqslant \|f\|_{L^1}$ and $\int |w_\varepsilon(x+h) - w_\varepsilon(x)|dx \leqslant \int |f(x+h) - f(x)|dx$ for $h \in \boldsymbol{R}^N$. Thus $\{w_\varepsilon : \varepsilon > 0\}$ is precompact in $L^1_{\text{loc}}(\boldsymbol{R}^N)$. Also, by (III) for ($\mathrm{P}_\varepsilon$), $\int j(w_\varepsilon) \leqslant \int j(f)$ for $j \in \mathfrak{J}_0$. If $\varepsilon_n \to 0+$ and $w_{\varepsilon_n} \to w$ in $L^1_{\text{loc}}(\boldsymbol{R}^N)$ it then follows from Fatou's lemma that $\int j(w) \leqslant \int j(f)$ for $j \in \mathfrak{J}_0$. In particular, $w \in L^1(\boldsymbol{R}^N)$. Next, using Lemma A.5 if $N \geqslant 3$ and Lemma A.14 if $N = 2$ one finds

$$\|u_\varepsilon\|_{M^{N/(N-2)}} \leqslant c_N \|\Delta u_\varepsilon\|_{L^1} \leqslant 2c_N \|f\|_{L^1} \quad \text{if } N \geqslant 3$$

and

$$\|\operatorname{grad} u_\varepsilon\|_{M^{N/(N-1)}} \leqslant 2d_N \|f\|_{L^1} \quad \text{if } N \geqslant 2\,.$$

If $N \geqslant 3$ these estimates imply that u_ε is bounded in $W^{1,1}_{\text{loc}}(\boldsymbol{R}^N)$ and hence $\{u_\varepsilon : \varepsilon > 0\}$ is precompact in $L^1_{\text{loc}}(\boldsymbol{R}^N)$. $\big(M^p(\boldsymbol{R}^N) \subset L^1_{\text{loc}}(\boldsymbol{R}^N)$ with continuous injection if $1 < p < \infty\big)$. If $N = 2$, the same is true since u_ε is assumed to be bounded in $L^1_{\text{loc}}(\boldsymbol{R}^N)$. In addition, $\{\operatorname{grad} u_\varepsilon\}$ is also precompact in $L^1_{\text{loc}}(\boldsymbol{R}_N)$ since

$$\|\operatorname{grad} u_\varepsilon(\cdot + h) - \operatorname{grad} u_\varepsilon(\cdot)\|_{M^{N/(N-1)}} \leqslant 2d_N \|f(\cdot + h) - f(\cdot)\|_{L^1} \tag{1.9}$$

for $h \in \boldsymbol{R}^N$. Hence properties (1.4) and (1.5) are easily obtained from Fatou's lemma (see the remark following Definition A.1). The fact that $u = \lim_n u_{\varepsilon_n}$ is a solution of (P) is clear.

Finally, if $N = 1$ we have

$$\left\|\frac{d}{dx}u_\varepsilon\right\|_{L^\infty} \leqslant \left\|\frac{d^2}{dx^2}u_\varepsilon\right\|_{L^1} \leqslant 2\|f\|_{L^1} \left(\frac{du_\varepsilon}{dx}(\pm\infty) = 0 \text{ since } u_\varepsilon \in L^1(\boldsymbol{R})\right).$$

Therefore, $\{u_\varepsilon\}$ is precompact in $L^1_{\text{loc}}(\boldsymbol{R})$ as soon as it is bounded in $L^1_{\text{loc}}(\boldsymbol{R})$, and (1.6) is clear. The proof is complete.

Lemma 1.3 reduces the problem of showing (P) is well-posed in a class $\mathfrak{L}$ considerably. For $N \geqslant 3$ and $\mathfrak{L} = M^{N/(N-2)}(\boldsymbol{R}^N)$ it will suffice to show that solutions of $u \in \mathfrak{L}$ are unique. Then $T_\beta f = f + \Delta u$ is also unique and hence $w_\varepsilon = T_{\varepsilon I+\beta}f \to f + \Delta u$ in $L^1_{\text{loc}}(\boldsymbol{R}^N)$. IV then follows from Fatou's lemma. If $N = 2$ and $\mathfrak{L} = \{u \in W^{1,1}_{\text{loc}}(\boldsymbol{R}^2) : \operatorname{grad} u \in M^2(\boldsymbol{R}^2)\}$, or $N = 1$ and $\mathfrak{L} = L^1_{\text{loc}}(\boldsymbol{R})$, a bound on u_ε will first have to be obtained. Then it will suffice to show that two solutions of (P) in $\mathfrak{L}$ differ by a constant. For in this case $T_\beta f = f + \Delta u$ is still unique, and IV holds as above. The cases $N \geqslant 3$, $N = 2$ and $N = 1$ are treated separately below.

2. – $N \geqslant 3$.

The main result of this section is

THEOREM 2.1. *The problem* (P) *is well-posed in* $\mathcal{L} = M^{N/(N-2)}(\boldsymbol{R}^N)$ *and the solution* u *of* (P) *in* $\mathcal{L}$ *is unique* (i.e. G_β *is single-valued*). *There is a constant* C_N *depending only on* N *such that*

$$\|G_\beta f - G_\beta \hat{f}\|_{M^{N/(N-2)}} + \|\operatorname{grad}(G_\beta f - G_\beta \hat{f})\|_{M^{N/(N-1)}} \leqslant C_N \|f - \hat{f}\|_{L^1} \tag{2.2}$$

for f, $\hat{f} \in L^1(\boldsymbol{R}^N)$. *Moreover,* G_β *is order preserving.*

PROOF OF THEOREM 2.1. By the preceding remarks (P) is well-posed in $\mathcal{L}$ if solutions $u \in \mathcal{L}$ are unique. Let u_1, $u_2 \in \mathcal{L}$ be solutions of (P), $u = u_1 - u_2$ and $w = \Delta(u_1 - u_2)$. Then $u \in \mathcal{L}$ and $w \in L^1(\boldsymbol{R}^N)$ and $uw \geqslant 0$ a.e. on $\boldsymbol{R}^N$ (by the monotonicity of β). It follows from Lemma A.10 that for $p \in \mathcal{T}_0$

$$\int p'(u)|\operatorname{grad} u|^2 + \int wp(u) \leqslant 0 .$$

Since $wp(u) \geqslant 0$, $\operatorname{grad} u = 0$ and u is a constant function in $M^{N/(N-2)}(\boldsymbol{R}^N)$. But then $u = 0$.

If $u = G_\beta f$, $\hat{u} = G_\beta \hat{f}$, IV implies

$$\|\Delta(u - \hat{u})\|_{L^1} \leqslant 2\|f - \hat{f}\|_{L^1}$$

and then (2.2) is a consequence of Lemma A.5. Finally G_β is order preserving since $G_\beta f = \lim_{\varepsilon \downarrow 0} G_{\varepsilon I + \beta} f$ in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ and $G_{\varepsilon I + \beta}$ is order preserving (see [2]).

REMARK. (P) is well-posed in any subspace $\mathcal{L}$ of $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ such that

$$\text{(i)} \qquad M^{N/(N-2)}(\boldsymbol{R}^N) \subset \mathcal{L}$$

$$\text{(ii)} \qquad u \in \mathcal{L} \text{ and } \Delta u = 0 \text{ implies } u = 0 .$$

Indeed, it suffices to show a solution $u \in \mathcal{L}$ in fact lies in $M^{N/(N-2)}(\boldsymbol{R}^N)$. Let $u \in \mathcal{L}$, $\hat{u} \in M^{N/(N-2)}(\boldsymbol{R}^N)$ be solutions and $v \in M^{N/(N-2)}(\boldsymbol{R}^N)$ satisy $\Delta v = \Delta(u - \hat{u})$. Then $(v - u + \hat{u}) \in \mathcal{L}$ and $\Delta(v - u + \hat{u}) = 0$, so $u = v + \hat{u} \in M^{N/(N-2)}(\boldsymbol{R}^N)$. Interesting examples of choices $\mathcal{L}$ satisfying (i) and (ii) are the following:

$$\mathcal{L}_1 = \left\{ u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N) : \lim_{n \to \infty} \int\limits_{1 \leqslant |x| \leqslant 2} |u(nx)|\, dx = 0 \right\} .$$

To check (i) observe that $M^p(\boldsymbol{R}^N)\subset\mathfrak{L}_1$ for every $1<p<\infty$, while (ii) follows from Lemma A.8. Another class is

$$\mathfrak{L}_2=\left\{u\in L^1_{\mathrm{loc}}(\boldsymbol{R}^N):\int\frac{1}{(1+|x|)^\alpha}|u(x)|\,dx<\infty\right\}$$

where $2<\alpha\leqslant N$. (Related spaces are considered in Nirenberg and Walker [5]). Indeed, to check (i) note that

$$\int_{\boldsymbol{R}^n}\frac{1}{(1+|x|)^\alpha}|u(x)|\,dx\leqslant\int_{|x|\leqslant 1}|u(x)|\,dx+$$
$$+\sum_{k=0}^{\infty}\int_{2^k\leqslant|x|\leqslant 2^{k+1}}\frac{1}{2^{k\alpha}}|u(x)|\,dx\leqslant$$
$$\leqslant C\|u\|_{M^{N/(N-2)}}\left(1+\sum_{k=0}^{\infty}\frac{2^{(k+1)2}}{2^{k\alpha}}\right)\leqslant C_1\|u\|_{M^{N/(N-2)}}\,.$$

On the other hand $\mathfrak{L}_2\subset\mathfrak{L}_1$ since

$$\int_{1\leqslant|x|\leqslant 2}|u(nx)|\,dx\leqslant\frac{C}{n^N}\int_{n\leqslant|y|\leqslant 2n}|u(y)|\,dy\leqslant C_2\int_{n\leqslant|y|\leqslant 2n}\frac{1}{(1+|y|)^N}|u(y)|\,dy$$

and the right hand side tends to zero as $n\to\infty$ if $u\in\mathfrak{L}_2$.

3. - $N=2$.

The main result of this section is

THEOREM 3.1. *Assume* $0\in\operatorname{int}\beta(\boldsymbol{R})$. *Then* (P) *is well-posed in the class*

$$\mathfrak{L}=\{u\in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2):|\operatorname{grad}u|\in M^2(\boldsymbol{R}^2)\}\,.$$

In addition, two solutions of (P) *in* $\mathfrak{L}$ *differ by a constant and there exists* C *such that*

$$\|\operatorname{grad}(G_\beta f-G_\beta\hat f)\|_{M^2}\leqslant C\|f-\hat f\|_{L^1}\quad\text{for } f,\hat f\in L^1(\boldsymbol{R}^2)\,. \tag{3.2}$$

Also G_β *maps bounded subsets of* $L^1(\boldsymbol{R}^2)$ *into bounded subsets of* $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^N)$ *for* $1\leqslant p<2$. *Finally we have*

$$\int T_\beta f=\int f\quad\text{for } f\in L^1(\boldsymbol{R}^2)\,. \tag{3.3}$$

PROOF OF THEOREM 3.1. We begin by showing the uniqueness up to a constant. Let $\lambda > 0$ be large enough so that $0 \notin \beta(\lambda)$ and $0 \notin \beta(-\lambda)$. Suppose $u_1, u_2 \in \mathcal{L}$ are two solutions of (P). We are going to prove that $\operatorname{grad}(u_1 - u_2) = 0$. First observe that meas $[|u_i| > \lambda] < \infty$ for $i = 1, 2$, (since $f + \Delta u_i \in \beta(u_i)$ a.e. and $f + \Delta u_i \in L^1(\boldsymbol{R}^2)$) so that meas $[|u_1 - u_2| > 2\lambda] < \infty$. If $u = u_1 - u_2$, $w = \Delta(u_1 - u_2)$ we have $u \in \mathcal{L}$, meas $[|u| > 2\lambda] < \infty$, $w \in L^1(\boldsymbol{R}^2)$ and $u \cdot w \geqslant 0$ a.e. It follows from Lemma A.10 that $\operatorname{grad} u = 0$. To prove that (P) is well-posed in $\mathcal{L}$ it remains to show (in view of the remarks after Lemma 1.3) that the solution u_ε of (P$_\varepsilon$) remains bounded in $L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$ as $\varepsilon \to 0+$. However, with the same reasoning and λ as above, meas $[|u_\varepsilon| > \lambda]$ is bounded by a constant μ independent of ε. Therefore, by Lemma A.16 and the fact that $\operatorname{grad} u_\varepsilon$ is bounded in $M^2(\boldsymbol{R}^2)$ we conclude that $\|u_\varepsilon\|_{L^1(B)}$ is bounded provided B is a ball such that meas $B > \mu$. The inequality (3.2) is a consequence of IV and Lemma A.11 while (3.3) follows from Lemma A.13. Finally, suppose f lies in a bounded subset of $L^1(\boldsymbol{R}^2)$ and let $u \in G_\beta(f)$. Then we have

$$\|T_\beta f\|_{L^1} \leqslant \|f\|_{L^1} \leqslant C \qquad \text{and} \qquad \|\operatorname{grad} u\|_{M^2} \leqslant 2d_2 \|f\|_{L^1} \leqslant C_1$$

for some C, C_1. The same argument as above shows that u is bounded in $L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$. Moreover, $|\operatorname{grad} u|$ is bounded in $L^p_{\mathrm{loc}}(\boldsymbol{R}^2)$ for $1 \leqslant p < 2$ by Lemma A.2. It follows that u is bounded in $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^2)$ for $1 \leqslant p < 2$.

REMARK. It is clear that (P) is well-posed in any subspace $\mathcal{L}$ of $W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2)$ such that

$$\text{(i)} \qquad \mathcal{L} \subset \{u \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2) : |\operatorname{grad} u| \in M^2(\boldsymbol{R}^2)\}$$

and

$$\text{(ii)} \qquad u \in \mathcal{L} \text{ and } \Delta u \in L^1(\boldsymbol{R}^2) \text{ imply } |\operatorname{grad} u| \in M^2(\boldsymbol{R}^2)\,.$$

Examples of such classes are

$$\mathcal{L}_1 = \Big\{u \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2) : \lim_{n \to \infty} \int_{1 \leqslant |x| \leqslant 2} |(\operatorname{grad} u)(nx)|\, dx = 0\Big\}$$

(see Lemma A.11) and

$$\mathcal{L}_2 = \Big\{u \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2) : \int \frac{1}{(1 + |x|)^\alpha} |\operatorname{grad} u(x)|\, dx < \infty\Big\}$$

where $1<\alpha\leqslant 2$. To check (i) for $\mathfrak{L}_2$, note that

$$\int_{\boldsymbol{R}^2}\frac{1}{(1+|x|)^\alpha}\,|v(x)|\,dx\leqslant\int_{|x|\leqslant 1}|v(x)|\,dx+\sum_{k=0}^{\infty}\int_{2^k\leqslant|x|\leqslant 2^{k+1}}\frac{1}{2^{\alpha k}}\,|v(x)|\,dx\leqslant C\|v\|_{M^2}\,.$$

On the other hand, $\mathfrak{L}_2\subset\mathfrak{L}_1$ so (ii) holds for $\mathfrak{L}_2$.

We now take a more detailed look at the question of uniqueness of solutions u of (P) in the $\mathfrak{L}$ of Theorem 3.1. While this is settled completely below, we first state a result giving two interesting criteria under which solutions of (P) are unique:

PROPOSITION 3.4. *Under the assumptions of Theorem* 3.1, (P) *has a unique solution* $u=G_\beta f\in\mathfrak{L}$ *provided either* $\int f\neq 0$ *or* $\beta^{-1}(0)=\{0\}$.

For the proof we will need the following lemma:

LEMMA 3.5. *Let* β *be a maximal monotone graph in* $\boldsymbol{R}$, $0\in\beta(0)$, $p>1$, $u\in W^{1,p}_{\text{loc}}(\boldsymbol{R}^M)$, $M\geqslant 1$, $c\in\boldsymbol{R}$ *and* $w(x)\in\beta\big(u(x)\big)\cap\beta\big(u(x)+c\big)$ a.e. *If* $w\in L^1(\boldsymbol{R}^M)$, *then either* $w=0$ *or* $c=0$.

PROOF. Let $j\in\mathfrak{J}_0$ be such that $\partial j=\beta$ where ∂j is the subdifferential of j. By the definition of subdifferential

$$j\big(u(x)+c\big)-j\big(u(x)\big)\geqslant w(x)c \qquad \text{a.e. } x\in\boldsymbol{R}^M$$

$$j\big(u(x)\big)-j\big(u(x)+c\big)\geqslant w(x)(-c) \quad \text{a.e. } x\in\boldsymbol{R}^M\,.$$

Thus $j(u+c)-j(u)=wc$. Next we show that $j(u+c)-j(u)$ is constant. Since $w\in L^1(\boldsymbol{R}^M)$ this completes the proof. First assume that $\beta(\boldsymbol{R})$ is bounded. Then j is Lipschitz continuous and $j(u+c)$, $j(u)\in W^{1,p}_{\text{loc}}(\boldsymbol{R}^M)$. Moreover, $\operatorname{grad}\big(j(u+c)-j(u)\big)=w\operatorname{grad}(u+c-u)=0$ a.e. $\big($since $u\in W^{1,p}_{\text{loc}}(\boldsymbol{R}^M)$ implies u has partial derivatives in the usual sense a.e.$\big)$. If β is not bounded, let β_A be β truncated above of A and below at $-A$ $\big($an explicit formula is $\beta_A=(\partial I_{[-A,A]}+\beta^{-1})^{-1}$ where I_K is the indicator function of $K\big)$, and w_A the truncation of w. Then $w_A\in\big(\beta_A(u+c)\cap\beta_A(u)\big)$ a.e. By the above, $w_A=0$ or $c=0$. The proof is complete since $w_A=0$ for some $A>0$ implies $w=0$.

If u_1 and u_2 are two solutions of (P) then $w=f+\Delta u_1=f+\Delta u_2\in\beta(u_1)=$ $=\beta(u_2+c)$ a.e. where $c=u_1-u_2$ is a constant by Theorem 3.1. Since $w\in L^1(\boldsymbol{R}^2)$, either $w=0$ so $-\Delta u_i=f$ or $w\neq 0$ and $c=0$ by the preceding lemma. Thus solutions of (P) are not unique if and only if there exist $v\in L^\infty(\boldsymbol{R}^2)$ such that $\Delta v=f$ and $2\|v\|_{L^\infty}<\operatorname{meas}\beta^{-1}(0)$. Proposition 3.4 now follows from Lemmas A.15 and A.13.

The next result shows that G_β is as order preserving as it can be, given that it is not necessarily single-valued.

PROPOSITION 3.6. *Let* f, $\hat{f} \in L^1(\boldsymbol{R}^2)$ *with* $f \leqslant \hat{f}$ a.e. *and* $f \neq \hat{f}$, $u \in G_\beta f$, *and* $\hat{u} \in G_\beta \hat{f}$ *then* $u \leqslant \hat{u}$ a.e.

PROOF OF PROPOSITION 3.6. Let $p \in \mathcal{T}_0$ satisfy $p(r) = 0$ for $r \leqslant 0$ and $p'(r) > 0$ for $r > 0$. It follows from Lemma A.13 (applied to $u - \hat{u}$) that

$$\int p'(u-\hat{u})|\operatorname{grad}(u-\hat{u})|^2 + \int (w-\hat{w})p(u-\hat{u}) \leqslant \int (f-\hat{f})p(u-\hat{u})$$

where $w = T_\beta f$, $\hat{w} = T_\beta \hat{f}$. Since β is monotone $(w-\hat{w})p(u-\hat{u}) \geqslant 0$ and therefore $\int p'(u-\hat{u})|\operatorname{grad}(u-\hat{u})|^2 \leqslant 0$. Hence $\operatorname{grad} p(u-\hat{u}) = 0$ a.e. on $\boldsymbol{R}^2$ and so $p(u-\hat{u}) = C$ with $C \geqslant 0$. If $C = 0$ we conclude that $u \leqslant \hat{u}$. Otherwise $C > 0$ and so $u - \hat{u} = C' > 0$. Then $w \geqslant \hat{w}$ a.e. On the other hand $f \leqslant \hat{f}$ implies $w \leqslant \hat{w}$ since T_β is order-preserving. Thus $w = \hat{w}$ and $\Delta u = \Delta(\hat{u} + C') = \Delta \hat{u}$. We conclude $f = \hat{f}$, a contradiction.

To conclude this section we give two results related to the necessity of the condition $0 \in \operatorname{int}\beta(\boldsymbol{R})$ in Theorem 3.1. The first is:

THEOREM 3.7. *Let β be a maximal monotone graph in $\boldsymbol{R}$ with domain $D(\beta)$ bounded above and $\beta(\boldsymbol{R}) \subset [0, \infty)$. Then given $f \in L^1(\boldsymbol{R}^2)$ with $\int f < 0$ there is no function $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$ with the properties $\Delta u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$ and $f + \Delta u \in \beta(u)$* a.e.

PROOF OF THEOREM 3.7. First note that the nonexistence claim is stronger than saying (P) has no solution since $\Delta u \in L^1(\boldsymbol{R}^2)$ is not required. Assume, to obtain a contradiction, that u has the above properties. Set $M = \sup D(\beta)$. Then $u \leqslant M$ a.e., $u \in L^1_{\mathrm{loc}}$ and $\Delta u = -f + (\Delta u + f) \geqslant -f$ a.e. Let $\varrho \in \mathcal{D}^+(\boldsymbol{R}^2)$, $\int \varrho = 1$, $\tilde{u} = \varrho * u$. Then $\tilde{u} \leqslant M$ a.e. and $\Delta \tilde{u} \geqslant -\tilde{f} = -\varrho * f$. Since $\int f < 0$ we can assume $\int \tilde{f} < 0$ by an appropriate choice of ϱ.

Now $\tilde{u} \in C^\infty(\boldsymbol{R}^2)$. Let $v\colon [0, \infty) \to \boldsymbol{R}$ be given by

$$v(r) = \int_0^{2\pi} \tilde{u}(r\cos\theta, r\sin\theta)\, d\theta\,.$$

Then

$$v_{rr} + \frac{1}{r} v_r = \int_0^{2\pi} \tilde{u}_{rr} + \frac{1}{r}\tilde{u}_r\, d\theta = \int_0^{2\pi} \left(\Delta\tilde{u} - \frac{1}{r^2}\tilde{u}_{\theta\theta}\right) d\theta \geqslant \int_0^{2\pi} -\tilde{f}(r\cos\theta, r\sin\theta)\, d\theta\,.$$

Thus for $R > 0$

$$\int_0^R (r v_{rr} + v_r) \geqslant -\int_{|x| \leqslant R} \tilde{f}(x)\, dx\,.$$

If $-\int\limits_{|x|\leqslant R} \tilde{f}(x)\,dx \geqslant \varepsilon > 0$ for $R > R_0$ the above implies that $Rv_r(R) \geqslant \varepsilon$ for $R > R_0$. Then $v(R) \geqslant \varepsilon \log R + c$ for some c. On the other hand $\tilde{u} \leqslant M$ implies $v \leqslant 2\pi M$, a contradiction.

REMARKS.

1) By a similar proof we get easily the following (known) fact. Suppose $u \in L^1_{\text{loc}}(\boldsymbol{R}^2)$, $\Delta u \geqslant 0$ (in the sense of distributions) and u is bounded above; then u is a constant. Indeed it is sufficient to handle the case of a smooth u. As above we get that $\int \Delta u = 0$ and therefore $\Delta u = 0$. Next we can apply the previous result to $w = e^u$; since $\Delta w = e^u(\Delta u + |\operatorname{grad} u|^2) \geqslant 0$ and w is bounded above we conclude that $\Delta w = 0$ and so $\operatorname{grad} u = 0$.

2) Suppose now that $\lim\limits_{r\to+\infty} \inf \beta_0(r)/r > 0$; then for any given $f \in (\boldsymbol{R}^2)$ with $\int f < 0$, (P) has no solution. Indeed we get as above $v(R) \geqslant \varepsilon \log R + C$. On the other hand $\beta^0(u) \geqslant \delta u - C'$ $(\delta > 0)$ and thus

$$\varepsilon\delta \log r + C\delta \leqslant \int_0^{2\pi} \beta^0\big(u(r,\theta)\big)\,d\theta + 2\pi C'$$

which contradicts the fact that $\beta^0(u) \in L^1(\boldsymbol{R}^2)$. It is natural to raise the question whether a solution of (P) exists under the additional assumption $\int f > 0$.

In the case $N \geqslant 3$ the proof of existence relied heavily on the estimate that $\|u_\varepsilon\|_{M^{N/(N-1)}} \leqslant c_N \|\Delta u_\varepsilon\|_{L^1} \leqslant 2c_N \|f\|_{L^1}$. In particular, u_ε remained bounded in $L^1_{\text{loc}}(\boldsymbol{R}^2)$ as $\varepsilon \to 0$, and this was the crucial ingredient in the existence of u. Such an estimate does not hold when $N = 2$. More precisely, let $B = \{x \in \boldsymbol{R}^2 \colon \|x\| \leqslant 1\}$. Then there is *no* C for which

$$\|u\|_{L^1(B)} \leqslant C \|\Delta u\|_{L^1(\boldsymbol{R}^2)} \quad \text{for } u \in \mathfrak{D}(\boldsymbol{R}^2)\,. \tag{3.8}$$

We give two proofs. First, if (3.8) holds then a similar estimate holds if B is replaced by rB, $r > 0$ (by scaling). Moreover, if (3.8) holds for $u \in \mathfrak{D}$ it holds for $u \in L^1(\boldsymbol{R}^2)$. But then we have existence of solutions of (P) for every β, contradicting Theorem 3.7. A direct proof may be obtained by choosing $u(x) = v(kx)$ for fixed $v \in \mathfrak{D}$ and $k \in \boldsymbol{R}$. Then (3.8) may be rewritten as

$$\frac{1}{k^2} \int\limits_{|y|\leqslant k} |v(y)|\,dy \leqslant C \int\limits_{\boldsymbol{R}^2} |\Delta v(y)|\,dy\,.$$

As $k \to 0$ we find

$$2\pi|v(0)| \leqslant C\|\Delta v\|_{L^1} \quad \text{for } v \in \mathfrak{D}.$$

Now set $v = \zeta_0(\varrho_n * \log)$ where $\varrho_n \to \delta$ and ζ_0 is the cut-off function of Section 1. This yields

$$2\pi|(\varrho_n * \log)(0)| \leqslant C\|\Delta\zeta_0(\varrho_n * \log)\|_{L^1} + 2C\left\|\operatorname{grad}\zeta_0\left(\varrho_n * \frac{x}{|x|^2}\right)\right\|_{L^1} + 2\pi C\|\zeta_0\varrho_n\|_{L^1}.$$

However, $|(\varrho_n * \log)(0)| \to \infty$ as $n \to \infty$, and we have a contradiction.

REMARK. The case $\beta = 0$ is a special one with regard to existence. The problem $-\Delta u = f \in L^1(\boldsymbol{R}^2)$ always has solutions u in the class BMO of functions of bounded mean oscillation. If $u \in \mathrm{BMO}$ and $\Delta u \in L^1(\boldsymbol{R}^2)$, then $\operatorname{grad} u \in M^2(\boldsymbol{R}^2)$. We have not employed these facts in our presentation as we have not needed them.

4. – $N = 1$.

The main result of this section is:

THEOREM 4.1. *Assume* $0 \in \operatorname{int}\beta(\boldsymbol{R})$. *Then* (P) *is well-posed in the class* $\mathfrak{L} = L^1_{\mathrm{loc}}(\boldsymbol{R})$. *In addition, two solutions of* (P) *in* $\mathfrak{L}$ *differ by a constant and*

$$\left\|\frac{d}{dx}(G_\beta f - G_\beta \hat{f})\right\|_{L^\infty} \leqslant 2\|\hat{f} - f\|_{L^1} \quad \text{for } f, \hat{f} \in L^1(\boldsymbol{R}). \tag{4.2}$$

Also, G_β *maps bounded subsets of* $L^1(\boldsymbol{R})$ *into bounded subsets of* $W^{1,\infty}(\boldsymbol{R})$. *Finally, we have*

$$\int T_\beta f = \int f \quad \text{for } f \in L^1(\boldsymbol{R}). \tag{4.3}$$

PROOF OF THEOREM 4.1. We first obtain some simple estimates on a solution u of (P). We write $u' = du/dx$, etc. It follows from $u'' \in L^1(\boldsymbol{R})$ that $u' \in L^\infty(\boldsymbol{R})$ and the limits $u'(\pm\infty)$ exist. If, e.g., $u'(+\infty) \neq 0$ then $|u(x)| \to \infty$ as $x \to \infty$. However, since $0 \in \operatorname{int}\beta(\boldsymbol{R})$ this contradicts the properties $f + u'' \in \beta(u)$ a.e. and $f + u'' \in L^1(\boldsymbol{R})$ of the function $w = f + u''$. Thus $u'(\pm\infty) = 0$ and so $\|u'\|_{L^\infty} \leqslant \|u''\|_{L^1}$. Next, if $j \in \mathfrak{J}_0$ and $\partial j = \beta$ we have $j(u)' = wu'$ a.e., so $\|j(u)'\|_{L^1} \leqslant \|w\|_{L^1}\|u'\|_{L^\infty}$. Once again the properties of w used above imply that $j(u)(\pm\infty) = 0$ and $\|j(u)\|_{L^\infty} \leqslant \|w\|_{L^1}\|u'\|_{L^\infty}$. But

$j(r) \to \infty$ as $|r| \to \infty$ since $0 \in \operatorname{int}\beta(\boldsymbol{R})$ and therefore $u \in L^\infty(\boldsymbol{R})$. It is trivial to show that if $u'' \in L^1(\boldsymbol{R})$ and $u \in L^\infty(\boldsymbol{R})$, then $p'(u)u'^2 \in L^1(\boldsymbol{R})$ and

$$\int p'(u)u'^2 + \int p(u)u'' \leqslant 0 \qquad \text{for } p \in \mathfrak{P} . \tag{4.2}$$

Using (4.2) in the same way as Lemma 1.3 was employed in the proof of Theorem 3.1 we find that solutions of (P) are unique up to a constant and that (4.3) holds. Moreover, the above arguments applied to the solution u_ε of (P_ε) yield $\|(\varepsilon/2)u_\varepsilon^2 + j(u_\varepsilon)\|_{L^\infty} \leqslant \|f + u_\varepsilon''\|_{L^1}\|u_\varepsilon'\|_{L^\infty} \leqslant 2\|f\|_{L^1}^2$ so u_ε is bounded in $L^\infty(\boldsymbol{R})$. Thus (P) is well-posed in $\mathfrak{L}$. We summarize the estimates established for a solution u of (P):

$$u'(\pm\infty) = 0\,, \qquad \|u'\|_{L^\infty} \leqslant \|u''\|_{L^1} \leqslant 2\|f\|_{L^1} \tag{4.3}$$

and

$$\|j(u)\|_{L^\infty} \leqslant \|f + u''\|_{L^1}\|u'\|_{L^\infty} \leqslant 2\|f\|_{L^1}^2 . \tag{4.4}$$

The inequality (4.2) follows from (4.3) since if $u \in G_\beta f$, $\hat{u} \in G_\beta \hat{f}$ then $(u - \hat{u})'(\pm\infty) = 0$ and so $\|(u-\hat{u})'\|_{L^\infty} \leqslant \|(u-\hat{u})''\|_{L^1} \leqslant 2\|f - \hat{f}\|_{L^1}$: Moreover, (4.3) shows $f \mapsto (G_\beta f)'$ bounded from $L^1(\boldsymbol{R})$ to $L^\infty(\boldsymbol{R})$ and (4.4) shows that $f \mapsto G_\beta f$ is bounded from $L^1(\boldsymbol{R})$ to $L^\infty(\boldsymbol{R})$.

REMARK. We cannot ask that (P) be well-posed in a class $\mathfrak{L}_1$ larger than $L^1_{\mathrm{loc}}(\boldsymbol{R})$. It is obviously well-posed in $\mathfrak{L}_1$ if $L^1_{\mathrm{loc}}(\boldsymbol{R}) \supset \mathfrak{L}_1 \supset W^{1,\infty}(\boldsymbol{R})$.

The situation as regards uniqueness of solutions of (P) is precisely as in Section 3, and is established by the same argument. Solutions of (P) are not unique if and only if there exists $v \in L^\infty(\boldsymbol{R})$ such that $v'' = f$ and $2\|v\|_{L^\infty} < \operatorname{meas}\beta^{-1}(0)$. Moreover, if $v \in L^\infty(\boldsymbol{R})$ and $v'' \in L^1(\boldsymbol{R})$ then $v'(\pm\infty) = 0$ so $\int_{-\infty}^{\infty} v'' = 0$. Thus we have the analogue of Proposition 3.4:

PROPOSITION 4.5 *If either* $\beta^{-1}(0) = \{0\}$ *or* $\int f \neq 0$ *solutions of* (P) *are unique.*

Finally we state the analogues of Proposition 3.6 and Theorem 3.7. The proofs are simpler where they differ from those for $N = 2$ and are omitted.

PROPOSITION 4.6. *Let* f, $\hat{f} \in L^1(\boldsymbol{R})$ *with* $f \leqslant \hat{f}$ a.e. *and* $f \neq \hat{f}$. *If* $u \in G_\beta f$ *and* $\hat{u} \in G_\beta \hat{f}$ *then* $u \leqslant \hat{u}$ a.e.

THEOREM 4.7. *Let* β *be as in Theorem* 3.7. *Then given* $f \in L^1(\boldsymbol{R})$ *with* $\int f < 0$ *there is no function* $u \in L^1_{\mathrm{loc}}(\boldsymbol{R})$ *with the properties* $u'' \in L^1_{\mathrm{loc}}(\boldsymbol{R})$ *and* $f + u'' \in \beta(u)$ a.e.

5. – Problems well-posed in $L^p(\boldsymbol{R}^N)$.

If β is a maximal monotone graph in $\boldsymbol{R}$, then β^0 denotes the function with domain $D(\beta)$ such that $\beta^0(r)$ is the element of $\beta(r)$ of least modulus. If u is measurable and $w \in \beta(u)$ a.e., then $|w| \geqslant |\beta^0(u)|$ a.e. Thus if u is a solution of (P), $\beta^0(u) \in L^1(\boldsymbol{R}^N)$. In this section, under various conditions on β, we are interested in the consequences of this additional information about u. A main result of this section is:

THEOREM 5.1. *Let β be a maximal monotone graph in $\boldsymbol{R}$ satisfying $0 \in \beta(0)$ and*

$$(5.2) \qquad \begin{cases} \textit{there are numbers } k,\ A > 0 \textit{ such that} \\ |r| \leqslant k|\beta^0(r)| \quad \textit{for } r \in D(\beta),\ |r| \leqslant A\,. \end{cases}$$

Then (P) *is well-posed in $L^1(\boldsymbol{R}^N)$ for $N \geqslant 1$. Moreover, G_β is a bounded map from $L^1(\boldsymbol{R}^N)$ to $L^1(\boldsymbol{R}^N)$ which is continuous if $N \geqslant 3$.*

PROOF OF THEOREM 5.1. The arguments differ for $N \geqslant 3$, $N = 2$ and $N = 1$. We first give the simple estimates common to the three cases. Note that (5.2) implies $0 \in \operatorname{int} \beta(\boldsymbol{R})$ and $\beta^{-1}(0) = \{0\}$ so $u = G_\beta f$ is uniquely defined for $f \in L^1(\boldsymbol{R}^N)$, $N \geqslant 1$. It follows from (5.2) that

$$(5.3) \qquad \int\limits_{[|u| \leqslant A]} |u| \leqslant k \int\limits_{[|u| \leqslant A]} |\beta^0(u)| \leqslant k \int |f + \Delta u| \leqslant k \|f\|_{L^1}$$

since $T_\beta u = f + \Delta u \in \beta(u)$ a.e. Moreover, $|\beta^0(r)| \geqslant (A/k)$ for $|r| \geqslant A$ by the monotonicity of β. Thus

$$(A/k)\operatorname{meas}[|u| > A] \leqslant \int\limits_{[|u| \leqslant A]} |\beta^0(u)| \leqslant \int |f + \Delta u| \leqslant \|f\|_{L^1}$$

and we have

$$(5.4) \qquad \operatorname{meas}[|u| > A] \leqslant (k/A)\|f\|_{L^1}\,.$$

In each of the cases $N \geqslant 3$, $N = 2$ and $N = 1$ (5.3) implies $u \in L^1([|u| \leqslant A])$ while (5.4) implies $\operatorname{meas}[|u| > A] < \infty$. It will remain to show $u \in L^1([|u| > A])$, the reason for which varies with the case.

$N \geqslant 3$. Since the $\mathfrak{L}_1$ in the remark following Theorem 2.1 includes $L^1(\boldsymbol{R}^N)$, in order to show (P) is well-posed in $L^1(\boldsymbol{R}^N)$ it suffices to prove $G_\beta L^1(\boldsymbol{R}^N) \subseteq$

$\subseteq L^1(\boldsymbol{R}^N)$. Now $u = G_\beta f$ satisfies $\|u\|_{M^{N/(N-2)}} \leqslant 2c_N\|f\|_{L^1}$. This and (5.3), (5.4) imply that

$$\int |u| = \int_{[|u|\leqslant A]} |u| + \int_{[|u|>A]} |u| \leqslant k\|f\|_1 + (\operatorname{meas}[|u|>A])^{2/N}\|u\|_{M^{N/(N-2)}} \leqslant k\|f\|_{L^1} + 2C_N(k/A)^{2/N}\|f\|_{L^1}^{(N+2)/N}. \tag{5.5}$$

Thus $G_\beta\colon L^1(\boldsymbol{R}^N)) \to L^1(\boldsymbol{R}^N))$ and it is bounded. To see that G_β is continuous into $L^1(\boldsymbol{R}^N)$ let $f_n \to f$ in $L^1(\boldsymbol{R}^N)$ and $w_n = T_\beta f_n$. Then $w_n \to w = T_\beta f$ since T_β is an L^1 contraction. For $l > 0$ set $K_{nl} = [|w_n| \geqslant l$ or $|w| \geqslant l]$ so $\boldsymbol{R}^N \setminus K_{nl} = [|w_n| < l$ and $|w| < l]$. We have

$$l \operatorname{meas} K_{nl} \leqslant \int (|w_n| + |w|) \leqslant C \tag{5.6}$$

where C is independent of n. Now

$$\int |G_\beta f_n - G_\beta f| = \int_{K_{nl}} |G_\beta f_n - G_\beta f| + \int_{[|w_n|, |w|<l]} |G_\beta f_n - G_\beta f| \leqslant 2c_N\|f_n - f\| \operatorname{meas}(K_{nl})^{2/N} + \int_{[|w_n|, |w|<l]} |G_\beta f_n - G_\beta f|. \tag{5.7}$$

If $l < \min(|\beta^0(A)|, |\beta^0(-A)|)$, then $u_n = G_\beta f_n$ and $u = G_\beta f$ satisfy $|u_n| \leqslant k|w_n|$, $|u| < k|w|$ on $[|w_n|, |w| < l]$ by (5.2). For such l

$$\int_{[|w_n|, |w|<l]} |G_\beta f_n - G_\beta f| \leqslant k \int_{[|w_n|, |w|<l]} (|w_n| + |w|). \tag{5.8}$$

Taking the lim sup in (5.7) as $n \to \infty$ and using (5.6) and (5.8) yields

$$\limsup_{n\to\infty} \|G_\beta f_n - G_\beta f\|_{L^1} \leqslant 2k \int_{[|w|<l]} |w|$$

and the result follows by sending l to zero.

$N = 1$. By Theorem 4.1 G_β maps bounded subsets of $L^1(\boldsymbol{R})$ into bounded subsets of $L^\infty(\boldsymbol{R})$. Thus (5.3) and (5.4) imply

$$\int |u| \leqslant \int_{[|u|\leqslant A]} |u| + \int_{[|u|>A]} |u| \leqslant k\|f\|_{L^1} + (k/A)\|f\|_{L^1}\|u\|_{L^\infty}$$

and $G_\beta\colon L^1(\boldsymbol{R}) \to L^1(\boldsymbol{R})$ and is bounded. Since $L^1(\boldsymbol{R}) \subset L^1_{\mathrm{loc}}(\boldsymbol{R})$, (P) is well-posed in $L^1(\boldsymbol{R})$.

$N = 2$. This case is somewhat more delicate. We need to estimate $u \in L^1([|u| > A])$, which is the point of the next lemma.

LEMMA 5.9. *Let* $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$, $\operatorname{grad} u \in M^2(\boldsymbol{R}^2)$, $\lambda \geqslant 0$ *and* $\operatorname{meas}[|u| > \lambda] < \infty$. *Then*

$$\int (|u| - \lambda)^+ \leqslant C \|\operatorname{grad} u\|_{M^2} \operatorname{meas}[|u| > \lambda] .$$

where C *is independent of* u *and* λ.

Assuming Lemma 5.9 for the moment, we complete the proof of Theorem 5.1. If $u = G_\beta f$ then $\|\operatorname{grad} u\|_{M^2} \leqslant 2d_2 \|f\|_{L^1}$. Moreover, by (5.2), $\operatorname{meas}[|u| > \lambda] < \infty$ for all $\lambda > 0$. Thus Theorem 3.1, Lemma 5.9, (5.3) and (5.4) yield

$$\begin{aligned}\int |u| \leqslant \int_{[|u| \leqslant A]} |u| + \int_{[|u| > A]} |u| &\leqslant k\|f\|_1 + \int (|u| - A)^+ + A \operatorname{meas}[|u| > A] \\ &\leqslant k\|f\|_1 + \big(C2d_2\|f\|_{L^1} + A\big) \operatorname{meas}[|u| > A] \\ &\leqslant \big(2 + (2Cd_2\|f\|_{L^1} + A)(k/A)\big) \|f\|_{L^1} .\end{aligned}$$

At this point we know that $G_\beta L^1(\boldsymbol{R}^2) \subset L^1(\boldsymbol{R}^2)$. The fact that then (P) is well-posed in $L^1(\boldsymbol{R}^2)$ follows from Lemma A.14, which implies that a solution u of (P) in $L^1(\boldsymbol{R}^2)$ lies in the $\mathfrak{L}$ of Theorem 3.1.

PROOF OF LEMMA 5.9. We actually show a little more, namely $\operatorname{meas}[u > \lambda] < \infty$ implies

$$\int (u - \lambda)^+ \leqslant C \|\operatorname{grad} u\|_{M^2} \operatorname{meas}[u > \lambda] .$$

Applying this result to $-u$ and summing gives the result of the lemma. Now $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$ and $|\operatorname{grad} u| \in M^2(\boldsymbol{R}^2)$ implies $u \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^2)$ for $1 \leqslant p < 2$ by Lemma A.2. Given $\lambda_1 > \lambda$ set

$$p(u) = \begin{cases} \lambda_1 - \lambda & [u > \lambda_1] \\ u - \lambda & \text{on } [\lambda_1 \geqslant u \geqslant \lambda] \\ 0 & \text{on } [u < \lambda] . \end{cases}$$

Then $p(u) \in W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^2)$ for $1 \leqslant p < 2$ and

$$\operatorname{grad} p(u) = \begin{cases} 0 & \text{a.e. on } [u \geqslant \lambda_1] \\ \operatorname{grad} u & \text{a.e. on } [\lambda_1 > u > \lambda] \\ 0 & \text{a.e. on } [u \leqslant \lambda] . \end{cases}$$

Thus $\|\operatorname{grad} p(u)\|_{M^2} \leqslant \|\operatorname{grad} u\|_{M^2}$. Moreover, $p(u) \in L^\infty(\boldsymbol{R}^2)$ is supported in a set of finite measure. Hence $p(u) \in L^q(\boldsymbol{R}^2)$, $1 \leqslant q \leqslant \infty$. Now by the Sobolev-Nirenberg-Gagliardo inequality (e.q. [7, section 1.9]) if $v \in L^p(\boldsymbol{R}^2)$, $\operatorname{grad} v \in L^r(\boldsymbol{R}^2)$, $1/p = 1/r - \frac{1}{2}$ and $1 \leqslant r < 2$, then there is a constant C such that

$$\|v\|_{L^p} \leqslant C\|\operatorname{grad} v\|_{L^r}.$$

The constant C depends only on p and r. For the purposes of this lemma, we choose $p = 2$, $r = 1$, $v = p(u)$. This yields

$$\begin{aligned}\int p(u) = \int_{[u>\lambda]} p(u) &\leqslant \left(\int p(u)^2\right)^{\frac{1}{2}}(\operatorname{meas}[u>\lambda])^{\frac{1}{2}}\\ &\leqslant C\Big(\int_{[u>\lambda]} |\operatorname{grad} p(u)|\Big)(\operatorname{meas}[u>\lambda])^{\frac{1}{2}}\\ &\leqslant C\|\operatorname{grad} p(u)\|_{M^2}\operatorname{meas}[u>\lambda]\\ &\leqslant C\|\operatorname{grad} u\|_{M^2}\operatorname{meas}[u>\lambda].\end{aligned}$$

Now let $\lambda_1 \to \infty$. The conclusion follows from Fatou's lemma.

REMARK. If in addition to the assumptions of Lemma 5.9 we have $\Delta u \in L^1(\boldsymbol{R}^2)$, then we have $\int(|u| - \lambda)^+ \leqslant c d_2 \|\Delta u\|_{L^1} \operatorname{meas}[|u| > \lambda]$ by Lemma A.11. It is interesting to note that an equality of this type does not hold if only $\operatorname{meas}[|u| > \lambda] < \infty$ and $\Delta u \in L^1(\boldsymbol{R}^2)$ are assumed. W. Rudin has given us an example of a nonconstant harmonic function u satisfying $\operatorname{meas}[|u| > \lambda] < M < \infty$ for all $\lambda > 1$.

REMARK. The condition (5.2) seems fairly sharp as a criterion for well-posedness in $L^1(\boldsymbol{R}^N)$. Indeed, let $\alpha > 1$ and $\varlimsup_{r\to 0^+} \beta^0(r)/r^\alpha < \infty$. Choose $r_0 > 0$, $k > 0$ so that $|\beta^0(r)| \leqslant k r^\alpha$ for $0 < r < r_0$. Let

$$u(x) = \begin{cases} \dfrac{1}{|x|^N} & \text{for } |x| > R\,, \\ -\dfrac{\lambda}{2}|x|^2 + r_0 & \text{for } |x| \leqslant R\,, \end{cases}$$

where λ, R are chosen so that $u \in C^1(\boldsymbol{R}^N)$. Then $u \in M^{N/(N-2)}(\boldsymbol{R}^N) \setminus L^1(\boldsymbol{R}^N)$ if $N \geqslant 3$, $|\operatorname{grad} u| \in M^2(\boldsymbol{R}^2)$ and $u \notin L^1(\boldsymbol{R}^2)$ if $N = 2$, $u \in L^\infty(\boldsymbol{R}) \setminus L^1(\boldsymbol{R})$ if $N = 1$ while $\beta^0(u)$, $\Delta u \in L^1(\boldsymbol{R}^N)$.

One can generalize Theorem 5.1 suitably to include the cases:

There exist p, $1 \leqslant p < \infty$, $A, k > 0$ such that

$$|u|^p \leqslant k|\beta^0(u)| \text{ for } u \in D(\beta) \text{ and } |u| \leqslant A\,. \tag{5.10}$$

We explicitly allow $A = \infty$ which means $|u|^p \leqslant k|\beta^0(u)|$ for $u \in D(\beta)$.

THEOREM 5.11. *Let* (5.10) *hold. Then*

(i) *If* $N \geq 3$ *and* $1 \leq p < N/(N-2)$ (P) *is well-posed in* $L^p(\boldsymbol{R}^N)$. *If* $N \geq 3$ *and* $A = \infty$, (P) *is well-posed in* $L^p(\boldsymbol{R}^N)$.

(ii) *If* $N = 1$, (P) *is well-posed in* $L^p(\boldsymbol{R})$.

(iii) *If* $N = 2$, (P) *is well-posed in* $L^p(\boldsymbol{R}^2)$.

PROOF OF THEOREM 5.11. The proofs resemble the arguments used in obtaining Theorem 5.1, so we only sketch them. As (5.2) gave bounds on $\int_{[|u|<A]} |u|^p$ and $\text{meas}[|u| > A]$ for $u = G_\beta f$ in the case $p = 1$, so does (5.10) give similar bounds here. The point is then to see that $u \in L^p([|u| > A])$. If $N = 3$, $u \in M^{N/(N-2)}(\boldsymbol{R}^N)$ supplies this information if $1 \leq p < N/(N-2)$ (by Lemma A.5), while $u \in L^\infty(\boldsymbol{R})$ if $N = 1$. If $N = 2$, Lemma 5.9 is replaced by:

LEMMA 5.12. *Let* $u \in L^1_{\text{loc}}(\boldsymbol{R}^2)$, $|\text{grad}\, u| \in M^2(\boldsymbol{R}^2)$, $\lambda \geq 0$ *and* $\text{meas}[|u| > \lambda] < \infty$. *Then*

$$\|(|u| - \lambda)^+\|_{L^p} \leq c_p \|\text{grad}\, u\|_{M^2} (\text{meas}[|u| > \lambda])^{1/p}$$

for $1 \leq p < \infty$ *where* c_p *depends only on* p.

PROOF OF LEMMA 5.12. Form the same function $p(u)$ as in the proof of Lemma 5.9. If $1 \leq p < 2$, use

$$\left[\int p(u)^p\right]^{1/p} \leq \Big(\int_{[u>\lambda]} p(u)^2\Big)^{\frac{1}{2}} (\text{meas}[u > \lambda])^{(2-p)/2p} \leq c \|\text{grad}\, p(u)\|_{M^2} (\text{meas}[u > \lambda])^{1/p}.$$

If $p \geq 2$, use the Sobolev inequality directly with $1/p = 1/r - \frac{1}{2}$ or $r = 2p/(p+2)$. The rest is the same as Lemma 5.9.

There are only two points remaining. First, if $A = \infty$, (5.10) itself guarantees that $u = G_\beta f \in L^p(\boldsymbol{R}^N)$ for $N \geq 1$. The final point is the question of uniqueness for the case $N = 2$. But again we may use Lemma A.14.

6. – Solutions with compact support.

Let β be a maximal monotone graph in $\boldsymbol{R}$ with $0 \in \beta(0)$ and $f \in L^1_{\text{loc}}(\boldsymbol{R}^N)$. In this section it is convenient to index (P) by β and f. Also, in this section, a solution of

$$(\text{P}_{\beta f}) \qquad -\Delta u + \beta(u) \ni f \quad \text{on } \boldsymbol{R}^N$$

is a function $u \in L^1_{\text{loc}}(\boldsymbol{R}^N)$ such that $\Delta u \in L^1_{\text{loc}}(\boldsymbol{R}^N)$ and $f + \Delta u \in \beta(u)$ a.e. The requirements that f and Δu lie in $L^1(\boldsymbol{R}^N)$ have been dropped. $L^1_0(\boldsymbol{R}^N)$ denotes $\{u \in L^1(\boldsymbol{R}^N)\colon \operatorname{supp} u \text{ is compact}\}$ where $\operatorname{supp} u$ is the support of u. We will prove, under various assumptions, that $(\mathrm{P}_{\beta f})$ has solutions $u \in L^1_0(\boldsymbol{R}^N)$. The main results are stated next.

THEOREM 6.1. *Let* $\varphi \in \mathfrak{J}_0$ *satisfy* $\partial\varphi = \beta$. *Then* $(\mathrm{P}_{\beta f})$ *has a solution* $u \in L^1_0(\boldsymbol{R}^N)$ *for all* $f \in L^1_0(\boldsymbol{R}^N)$ *if and only if*

$$\int_{-1}^{1} (\varphi(s))^{-\frac{1}{2}}\, ds < \infty . \tag{6.2}$$

By convention, $\varphi(s)^{-\frac{1}{2}} = 0$ if $\varphi(s) = \infty$ and $\varphi(s)^{-\frac{1}{2}} = \infty$ if $\varphi(s) = 0$. Observe that if $\beta(r) = |r|^\alpha \operatorname{sign} r$ for $0 < \alpha < 1$ or $0 \in \operatorname{int}\beta(0)$, then $\varphi^{-\frac{1}{2}}$ satisfies (6.2).

THEOREM 6.3. *Let* $\beta(0) = [\gamma_-, \gamma_+]$, $-\infty < \gamma_- < 0 < \gamma_+ < \infty$, *and* $f \in L^1_{\text{loc}}(\boldsymbol{R}^N)$. *Suppose* $R > 0$ *and there are functions* $g_\pm \in L^1_{\text{loc}}([0, \infty))$ *such that* $\nu(\gamma_\nu - f(x)) \geqslant g_\nu(|x|) \geqslant 0$ *for* $\nu \in \{+, -\}$ a.e. *on* $[|x| \geqslant R]$ *and which satisfy* $\int_0^\infty r^{N-1} g_\nu(r)\, dr = \infty$. *Then* $(\mathrm{P}_{\beta f})$ *has a solution* $u \in L^1_0(\boldsymbol{R}^N)$. *If* $N = 1$ *or* $N = 2$ *and* $\gamma_-, \gamma_+ \in \operatorname{int}\beta(\boldsymbol{R})$, *then* $\int_0^\infty r g_\nu(r)\, dr = \infty$ *for* $N = 2$ *and* $\int_0^\infty r \log(1 + r) g^\nu(r)\, dr = \infty$ *if* $N = 1$ *are sufficient to imply that* $(\mathrm{P}_{\beta f})$ *has a solution* $u \in L^1_0(\boldsymbol{R}^N)$.

REMARKS. Solutions u of $(\mathrm{P}_{\beta f})$ are unique in the class $L^1_0(\boldsymbol{R}^N)$. Indeed, if u is such a solution then u, $\Delta u \in L^1(\boldsymbol{R}^N)$ and we may use the proofs of the preceding sections (for $N = 1$ or 2, note that if functions in $L^1_0(\boldsymbol{R}^N)$ differ by a constant then they coincide).

The simplest case of Theorem 6.3 arises if $\gamma_+ > \alpha_+ \geqslant f \geqslant \alpha_- > \gamma_-$ for some constants α_+, α_-. This special case can be deduced from Theorem 6.2 with the aid of Lemmas 6.4 and 6.5 below, and extends a result of Brezis [1], as does Theorem 6.3. The generalization arises from allowing f to be unbounded on $[|x| \leqslant R]$. Our proofs are different from those in [1] however.

The proofs of these theorems will employ the next two simple comparison results.

LEMMA 6.4. *Let* $f_1, f, f_2 \in L^1_{\text{loc}}(\boldsymbol{R}^N)$ *and* $f_1 \leqslant f \leqslant f_2$ a.e. *If* $(\mathrm{P}_{\beta f_i})$ *has a compactly supported solution* $u_i \in L^1_0(\boldsymbol{R}^N)$ *for* $i = 1, 2$, *then* $(\mathrm{P}_{\beta f})$ *has a solution* u *satisfying* $u_1 \leqslant u \leqslant u_2$ (*so* $u \in L^1_0(\boldsymbol{R}_N)$). *Moreover*, $f_1 + \Delta u_1 \leqslant f + \Delta u \leqslant f_2 + \Delta u_2$.

PROOF OF LEMMA 6.4. Let Ω be an open ball containing $\operatorname{supp} u_1$ and $\operatorname{supp} u_2$. By a result of [2] there exists a unique $v \in W_0^{1,1}(\Omega)$ such that $\Delta v \in L^1(\Omega)$ and $f_1 + \Delta v \in \beta(v)$ a.e. on Ω. Moreover $u_1 \leqslant v \leqslant u_2$ a.e. on Ω and $f_1 + \Delta u_1 \leqslant f + \Delta v \leqslant f_2 + \Delta u_2$ a.e. on Ω. Since then $\operatorname{supp} v \subset \Omega$, the function u defined by $u = v$ on Ω and $u = 0$ on $\boldsymbol{R}^N \setminus \Omega$ has the desired properties.

LEMMA 6.5. *Let $f \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ and η be a maximal monotone graph in $\boldsymbol{R}$ with $0 \in \eta(0)$. Suppose $D(\eta) \supset D(\beta)$ and $|\eta^0(r)| \leqslant |\beta^0(r)|$ for $r \in D(\beta)$. If $f \geqslant 0$ or $f \leqslant 0$ and $(\mathrm{P}_{\eta f})$ has a compactly supported solution, then $(\mathrm{P}_{\beta f})$ also has a solution with compact support.*

PROOF OF LEMMA 6.5. Assume that $f \geqslant 0$ and v is a solution of $(\mathrm{P}_{\eta f})$ with compact support. As in the previous proof, let Ω be a ball containing $\operatorname{supp} v$ and $u \in W_0^{1,1}(\Omega)$ satisfy $\Delta u \in L^1(\Omega)$ and $f + \Delta u \in \beta(u)$ a.e. on Ω. Then $u, v \geqslant 0$ and so $\beta^0(u) \geqslant \eta^0(v)$ a.e. Setting $h = -\Delta u + \eta^0(u) \in f + \eta^0(u) - \beta(u)$ we therefore have $h \leqslant f$ a.e. on Ω. Moreover $h + \Delta u = \eta^0(u) \in \eta(u)$ a.e. on Ω. From the results of [2] we conclude that $v \geqslant u \geqslant 0$ a.e. on Ω. Again, extending u to be zero on $\boldsymbol{R}^N \setminus \Omega$ results in a compactly supported solution of $(\mathrm{P}_{\beta f})$. The case $f \leqslant 0$ is treated similarly.

REMARK. If $f \geqslant 0$ we only use $\beta^0(r) \geqslant \eta^0(r)$ for $r \geqslant 0$ while if $f \leqslant 0$, $\eta^0(r) \geqslant \beta^0(r)$ for $r \leqslant 0$ suffices.

PROOF OF THEOREM 6.1. Observe first that (6.2) implies that (5.2) holds. Indeed, if $r > 0$ and $r \in D(\beta)$ then $\varphi(r) \leqslant r\beta^0(r)$. Moreover, φ is nondecreasing on $\boldsymbol{R}^+$ so

$$\int_{\frac{1}{2}r}^{r} \varphi(s)^{-\frac{1}{2}}\, ds \geqslant \frac{r}{2}\, \varphi(r)^{-\frac{1}{2}} \geqslant \frac{1}{2} \sqrt{r/\beta^0(r)}\,.$$

Thus (6.2) implies $\lim\limits_{r \to 0+} \big(r/\beta^0(r)\big) = 0$, which implies (5.2) for $r \geqslant 0$. The case $r \leqslant 0$ follows similarly. Hence $(\mathrm{P}_{\beta f})$ is well-posed in $L^1(\boldsymbol{R}^N)$ by Theorem 5.1. We show the solution $u = G_\beta f$ of $(\mathrm{P}_{\beta f})$ has compact support if $f \in L^1_0(\boldsymbol{R}^N)$. The preceding two lemmas allow us to assume that f does not change sign and that β is bounded. Indeed, by Lemma 6.4, it is enough to treat $f^+ = \max(f, 0)$ and $-f^- = f - f^+$ in place of f, and by Lemma 6.5 we may truncate β (note that this preserves (6.2)). Hence we assume that $f \geqslant 0$ and $\beta(\boldsymbol{R}) \subset [-A, A]$ for some A. Let $\operatorname{supp} f \subset \{|x| < R\}$ and $u = G_\beta f$. Now $f + \Delta u \in \beta(u)$ implies that $|\Delta u| \leqslant A$ on $\{|x| > R\}$. But then $u \in W^{1,1}_{\mathrm{loc}}(\{|x| > R\})$ and $\Delta u \in L^\infty(\{|x| > R\})$. By standard arguments we conclude that $u \in W^{2,p}_{\mathrm{loc}}(\{|x| > R\})$ for $1 \leqslant p < \infty$. Choosing $p > N$, the Sobolev embedding theorem implies $u \in C^1(\{|x| > R\})$. Let $R_0 > R$ and fix $M = \max\{|u(x)|: |x| = R_0\}$. Next we build a radial comparison function v on $\{|x| > R_0\}$ with compact support such that $v \geqslant u$ on $\{|x| = R_0\}$ and there exists $g \geqslant 0$ such that $g + \Delta v \in \beta(v)$ a.e. on $\{|x| > R_0\}$. The function $\tau \to \int_0^\tau (2\varphi(s))^{-\frac{1}{2}}\, ds$ is a nondecreasing function from $\boldsymbol{R}^+$; it is onto because β is bounded. Let h be the inverse function so that $h'(r) = \sqrt{2\varphi(h(r))}$ and $h''(r) \in \beta(h(r))$ a.e. on $\boldsymbol{R}^+$. Set

$$v(x) = \begin{cases} h(R_1 - |x|) & \text{for } R_0 \leqslant |x| \leqslant R_1 \\ 0 & \text{for } |x| \geqslant R_1 \end{cases}$$

where $R_1 > R_0$. Then $v \in C^1(\{|x| \geqslant R_0\})$ and if

$$g(x) = \begin{cases} (N-1)h'(R_1 - |x|) & \text{for } R_0 < |x| < R_1 \\ 0 & \text{for } |x| > R_1 \end{cases}$$

then $g \geqslant 0 = f$ on $\{|x| > R_0\}$ and $\Delta v + g \in \beta(v)$ a.e. on $\{|x| > R_0\}$. If we choose $R_1 > R_0$ so that $h(R_1 - R_0) \geqslant M$, it follows that also $v = h(R_1 - R_0) \geqslant u$ on $\{|x| = R_0\}$. The next lemma will allow us to conclude that then $v \geqslant u$ on $\{|x| \geqslant R_0\}$.

LEMMA 6.6. *Let $R > 0$ and $u \in L^1(\{|x| > R\}) \cap C^1(\{|x| \geqslant R\})$ satisfy $\Delta u \in L^1(\{|x| > R\})$. If $u^+ \Delta u \leqslant 0$ a.e. on $\{|x| \geqslant R\}$ and $u \leqslant 0$ on $\{|x| = R\}$, then $u \leqslant 0$ on $\{|x| > R\}$.*

PROOF OF LEMMA 6.6. Since $u \in C^1(\{|x| \geqslant R\})$ and $\Delta u \in L^1(\{|x| \geqslant R\})$,

$$\int_{|x| \geqslant R} (\Delta u)\psi = -\int_{|x| \geqslant R} \operatorname{grad} u \operatorname{grad} \psi$$

for $\psi \in C^1(\{|x| \geqslant R\})$ provided that ψ has compact support and $\psi = 0$ on $\{|x| = R\}$. Choose $p \in \mathfrak{T}$ so that $p(r) = 0$ on $r \leqslant 0$ and $p' \in L^\infty(\boldsymbol{R})$. Setting $\psi = p(u)\zeta_n$ above with $\zeta_n = \zeta_0(x/n)$ we find (because $u_+ \Delta u \geqslant 0$ implies $p(u)\zeta_n \Delta u \geqslant 0$):

$$\begin{aligned} 0 \leqslant \int_{|x| \geqslant R} (\Delta u)(p(u)\zeta_n) &= -\int_{|x| \geqslant R} |\operatorname{grad} u|^2 p'(u)\zeta_n \\ &\quad - \int_{|x| \geqslant R} p(u) \operatorname{grad} u \operatorname{grad} \zeta_n \\ &= -\int_{|x| \geqslant R} |\operatorname{grad} u|^2 p'(u)\zeta_n + \int_{|x| \geqslant R} j(u)\, \Delta\zeta_n \end{aligned}$$

where $j(r) = \int_0^r p(s)\,ds$. Thus

$$\int_{|x|\geqslant R} |\operatorname{grad} u|^2 p'(u)\zeta_n \leqslant \frac{1}{n^2}\|p\|_\infty \|\Delta\zeta_0\|_\infty \|u\|_{L^1(n\leqslant|x|\leqslant 2n)}$$

and, letting $n\to\infty$, $|\operatorname{grad} u|^2 p'(u) = 0$ a.e. on $\{|x|\geqslant R\}$ by Fatou's lemma. It follows that $\operatorname{grad} u = 0$ a.e. on the open set $\{u>0\}$, whence the result.

Applying the lemma to $u-v$ above we conclude $u\leqslant v$ on $\{|x|\geqslant R_0\}$ and therefore u has compact support.

NECESSITY. Suppose for instance that $\int_0^1 (\varphi(s))^{-\frac12}\,ds = \infty$. By Lemma 6.5 we can assume $\beta^{-1}(0) = 0$. Let f be the characteristic function of $\{|x|\leqslant 1\}$. Assume, to obtain a contradiction, that u is a solution of $(P_{\beta f})$ with compact support. By uniqueness of solutions $u\in L^1(\boldsymbol{R}^N)$, u must be radial since f is radial and the problem is invariant under rotations. That is, u has the form $u(x) = v(|x|)$. The function v satisfies $v\in C^1((0,\infty))$, $v\geqslant 0$ and

$$\|f\|_\infty = 1 \geqslant h(r) = v''(r) + \frac{N-1}{r}v'(r) + g(r)\in\beta(v(r)) \quad \text{a.e. } r>0\,, \tag{6.7}$$

where $g(r) = 1$ for $0\leqslant r\leqslant 1$ and $g(r) = 0$ for $r>1$. Since $u\in L^1_0(\boldsymbol{R}^N)$ and $u\geqslant 0$, $R = \max\{r\colon v(r)>0\}$ is positive and finite. Clearly $R\geqslant 1$ for (6.7) implies that $g(r)\in\beta(v(r))$ for $r>R$ while $1\notin\beta(0)$. In fact $R>1$, because (6.7) implies that

$$\frac{d}{dr}\left(r^{N-1}v'(r)\right) \leqslant r^{N-1}\left(1-g(r)\right) \qquad \text{a.e. } r>0\,. \tag{6.8}$$

From (6.8), $r^{N-1}v'(r)$ is decreasing on $0<r\leqslant 1$. Hence if $R=1$, then $v(1) = v'(1) = 0$ and $v'(r)\geqslant 0$ for $0<r\leqslant 1$. Thus $v\leqslant 0$ on $(0,1)$. Since also $v\geqslant 0$ and v is not identically zero on $(0,1)$ this is impossible. Next we claim that $v(r)>0$ on $(1,R)$. Indeed, $h(r)\in\beta(v(r))$ so $h\geqslant 0$ and

$$\frac{d}{dr}\left(r^{N-1}v'(r)\right) = r^{N-1}h(r)\geqslant 0 \qquad \text{a.e. } 1<r<R\,.$$

Now $v'(R) = 0$, so $v'\leqslant 0$ on $1<r<R$. It follows that $v(r)>0$ and $h(r)>0$ on $1<r<R$, so $v'(r)<0$ on $1<r<R$. Thus

$$\int_0^{v(1)} \frac{ds}{\sqrt{2\varphi(s)}} = \int_1^R \frac{-v'(r)}{\sqrt{2\varphi(v(r))}}\,dr = \infty\,. \tag{6.9}$$

by the assumption on φ. We will obtain a contradiction by estimating $-v'(r)/\sqrt{\varphi(v(r))}$ on $[1, R]$. Now if $w = v'^2$ and $1 < r < R$ we have

$$\varphi(v)' = hv' = \left(v'' + \frac{N-1}{r}v'\right)v' \leqslant \frac{1}{2}\exp\left[-2(N-1)r\right]\left(\exp[2(N-1)r]w\right)'$$

and so $2\exp[2(N-1)R]\varphi(v)' \leqslant (\exp[2(N-1)r]w)'$. Since $w(R) = \varphi(v(R)) = 0$, integrating this inequality over the interval (r, R) leads us to conclude that

$$2\exp[2(N-1)R]\varphi(v) \geqslant \exp[2(N-1)r]w(r) \geqslant \exp[2(N-1)](v'(r))^2$$

for $1 \leqslant r \leqslant R$. Thus

$$\int_1^R \frac{-v'(r)}{\sqrt{2\varphi(v(r))}}\,dr \leqslant \int_1^R \exp[(N-1)(R-1)]\,dr < \infty\,,$$

contradicting (6.9).

Proof of Theorem 6.3. Adding the inequalities for g_+ and g_- we have $\gamma_+ - \gamma_- \geqslant g_+ + g_-$, so g_+ and g_- are bounded. Since g_+ is bounded, $\min(g_+, \gamma_+)$ satisfies the same integral condition as g_+ and $(\gamma_+ - f^+(x)) \geqslant \min(g_+, \gamma_+)$ on $\{|x| > R\}$. Dealing similarly with the minus case and recalling Lemma 6.4, we can suppose: $f \geqslant 0$ and there is an $R > 0$ such that $f > \gamma_+$ on $\{|x| < R\}$ while $f(x) = \gamma_+ - g_+(|x|)$ on $\{|x| > R\}$. Let $f_n = f$ on $\{|x| < n\}$ and $f_n = 0$ on $\{|x| > n\}$. By Theorem 6.2, $(\mathrm{P}_{\beta f_n})$ has a compactly supported solution u_n. Moreover, u_n and $w_n = f + \Delta u_n$ are nondecreasing in n since f_n is nondecreasing in n. (Note that we may assume $\int f_n \neq 0$ if $N = 1, 2$). At this point if $N = 1$ or $N = 2$ we assume $\sup\beta(\boldsymbol{R}) > \gamma_+$. Since

$$(w_n - \gamma_+) - \Delta u_n = f_n - \gamma_+ \leqslant (f - \gamma_+)^+\,,$$

$u_n \leqslant \bar{u}$, $w_n \leqslant \bar{w}$ where $\bar{u} = G_{\beta-\gamma_+}(f - \gamma_+)^+$ and $\bar{w} = (f - \gamma_+)^+ + \Delta\bar{u}$ are in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ (since $(f - \gamma_+)^+ \in L^1(\boldsymbol{R}^N)$). Thus $u_n \uparrow u \leqslant \bar{u}$ and $w_n \uparrow w \leqslant \bar{w}$ for some functions $u, w \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$. We have $w \in \beta(u)$ a.e. and $f + \Delta u = w$ in $\mathcal{D}'(\boldsymbol{R}^N)$, so u is a solution of $(\mathrm{P}_{\beta f})$. Thus it is enough to bound the supports of the u_n uniformly in n.

We make one further reduction. By Lemma 6.5, it suffices to assume that $\beta(\boldsymbol{R}) \subset [-A, A]$ for some $A > 0$. On $\{|x| > R\}$, $(f - \gamma_+)^+ = (-g_+(|x|))^+ = 0$, so $(f - \gamma_+)^+ + \Delta\bar{u} \in \beta(\bar{u})$ implies $\Delta\bar{u} \in L^\infty(\{|x| > R\})$. Also $\bar{u} \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^N)$, so $\bar{u} \in C^1(\{|x| > R\})$ as in the previous proof. Choose $R_0 > R$ and set

$M = \sup_{|x|=R_0} \{|\bar{u}(x)|\}$. Now if

$$h(r, r_0) = \int_{r_0}^{r} \left(\int_{\varrho}^{r} \left(\frac{s}{\varrho}\right)^{N-1} g_+(s)\, ds \right) d\varrho \,,$$

then $\lim_{r\to\infty} h(r, r_0) = \infty$ since

$$h(r, r_0) = \begin{cases} \displaystyle\int_{r_0}^{r} (s - r_0)\, g_+(s) ds & \text{if } N = 1\,, \\ \displaystyle\int_{r_0}^{r} s \left(\log\left(\frac{s}{r_0}\right)\right) g_+(s)\, ds & \text{if } N = 2\,, \\ \displaystyle\frac{1}{N-2} \int_{r_0}^{r} \left(\frac{s^{N-1}}{r^{N-2}} - s\right) g_+(s)\, ds & \text{if } N \geqslant 3\,. \end{cases}$$

Choose $\bar{R} > R_0$ so that $M = h(\bar{R}, R_0)$ and let

$$v(x) = \begin{cases} h(\bar{R}, |x|) & \text{if } R_0 \leqslant |x| \leqslant \bar{R} \\ 0 & \text{if } |x| > \bar{R}\,. \end{cases}$$

We have $v \in C^1(\{|x| \geqslant R_0\})$, $v = M \geqslant \bar{u} \geqslant u_n$ on $\{|x| = R_0\}$ and $\gamma_+ - \Delta v = \gamma_+ - g_+$ on $\{R_0 < |x| < \bar{R}\}$, and $-\Delta v = 0$ on $\{|x| > \bar{R}\}$. Thus if $z = \gamma_+$ on $\{R_0 < |x| < \bar{R}\}$ and $z = \gamma_+ - g_+$ on $\{|x| > \bar{R}\}$, then $z \in \beta(v)$ and $z - \Delta v = \gamma_+ - g_+ \geqslant f_n$. It now follows from Lemma 6.6 that $v \geqslant u_n$ on $\{|x| \geqslant R_0\}$ and so $\operatorname{supp} u_n \subset \{|x| \leqslant \bar{R}\}$.

Finally we treat the cases $N = 1, 2$ and $\gamma_+ = \sup \beta(\boldsymbol{R})$. The main difference here is that $\bar{u}$ is not available as an upper bound on the u_n. Assuming, however, that there is an $R_0 > R$ such that $u_n(x) \leqslant M < \infty$ for $|x| = R_0$ we can proceed as above. It remains then to obtain such a bound. We may assume $\beta(r) = \{\gamma_+\}$ if $r > 0$. Next observe that since u_n, $\Delta u_n \in L^1(\boldsymbol{R}^N)$, $\int_{[u_n>0]} - \Delta u_n \geqslant 0$. This follows from Lemma A.13 and A.14 if $N = 2$ (let p tend to the characteristic function of $(0, \infty)$ in Lemma A.13) and from (4.2) if $N = 1$. Thus

$$\int_{[u_n>0]} \gamma_+ \leqslant \int_{[u_n>0]} (\gamma_+ - \Delta u_n) = \int_{[u_n>0,\,|x|<R]} \gamma_+ - g_+ + \int_{[u_n>0,\,|x|>R]} (\gamma_+ - g_+)$$

which implies that

$$\int_{[u_n<0,\,|x|>R]} g_+(|x|)\, dx \leqslant \int_{[|x|<R]} f\,.$$

Now $|\Delta u_n| \leqslant \gamma_+ + g_+ \in L^\infty_{\text{loc}}(\{|x| > R\})$ so $u_n \in C^1(\{|x| > R\})$. Let $u = \lim u_n$. By the above and Fatou's lemma

$$\int_{[u<0,\,|x|>R]} g_+(|x|)\, dx \leqslant \int_{[|x|<R]} f\,.$$

Since $g_+(|x|) \notin L^1(\{|x| > R\})$, there exists x_0 such that $u(x_0) = 0$ (and hence $u_n(x_0) = 0$ for all n) and $R < |x_0|$. Pick R_1, R_2 so that $R < R_1 < |x_0| < R_2$ and let $v_n \in W^{1,1}_0(\{R_1 < |x| < R_2\})$ satisfy $\Delta v_n = \Delta u_n$. Since u_n is nondecreasing in n and $u_n \geqslant 0$, $u_n - v_n \geqslant 0$ and $u_n - v_n$ is nondecreasing in n. Also since $\{\Delta u_n\}$ is bounded in $L^\infty(\{R_1 < |x| < R_2\})$, $\{v_n\}$ is bounded in $C(\{R_1 \leqslant |x| \leqslant R_2\})$. By Harnack's theorem either $\{u_n - v_n\}$ is bounded on compact subsets of $\{R_1 < |x| < R_2\}$ or $\lim(u_n - v_n) = \infty$ on $\{R_1 < |x| < R_2\}$. Since $u_n(x_0) = 0$, the first alternative holds and the proof is complete.

REMARK. The hypotheses in Theorem 6.3 cannot be weakened. Let $f \in L^1_{\text{loc}}(\boldsymbol{R}^N)$ be radial, $\int_{|x|<1} (f-1)^+ > 0$ and $f \leqslant 1$ on $|x| \geqslant 1$. Then:

(1) If $\beta(r) = \{1\}$ for $r > 0$, $\beta(0) = [0, 1]$ and $\beta(r) = \{0\}$ for $r < 0$ and $(\mathrm{P}_{\beta f})$ has a solution $u \in L^1_0(\boldsymbol{R}^N)$, then $\int_{|x|>1} (1-f) = \infty$.

(2) Assume $\int_{|x|>1} q_N(1-f) < \infty$ where $q_N(x) = \log|x|$, $|x|$, 1 according as $N = 1, 2$ or $N \geqslant 3$. Then there is a maximal monotone β with $\beta(0) \supset [0, 1]$ and $\beta(\boldsymbol{R}) = \boldsymbol{R}$ such that $(\mathrm{P}_{\beta f})$ does not have a solution $u \in L^1_0(\boldsymbol{R}^N)$. The proofs use the methods introduced above and are left to the reader.

REMARK. Redheffer [6] has also obtained results related to those of this section while considering equations of a more general form. However, the results of [6] do not imply those presented here.

Appendix: What you always wanted to know about Δ^{-1} in $L^1(\boldsymbol{R}^N)$.

This appendix contains both known material which is presented somewhat differently than in other sources and results which appear to be new.

DEFINITION A.1. Let u be a measurable function on $\boldsymbol{R}^N$, $1 < p < \infty$ and $1/p' + 1/p = 1$. Then $\|u\|_{M^p} = \min\{C \in [0, \infty]: \int_K |u(x)|\, dx \leqslant C(\text{meas}\, K)^{1/p'}$ for all measurable $K \subset \boldsymbol{R}^N\}$. $M^p(\boldsymbol{R}^N)$ is the set of measurable functions u on $\boldsymbol{R}^N$ satisfying $\|u\|_{M^p} < \infty$.

It is easy to verify that $M^p(\boldsymbol{R}^N)$ is a Banach space under the norm $\|\,\|_{M^p}$. Furthermore, it follows at once from Fatou's lemma that if $\{u_n\} \subseteq M^p(\boldsymbol{R}^N)$ is a sequence satisfying $u_n \to u$ a.e., then $\|u\|_{M^p} \leqslant \liminf_{n\to\infty} \|u_n\|_{M^p}$.

LEMMA A.2. *Let $1 \leqslant q < p < \infty$. Then for every measurable function u on $\boldsymbol{R}^N$*

$$\text{(i)} \qquad \frac{(p-1)^p}{p^{p+1}} \|u\|_{M^p}^p \leqslant \sup_{\lambda>0} \{\lambda^p \operatorname{meas} [|u| > \lambda]\} \leqslant \|u\|_{M^p}^p .$$

Moreover

$$\text{(ii)} \qquad \int_K |u|^q \leqslant \frac{p}{p-q} \left(\frac{p}{q}\right)^{q/p} \|u\|_{M^p}^q (\operatorname{meas} K)^{(p-q)/p}$$

for every measurable subset $K \subseteq \boldsymbol{R}^N$. In particular, $M^p(\boldsymbol{R}^N) \subset L^q_{\mathrm{loc}}(\boldsymbol{R}^N)$ with continuous injection and $u \in M^p(\boldsymbol{R}^N)$ implies $|u|^q \in M^{p/q}(\boldsymbol{R}^N)$.

PROOF OF LEMMA A.2. We begin with the right-hand inequality of (i). Given u and $\lambda > 0$, set $K_i = \{x \in \boldsymbol{R}^N\colon |x| \leqslant i \text{ and } |u(x)| > \lambda\}$. Then

$$\lambda \operatorname{meas} K_i \leqslant \int_{K_i} |u(x)| dx \leqslant \|u\|_{M^p} (\operatorname{meas} K_i)^{1/p'} .$$

Thus $\lambda(\operatorname{meas} K_i)^{1/p} \leqslant \|u\|_{M^p}$ and as $i \to \infty$ we find $\lambda^p \operatorname{meas}[|u| > \lambda] \leqslant \|u\|_{M^p}^p$, which is the desired inequality. For the converse, set $\alpha(\lambda) = \operatorname{meas}[|u| > \lambda]$ and $B = \sup\limits_{\lambda>0} \lambda^p \alpha(\lambda)$. Given $\lambda_0 > 0$ we have

$$\int_K |u(x)|\, dx \leqslant \lambda_0 \operatorname{meas} K + \int_{[|u|>\lambda_0]} |u(x)|\, dx .$$

Now

$$\int_{[|u|>\lambda_0]} |u(x)|\, dx = -\int_{\lambda_0}^{\infty} \lambda\, d\alpha = \int_{\lambda_0}^{\infty} \alpha(\lambda)\, d\lambda + \alpha(\lambda_0)\, \lambda_0 \leqslant B \int_{\lambda_0}^{\infty} \frac{1}{\lambda^p}\, d\lambda + \frac{B}{\lambda_0^{p-1}} = B \frac{p}{p-1} \frac{1}{\lambda_0^{p-1}} .$$

Choosing λ_0 so that $\lambda_0^p \operatorname{meas} K = Bp$ we obtain

$$\int_K |u(x)|\, dx \leqslant \frac{p^{1+1/p}}{p-1} B^{1/p} (\operatorname{meas} K)^{1/p'}$$

or

$$\|u\|_{M^p}^p \leqslant \frac{p^{p+1}}{(p-1)^p} B .$$

Thus the first inequality in Lemma A.2 holds.

In order to prove (ii), observe that

$$B_1 = \sup_{\lambda>0} \lambda^{p/q} \operatorname{meas}\,[|u|^q > \lambda] = \sup_{\eta>0} \eta^p \operatorname{meas}\,[|u| > \eta] \leqslant \|u\|_{M^p}^p$$

and

$$\|\,|u|^q\|_{M^{p/q}}^{p/q} \leqslant \frac{p}{q}\left(\frac{p}{p-q}\right)^{p/q} B_1 \leqslant \frac{1}{-}\left(\frac{p}{p-q}\right)^{p/q} \|u\|_{M^p}^p$$

by (i). This inequality is a restatement of (ii).

As a direct application of Lemma A.2 we get:

LEMMA A.3. *For $N > \alpha > 0$ the function $|x|^{-\alpha}$ lies in $M^{N/\alpha}(\boldsymbol{R}^N)$.*

REMARKS. $M^p(\boldsymbol{R}^N)$ coincides with the space $L(p, \infty)$ of [9, Ch. V.3] and the norm $\|\,\|_{M^p}$ coincides with the norm $\|\,\|_{p\infty}$ of [9, p. 203]. However, the current definition is more direct. It has the disadvantage that $p = 1$ is not allowed however. It is clear that $L^p(\boldsymbol{R}^N) \subset M^p(\boldsymbol{R}^N)$ and $\|u\|_{M^p} \leqslant \|u\|_{L^p}$ (Hölder's inequality). Lemma A.3 shows this inclusion is strict.

LEMMA A.4. *If $E \in M^p(\boldsymbol{R}^N)$, $1 < p < \infty$, and $f \in L^1(\boldsymbol{R}^N)$, then $E * f \in$ $\in M^p(\boldsymbol{R}^N)$ and*

$$\|E * f\|_{M^p} \leqslant \|E\|_{M^p} \|f\|_{L^1}.$$

PROOF. We have

$$\begin{aligned}\int_K |(E * f)(x)| &\leqslant \int_K \Big(\int_{\boldsymbol{R}^n} |E(x-y)||f(y)|dy\Big)\, dx \\ &= \int_{\boldsymbol{R}^n} |f(y)|\Big(\int_K |E(x-y)|dx\Big)\, dy = \int_{\boldsymbol{R}^n} |f(y)|\Big(\int_{K-y} |E(z)|\, dz\Big)\, dy \\ &\leqslant \|E\|_{M^p} \|f\|_{L^1} (\operatorname{meas} K)^{1/p'}.\end{aligned}$$

(Note that the above and Fubini's theorem shows $\int_{\boldsymbol{R}^n} E(x-y) f(y)\, dy$ converges absolutely a.e. $x \in \boldsymbol{R}^N$.)

REMARK. This result is essentially (c) of Theorem 1 in [8, p. 119] (see the comment following the proof). However, our proof is simpler.

The spaces M^p enter our problem via the fundamental solutions for $-\Delta$. Let E_N be defined by

$$E_N(x) = \begin{cases} \dfrac{1}{(N-2) b_N |x|^{N-2}} & \text{if } N \geqslant 3\,, \\[2ex] \dfrac{1}{2\pi} \log \dfrac{1}{|x|} & \text{if } N = 2\,, \end{cases}$$

where b_N is the volume of the unit N-ball. Then $E_N \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^N)$ and $-\Delta E_N = \delta$ in $\mathcal{D}'(\boldsymbol{R}^N)$. Moreover, $E_N \in M^{N/(N-2)}(\boldsymbol{R}^N)$ for $N \geqslant 3$ and $|\mathrm{grad}\, E_N| \in M^{N/(N-1)}(\boldsymbol{R}^N)$ for $N \geqslant 2$. Thus if $N \geqslant 3$ and $f \in L^1(\boldsymbol{R}^N)$ then $u = E_N * f$ provides a solution in the space $M^{N/(N-2)}(\boldsymbol{R}^N)$ of the equation $-\Delta u = f$. Our next result asserts that any solution of $-\Delta u = f$ satisfying a certain decay condition of infinity must coincide with $E_N * f$ if $N \geqslant 3$.

LEMMA A.5. *Let $N \geqslant 3$, $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$, $\Delta u \in L^1(\boldsymbol{R}^N)$ and u satisfy*

$$\lim_{n\to\infty} \int\limits_{1 \leqslant |x| \leqslant 2} |u(nx)|\, dx = 0 . \tag{A.6}$$

Then $u = E_N * (-\Delta u)$. In particular, $u \in M_{M^{N/(N-2)}}(\boldsymbol{R}^N)$, $|\mathrm{grad}\, u| \in M^{N/(N-1)}(\boldsymbol{R}^N)$ and $\|u\|_{M^{N/(N-2)}} \leqslant c_N \|\Delta u\|_{L^1}$, $\|\mathrm{grad}\, u\|_{M^{N/(N-1)}} \leqslant d_N \|\Delta u\|_{L^1}$ for some constants c_N, d_N independent of u.

Changing variables in (A.6) by setting $y = nx$ one sees that (A.6) is equivalent to

$$\lim_{n\to\infty} n^{-N} \int\limits_{n \leqslant |y| \leqslant 2n} |u(y)|\, dy = 0 . \tag{A.7}$$

Thus (A.6) states that the average of $|u(y)|$ over the annulus $n \leqslant |y| < 2n$ tends to zero. It is obvious that $u \in L^1(\boldsymbol{R}^N)$ or $u \in M^p(\boldsymbol{R}^N)$ $1 < p < \infty$ implies (A.7) holds (for $N \geqslant 1$). Thus (since $E_N * (-\Delta u) \in M^{N/(N-2)}(\boldsymbol{R}^N)$) Lemma A.5 is a direct consequence of the next result.

LEMMA A.8. *Suppose $N \geqslant 1$, $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ and $\Delta u = 0$. If u satisfies* (A.6), *then $u = 0$.*

PROOF OF LEMMA A.8. The result is obvious if $N = 1$. We assume that $N \geqslant 2$. If v is integrable on the sphere $S_R = \{x\colon |x| = R\}$ we will denote the average of v over S_R by v_R. Since the average of $|u(y)|$ over $n \leqslant |y| \leqslant 2n$ may be expressed as a weighted average of $|u|_r$ over $n \leqslant r \leqslant 2n$, (A.7) implies that there is a sequence $r_n \to \infty$ such that $|u|_{r_n} \to 0$. Since u is harmonic on $\mathbb{R}^N$, Poisson's formula implies that $|u(x)| \leqslant 2^N |u|_{r_n}$ whenever $|x| < r_n/2$. Letting $n \to \infty$ with x fixed in this inequality we find $u(x) = 0$.

The next lemma is used in Section 2 to prove the uniqueness of solutions of (P) if $N \geqslant 3$.

LEMMA A.10. *Suppose $N \geqslant 3$, $u \in M^{N/(N-2)}(\boldsymbol{R}^N)$ and $\Delta u \in L^1(\boldsymbol{R}^N)$. Then for every $p \in \mathfrak{T}_0$ ($\mathfrak{T}_0$ is defined in Section 1)*

$$\sqrt{p'(u)}\,|\operatorname{grad} u| \in L^2(\boldsymbol{R}^N)$$

and

$$\int p'(u)|\operatorname{grad} u|^2 + \int \Delta u\, p(u) \leqslant 0 .$$

REMARK. Lemma A.10 implies, in particular, that for every $\lambda > 0$ $|\operatorname{grad} u| \in L^2([|u| < \lambda])$.

PROOF OF LEMMA A.10. Let $f = -\Delta u$, $u_n = \varrho_n * u$, $f_n = -\Delta u_n = \varrho_n * f$ where $\{\varrho_n\}$ is a sequence of mollifiers satisfying $\varrho_n \to \delta_0$ in $\mathfrak{D}'(\boldsymbol{R}^N)$. Multiplying the equation $f_n = -\Delta u_n$ by $p(u_n)$ and $\zeta \in \mathfrak{D}^+$ we obtain

$$\int p'(u_n)|\operatorname{grad} u_n|^2 \zeta + \int p(u_n) \operatorname{grad} u_n \operatorname{grad} \zeta = \int f_n p(u_n) \zeta .$$

Now $u_n \to u$ in $W^{1,1}_{\text{loc}}$ since $u \in W^{1,1}_{\text{loc}}$ (by Lemma A.5) and $f_n \to f$ in $L^1(\boldsymbol{R}^N)$. Thus Fatou's lemma allows us to conclude that $p'(u)|\operatorname{grad} u|^2 \zeta \in L^1(\boldsymbol{R}^N)$ and

$$\int p'(u)|\operatorname{grad} u|^2 \zeta + \int p(u) \operatorname{grad} u \operatorname{grad} \zeta \leqslant \int f p(u) \zeta .$$

Now choose $\zeta = \zeta_n = \zeta_0(x/n)$ as before. It remains to show that $X_n = \int p(u) \operatorname{grad} u \operatorname{grad} \zeta_n \to 0$ as $n \to \infty$. Recall $u \in M^{N/(N-2)}(\boldsymbol{R}^N)$ so $\operatorname{grad} u \in M^{N/(N-1)}(\boldsymbol{R}^N)$ by Lemma A.5. For $\lambda > 0$ one has

$$\begin{aligned} |X_n| \leqslant \frac{1}{n} \int_{[|u| \leqslant \lambda]} |\operatorname{grad} u(x)| \left| (\operatorname{grad} \zeta_0)\left(\frac{x}{n}\right) \right| |p(u(x))|\, dx \,+ \\ + \frac{\|p\|_{L^\infty}}{n} \int_{[|u| > \lambda]} |\operatorname{grad} u(x)| \left| (\operatorname{grad} \zeta_0)\left(\frac{x}{n}\right) \right| dx = Y_n + Z_n . \end{aligned}$$

Now $|p(u)| \leqslant p(\lambda) - p(-\lambda)$ when $|u| \leqslant \lambda$ since $p \in \mathfrak{T}_0$. Thus the first term Y_n above satisfies

$$|Y_n| \leqslant \frac{C_N}{n} \big(p(\lambda) - p(-\lambda)\big) \|\operatorname{grad} \zeta_0\|_{L^\infty} \|\operatorname{grad} u\|_{M^{N/(N-1)}} n^{N/p'}$$

where $1/p' = 1 - (N-1)/N = 1/N$ and C_N depends only on N. We conclude that

$$Y_N \leqslant C_N\big(p(\lambda) - p(-\lambda)\big) \|\operatorname{grad} u\|_{M^{N/(N-1)}} \|\operatorname{grad} \zeta_0\|_{L^\infty} .$$

On the other hand

$$|Z_n| \leqslant \frac{\|p\|_{L^\infty}}{n} \|\operatorname{grad} \zeta_0\|_{L^\infty} \int\limits_{[|u|>\lambda]} |\operatorname{grad} u| \leqslant$$

$$\leqslant \frac{\|p\|_{L^\infty} \|\operatorname{grad} \zeta_0\|_{L^\infty}}{n} \|\operatorname{grad} u\|_{M^{N/(N-1)}} (\operatorname{meas}\,[|u| > \lambda])^{1/N}$$

and $\operatorname{meas}[|u| > \lambda] < \infty$ since $u \in M^{N/(N-2)}$. Thus $\lim\limits_{n\to\infty} \sup |X_n| \leqslant C_N\big(p(\lambda) - p(-\lambda)\big) \|\operatorname{grad} u\|_{M^{N/(N-1)}} \|\operatorname{grad} \zeta_0\|_{L^\infty}$ for all $\lambda > 0$. Since $p(0) = \lim\limits_{\lambda\to 0} p(\lambda) = 0$, $\lim\limits_{n\to\infty} |X_n| = 0$.

The results corresponding to Lemma A.5 and A.10 in the case $N = 2$ are presented next.

LEMMA A.11. *Let* $u \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2)$, $\Delta u \in L^1(\boldsymbol{R}^2)$ *and*

$$\text{(A.12)} \qquad \lim_{n\to\infty} \int\limits_{1\leqslant|x|\leqslant 2} |(\operatorname{grad} u)(nx)|\, dx = 0 .$$

Then $\operatorname{grad} u = \operatorname{grad} E_2 * (-\Delta u)$. *In particular,*

$$|\operatorname{grad} u| \in M^2(\boldsymbol{R}^2) \quad \text{and} \quad \|\operatorname{grad} u\|_{M^2} \leqslant d_2 \|\Delta u\|_{L^1}$$

for some d_2 *independent of* u.

PROOF OF LEMMA A.11. Let $v = \operatorname{grad} u + \operatorname{grad} E_2 * \Delta u$, $v = (v_1, v_2)$. Clearly $v_i \in L^1_{\mathrm{loc}}(\boldsymbol{R}^2)$ satisfies (A.6) for $i = 1, 2$. Moreover, $\Delta v_i = 0$ in $\mathcal{D}'(\boldsymbol{R}^2)$. Thus $v_i = 0$ by Lemma A.8, and the result follows.

LEMMA A.13. *Let* $u \in W^{1,1}_{\mathrm{loc}}(\boldsymbol{R}^2)$, $|\operatorname{grad} u| \in M^2(\boldsymbol{R}^2)$ *and* $\Delta u \in L^1(\boldsymbol{R}^2)$. *Then* $p'(u)|\operatorname{grad} u|^2 \in L^1(\boldsymbol{R}^2)$ *for all* $p \in \mathcal{T}$ *and, in particular,* $|\operatorname{grad} u| \in L^2\big([|u| \leqslant \lambda]\big)$ *for* $\lambda > 0$. *If, in addition, there is a* $k > 0$ *for which* $\operatorname{meas}[|u| > k] < \infty$, *then*

$$\int p'(u)|\operatorname{grad} u|^2 + \int \Delta u\, p(u) \leqslant 0$$

for all $p \in \mathcal{T}$. *In particular,* $\int \Delta u = 0$.

PROOF OF LEMMA A.13. Let $\varrho_n \in \mathfrak{D}$, $\varrho_n \to \delta$, $u_n = \varrho_n * u$ and $f = -\Delta u$ so that $f_n = \varrho_n * f = -\Delta u_n$. Clearly $u_n \to u$ in $W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2)$ and $f_n \to f$ in $L^1(\mathbf{R}^2)$. Let $\zeta \in \mathfrak{D}^+$, $p \in \mathcal{T}$ and multiply $-\Delta u_n = f_n$ by $p(u_n)\zeta$ to obtain

$$\int p'(u_n)|\operatorname{grad} u_n|^2 \zeta = \int f_n p(u_n)\zeta + \int p(u_n) \operatorname{grad} u_n \operatorname{grad} \zeta .$$

Letting $n \to \infty$ we find, as before, $p'(u)|\operatorname{grad} u|^2 \in L^1_{\mathrm{loc}}(\mathbf{R}^2)$ and

$$\int p'(u)|\operatorname{grad} u|^2 \zeta \leqslant \int f p(u)\zeta + \int p(u) \operatorname{grad} u \operatorname{grad} \zeta$$

for $\zeta \in \mathfrak{D}^+(\mathbf{R}^2)$. Set $\zeta = \zeta_n = \zeta_0(x/n)$. We will show that $X_n = \int p(u) \operatorname{grad} u \cdot \operatorname{grad} \zeta_n$ remains bounded since $\operatorname{grad} u \in M^2(\mathbf{R}^2)$ while $X_n \to 0$ if also $\operatorname{meas}[|u| > k]$ is finite for some k. The proof will then be complete. We have

$$\begin{aligned}\left|\int (\operatorname{grad} u \operatorname{grad} \zeta_n) p(u)\right| &\leqslant \|p\|_{L^\infty} \|\operatorname{grad} \zeta_0\|_{L^\infty}\, 1/n \int_{n \leqslant |x| \leqslant 2n} |\operatorname{grad} u| \\ &\leqslant C \|\operatorname{grad} u\|_{M^2},\end{aligned}$$

so the first claim is established. For the second write

$$\begin{aligned}\left|\int (\operatorname{grad} u \operatorname{grad} \zeta_n) p(u)\right| \leqslant &\int_{[|u| \leqslant k]} |\operatorname{grad} u||\operatorname{grad} \zeta_n||p(u)| \\ &+ \int_{[|u| > k]} |\operatorname{grad} u||\operatorname{grad} \zeta_n||p(u)| = K_n + L_n .\end{aligned}$$

we have

$$K_n \leqslant \|p\|_{L^\infty} \Big(\int_{\substack{[|u| \leqslant k] \\ n \leqslant |x| \leqslant 2n}} |\operatorname{grad} u|^2 \Big)^{\frac12} \|\operatorname{grad} \zeta_0\|_{L^2}$$

since $\|\operatorname{grad} \zeta_n\|_{L^2} = \|\operatorname{grad} \zeta_0\|_{L^2}$. Now $|\operatorname{grad} u| \in L^2\big([|u| \leqslant k]\big)$ implies $K_n \to 0$. Finally,

$$\begin{aligned}L_n &\leqslant \|p\|_{L^\infty} \frac{\|\operatorname{grad} \zeta_0\|_{L^\infty}}{n} \int_{[|u| > k]} |\operatorname{grad} u| \leqslant \\ &\leqslant \frac{1}{n} \|p\|_{L^\infty} \|\operatorname{grad} \zeta_0\|_{L^\infty} \|\operatorname{grad} u\|_{M^2} (\operatorname{meas}[|u| > k])^{\frac12}\end{aligned}$$

so $L_n \to 0$ as $n \to \infty$. Choosing $p = \pm 1$ we see that $\pm \int \Delta u \leqslant 0$, and the proof is complete.

LEMMA A.14. *Let* $1 \leqslant p < \infty$ *and* $u \in L^p(\mathbf{R}^2)$ *be such that* $\Delta u \in L^1(\mathbf{R}^2)$. *Then* $u \in W^{1,1}_{\mathrm{loc}}(\mathbf{R}^2)$ *and* $\operatorname{grad} u = \operatorname{grad} E_2 * (-\Delta u)$. *In particular,* $|\operatorname{grad} u| \in M^2(\mathbf{R}^2)$ *and* $\|\operatorname{grad} u\|_{M^2} \leqslant d_2 \|\Delta u\|_{L^1}$.

PROOF OF LEMMA A.14. If $\varrho \in \mathcal{D}(\boldsymbol{R}^2)$, $\tilde{u} = \varrho * u$ has the properties assumed for u as well as $\operatorname{grad} \tilde{u} = (\operatorname{grad} \varrho) * u \in L^p(\boldsymbol{R}^2)$. Thus $\operatorname{grad} \tilde{u}$ satisfies (A.12) and by Lemma A.11

$$\operatorname{grad} \tilde{u} = \operatorname{grad} E_2 * (-\Delta \tilde{u}) = \operatorname{grad} E_2 * \big(\varrho * (-\Delta u)\big) .$$

Choose $\varrho = \varrho_n$ so that $\tilde{u} \to u$ in $L^p(\boldsymbol{R}^2)$ and $\Delta \tilde{u} \to \Delta u$ in $L^1(\boldsymbol{R}^2)$. Then, by the above, $\operatorname{grad} \tilde{u} \to \operatorname{grad} E_2 * (-\Delta u)$ in $M^2(\boldsymbol{R}^2)$ (so also in $L^1_{\text{loc}}(\boldsymbol{R}^2)$) and the result follows.

LEMMA A.15. *Let $u \in L^\infty(\boldsymbol{R}^2)$ be such that $\Delta u \in L^1(\boldsymbol{R}^2)$. Then $u \in W^{1,1}_{\text{loc}}(\boldsymbol{R}^2)$ and $|\operatorname{grad} u| \in L^2(\boldsymbol{R}^2)$. Moreover, there is a constant C such that*

$$\|\operatorname{grad} u\|^2_{L^2} \leqslant C\big(\|u\|_{L^\infty} + \|\Delta u\|_{L^1}\big)\|u\|_{L^\infty} .$$

PROOF OF LEMMA A.15. Using mollification again it suffices to treat $u \in C^\infty(\boldsymbol{R}^2) \cap L^\infty(\boldsymbol{R}^2)$. Let $f = -\Delta u$ and multiply by ζu for $\zeta \in \mathcal{D}^+(\boldsymbol{R}^2)$. One finds

$$\int \zeta |\operatorname{grad} u|^2 - \tfrac{1}{2}\int u^2 \Delta \zeta = \int f \zeta u \leqslant \|f\|_{L^1} \|u\|_{L^\infty} .$$

Setting $\zeta = \zeta_n = \zeta_0(x/n)$ leads to

$$\int \zeta_n |\operatorname{grad} u|^2 \leqslant \tfrac{1}{2}\|u\|^2_{L^\infty} \int |\Delta \zeta_n| + \|f\|_{L^1} \|u\|_{L^\infty} .$$

But $\|\Delta \zeta_n\|_{L^1} = \|\Delta \zeta_0\|_{L^1}$ and the result is obtained by letting $n \to \infty$:
The final result of this Appendix is:

LEMMA A.16. *Let B be a ball of radius R in $\boldsymbol{R}^N$ and $u \in W^{1,p}(B)$ with $1 < p < N$. Then there is a constant C depending only on p and N such that if $\sigma = \operatorname{meas}[|u| < \lambda] > 0$ then*

$$\|u\|_{L^{p^*}(B)} \leqslant \lambda (\operatorname{meas} B)^{1/p^*} + C\left(\left(\frac{\operatorname{meas} B}{\sigma}\right)^{1/p^*} + 1\right) \|\operatorname{grad} u\|_{L^p(B)}$$

$$\text{where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} .$$

PROOF OF LEMMA A.16. Let $u_B = (1/\operatorname{meas} B)\int_B u(x)\, dx$. By Poincaré's inequality (see e.g. [7]) we have $\|u - u_B\|_{L^{p^*}(B)} \leqslant C\|\operatorname{grad} u\|_{L^p(B)}$. Thus

$$\Big[\int_{[|u|<\lambda]\cap B} |u - u_B|^{p^*} dx\Big]^{1/p^*} \leqslant C\|\operatorname{grad} u\|_{L^p(B)}$$

and hence

$$|u_B|\sigma^{1/p^*} \leqslant \lambda\sigma^{1/p^*} + C\|\operatorname{grad} u\|_{L^p(B)}\,.$$

Therefore

$$\begin{aligned}\|u\|_{L^{p^*}(B)} &\leqslant |u_B|(\operatorname{meas} B)^{1/p^*} + C\|\operatorname{grad} u\|_{L^p(B)} \leqslant \\ &\leqslant \lambda(\operatorname{meas} B)^{1/p^*} + C\left[\left(\frac{\operatorname{meas} B}{\sigma}\right)^{1/p^*} + 1\right]\|\operatorname{grad} u\|_{L^p(B)}\,.\end{aligned}$$

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