# A Universal Logic Approach to Adaptive Logics*

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**Abstract**

In this paper, adaptive logics are studied from the viewpoint of universal logic (in the sense of the study of common structures of logics). The common structure of a large set of adaptive logics is described. It is shown that this structure determines the proof theory as well as the semantics of the adaptive logics, and moreover that most properties of the logics can be proved by relying solely on the structure, viz. without invoking any specific properties of the logics themselves.

## 1 Aim and Preliminaries

In this paper the common features of a wide variety of logics is studied. The logics, viz. adaptive logics, are very different both in nature and in application context. Of the adaptive logics studied until now, some are close to **CL** (Classical Logic), others are many valued, still others modal, and there clearly are adaptive logics of a still very different nature. The application contexts too are very varied: handling inconsistency, inductive generalization, abduction, handling plausible inferences, interpreting a person's changing position in an ongoing discussion, compatibility, etc. I shall show that all these logics have a common structure, which determines their proofs as well as their semantics, and moreover their metatheory. Specific adaptive logics will not even be mentioned, except as illustrative examples.

Adaptive logics adapt themselves to specific premise sets. To be more precise, they interpret a premise set "as normally as possible" with respect to some standard of normality. They explicate reasoning processes that display an internal and possibly an external dynamics. The *external* dynamics provides from the non-monotonicity of the inference relation: if premises are added, some consequences may not be derivable any more—formally: there are $\Gamma$, $\Delta$ and $A$ such that $\Gamma \vdash A$ and $\Gamma \cup \Delta \nvdash A$. The *internal* dynamics plays at the level of

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proofs (sequences of inferential steps): as insights in the premises grow, earlier drawn conclusions may be withdrawn, and conclusions withdrawn earlier may be classified again as derivable.

The origin of adaptive logics does not lie in any technical insights, but in an attempt to explicate reasoning processes that occur in actual reasoning, both everyday reasoning and scientific reasoning. As far as scientific reasoning is concerned, the processes are not located in contexts in which finished theories are formulated, but in contexts in which theories are forged or modified, and sometimes in contexts in which theories are applied. So adaptive logics are not intended to be used as the underlying logic of scientific theories,[1] but are intended for explicating problem-solving processes, especially creative processes or discovery processes.

So the actual reasoning processes were there first, and adaptive logics are an attempt to explicate them. The requirements on good explications—the *locus classicus* is obviously [19]—entail that, in comparison to more usual logics, adaptive logics have some unusual properties. The central cause of the unusual properties is that the explicated reasoning processes define a consequence relation for which there is no positive test (that is not even partially recursive)—see [18] or [17]. This need not prevent one from studying adaptive logics in a formally decent way, viz. in agreement with the usual metatheoretic standards.

A list of reasoning processes for which adaptive logics have been devised can be found in [7]. That paper contains also a list of inference relations that originated from without the adaptive tradition, but have been characterized by an adaptive logic, often under a translation.[2]

Many (not all) adaptive logics seem to have a common structure. Some of the others can be given this structure under a translation. This structure moreover seems to be central for the proof theory as well as for the semantics of an adaptive logic. It seems equally central for the soundness and completeness of the proofs with respect to the semantics, for the proofs of further metatheoretic properties, for computational aspects, and so on. In view of this originated the plan to describe this common structure, which was labelled the *standard format* (for adaptive logics). The first steps in that direction were taken in [5].

In the present paper, I shall first present a slightly improved version of the standard format (Section 2) and I shall describe the way in which the standard format determines the proof theory (Section 3) as well as the semantics (Section 4). From Section 5 to Section 8, I shall present a large list of theorems, including soundness and completeness, and prove them in terms of the standard format, viz. without referring to the specific properties of any adaptive logic. In Section 9, I briefly consider criteria for final derivability for all adaptive logics that use the Reliability strategy.

For certain purposes, it is necessary to combine adaptive logics. This too may be described and studied without relying on the properties of the specific adaptive logics that are combined, but the results cannot be presented in this paper.

The results of the present paper are provisional. This is so because every-

[1] Where **AL** is an adaptive logic, $Cn_{\mathbf{AL}}(\Gamma) = \{A \mid \Gamma \vdash_{\mathbf{AL}} A\}$ may still be taken to be a theory (of a sort). Occasionally, it might be useful to study such theory as a (sometimes provisional) alternative for an existing theory.

[2] At `http://logica.ugent.be/adlog/` more recent lists and references to the relevant papers are available.

thing is provisional, but also for a more specific reason. It is very well possible that a different, presumably more general 'standard format' is possible, which would apply to more adaptive logics in the sense that it imposes less requirements on them. During the last ten years, especially under the influence of young logicians in close contact with philosophers of science and other philosophers, the number and variety of adaptive logics has constantly been growing and more and more different domains were explored. The present version of the standard format is apparently sufficient for nearly all of these. Still, there is no warrant that the general idea of an adaptive logic has been fully explored. So it seems better to define the notion of an adaptive logic in an intuitive way and to consider the standard format as open for revision.

## 2 The Standard Format

A (non-combined) adaptive logic **AL** is characterized by a triple:
(1) A *lower limit logic* **LLL**: a reflexive, transitive, monotonic, and compact logic that has a characteristic semantics (with no trivial models) and contains **CL**—see below.
(2) A *set of abnormalities* $\Omega$: a set of formulas characterized by a (possibly restricted) logical form $\mathsf{F}$, which is **LLL**-contingent and contains at least one logical symbol.
(3) An *adaptive strategy*.

In this paper, the lower limit logic **LLL** will be taken to contain **CL**. If, for example, **LLL** is a paraconsistent logic, then the language will be extended with a new negation connective, and possibly with some further new connectives, which are all given their **CL**-meaning.[3] This does not hamper the paraconsistency of the logic. The standard negation will still be paraconsistent and one may require that the premise set be formulated in the non-extended language. The presence of the classical connectives greatly simplifies the metalinguistic proofs, and sometimes also the formulation of the adaptive logic. So, for the sake of generality, I introduce an important convention.

**Convention** From now on, $\neg$ will denote *classical negation* in all contexts (whereas $\sim$ is the standard negation) and $\sqcup$ will denote *classical disjunction* in all contexts (whereas $\vee$ is the standard disjunction).

Typical for adaptive logics is that their consequence set extends the **LLL**-consequence set by presupposing that 'as many' members of $\Omega$ are false as the premise set permits. If the logical form $\mathsf{F}$ that characterizes the set of abnormalities would not be **LLL**-contingent, then either $\vdash_{\mathbf{LLL}} \mathsf{F}$ or $\vdash_{\mathbf{LLL}} \neg\mathsf{F}$. In the former case, no member of $\Omega$ can possibly be false; in the latter case all members of $\Omega$ lead to triviality on **LLL**. In both cases, **AL** would reduce to **LLL**. Exactly the same situation arises if the logical form is $A$ (the **LLL**-consequences are unavoidable and every other formula would be a non-consequence). I qualify the restriction further when I come to the upper limit logic.

[3] In some paraconsistent logics, for example da Costa's $\mathbf{C}_i$ systems, classical negation can be defined—see [21] and elsewhere. Even where this is not the case, classical negation can simply be added, were it only for technical purposes.

The logical form $\mathsf{F}$ that characterizes the set of abnormalities $\Omega$ may be restricted. This means that the metavariables that occur in the logical form may be required to denote formally specific entities. Let us consider an example. Some inconsistency-adaptive logics have $\{\exists(A \wedge {\sim}A) \mid A \in \mathcal{F}\}$ as their set of abnormalities, in which $\mathcal{F}$ is the set of (open and closed) formulas and $\exists(A \wedge {\sim}A)$ is the existential closure of $A \wedge {\sim}A$. Other inconsistency-adaptive logics have $\{\exists(A \wedge {\sim}A) \mid A \in \mathcal{F}^p\}$ as their set of abnormalities, in which $\mathcal{F}^p$ is the set of primitive formulas (those not containing any logical symbols other than identity).

In the expression $Dab(\Delta)$, $\Delta$ will always be a finite subset of $\Omega$, and $Dab(\Delta)$ will denote the *classical* disjunction of the members of $\Delta$—the disjuncts may be defined to occur in a certain order or not, all such disjunctions being logically equivalent anyway. If $\Delta$ is a singleton, $Dab(\Delta)$ is an abnormality (a member of $\Omega$) and no classical disjunction occurs. If $\Delta = \emptyset$, $Dab(\Delta)$ is the empty string and $A \sqcup Dab(\Delta)$ is $A$.

The need for a strategy is best seen as follows. For some premise sets $\Gamma$ and lower limit logics **LLL**, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$ will obtain for some $\Delta$s that are not singletons. The strategy then determines what it means to interpret the premises 'as normally as possible' in such cases. In this paper, I shall only consider the Reliability strategy and the Minimal Abnormality strategy. These are the most basic strategies and will be clarified when we come to the proofs and semantics. For some lower limit logics and sets of abnormalities it holds that, whenever $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta)$, then there is an abnormality $D \in \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} D$. Where this is the case, both strategies come to the same, which is then called the Simple strategy. Other strategies seem less attractive and were only devised in order to characterize in terms of an adaptive logic consequence relations that were described in the literature.[4]

An adaptive logic **AL** can now be described in a different way. The **AL**-consequences of $\Gamma$ are all those formulas that can be derived from $\Gamma$ by **LLL**-means *and* by relying on the supposition that "the members of $\Omega$ are false in as far as $\Gamma$ permits them to be false". This expression is ambiguous, but the strategy disambiguates it.

The lower limit logic **LLL** and the set of abnormalities $\Omega$ jointly determine a so-called *upper limit logic* **ULL**. Syntactically **ULL** is obtained by adding an axiom (or rule) that connects abnormality to triviality. More generally, the upper limit logic **ULL** is exactly like the lower limit logic, except that it trivializes abnormalities. So, where $\Delta^{\neg} = \{\neg A \mid A \in \Delta\}$, we can define:

**Definition 1** $\Gamma \vdash_{\mathbf{ULL}} A$ *iff* $\Gamma \cup \Omega^{\neg} \vdash_{\mathbf{LLL}} A$

An **ULL**-model is an **LLL**-model that verifies no member of $\Omega$.

**Definition 2** $\Gamma \vDash_{\mathbf{ULL}} A$ *iff* $A$ *is verified by the* **LLL***-models of* $\Gamma$ *that verify no member of* $\Omega$.

A *normal premise set* is one that requires no abnormality to be true, in other words a premise set that has **ULL**-models.

[4] Moreover, some of these strategies may themselves be reduced to the Reliability or Minimal Abnormality strategy under a modal translation. An example is the Normal Selections strategy, which was invoked in [4] and [14] to characterize consequence relations that validate all classical consequences of all maximal consistent subsets of a premise set (or validate all classical consequences of the maximal extensions of a consistent premise set by defaults).

Even if $\mathsf{F}$, the form that characterizes $\Omega$, is **LLL**-contingent, some formulas of this form may be not **LLL**-contingent. If some of them are **LLL**-theorems, there would be no **ULL**-models and $Cn_{\mathbf{ULL}}(\Gamma)$ would be trivial for all $\Gamma$. So in such a case $\Omega$ should be defined as $\{\mathsf{F} \mid \nvdash_{\mathbf{LLL}} \mathsf{F}; \ldots\}$, in which $\ldots$ denotes further restrictions on $\mathsf{F}$. An example, taken from [10], is $\Omega = \{\neg\Diamond A \mid \nvdash_{\mathbf{S5}} \neg\Diamond A; A \in \mathcal{F}\}$, in which $\mathcal{F}$ is the set of non-modal formulas and a predicative version of **S5** is the lower limit logic. So the set of abnormalities comprises all formulas of the form $\neg\Diamond A$ that are not **S5**-theorems. From now on I suppose that the proviso $\nvdash_{\mathbf{LLL}} \mathsf{F}$ occurs standardly in the definition of $\Omega$, but I do not mention it in the subsequent examples because it is useless in view of the specific $\mathsf{F}$.

Often some logic is considered the standard of deduction—which logic is regarded as such may depend on the context. If the standard of deduction is the upper limit logic of an adaptive logic, the adaptive logic is called *corrective*. If the standard of deduction is the lower limit logic of an adaptive logic, the adaptive logic is called *ampliative*. In the present paper, I shall take **CL** to be the standard of deduction. This is merely a pragmatic decision, not one I consider correct.

Let us consider some examples of adaptive logics. The inconsistency-adaptive $\mathbf{CLuN}^m$ is defined by:

(1) *lower limit logic*: **CLuN** (full positive **CL** together with excluded middle, for example $(A \supset \sim A) \supset \sim A$)
(2) *set of abnormalities*: $\Omega = \{\exists(A \wedge \sim A) \mid A \in \mathcal{F}\}$
(3) *adaptive strategy*: Minimal Abnormality

The inconsistency-adaptive $\mathbf{ACLuN}^r$ is defined by the same elements, except that Reliability is its strategy. The upper limit logic of both adaptive logics is **CL**, viz. syntactically obtained by extending **CLuN** with the axiom $(A \wedge \sim A) \supset B$ and semantically obtained by restricting the set of **CLuN**-models to those that verify no inconsistency. On the previous convention, $\mathbf{ACLuN}^m$ and $\mathbf{ACLuN}^r$ are corrective adaptive logics. If a theory that was intended to be consistent and was given **CL** as its underlying logic turns out to be inconsistent, one wants to interpret it 'as normally as possible' in order to forge a consistent replacement for it *by reasoning from it*.

The (ampliative) logic of inductive generalization: $\mathbf{IL}^m$ is defined by:

(1) *lower limit logic*: **CL**
(2) *set of abnormalities*: $\Omega = \{\exists A \wedge \exists \sim A \mid A \in \mathcal{F}^\circ\}$, in which $\mathcal{F}^\circ$ is the set of formulas that contain no individual constants and no quantifiers (the purely functional formulas)
(3) *adaptive strategy*: Minimal Abnormality

The upper limit logic is **UCL**, obtained syntactically by extending **CL** with the axiom $\exists\alpha A(\alpha) \supset \forall\alpha A(\alpha)$, which reduces non-uniformity to triviality, and obtained semantically by restricting the set of **CL**-models to the uniform **CL**-models. Uniformity is obviously an idea taken from [20]. In all **UCL**-models $v(\pi^r) \in \{\emptyset, D^{(r)}\}$: the extension of a predicate of rank $r$ is either empty or universal (the set of all $r$-tuples of members of the domain). Needless to say, applying **UCL** to the actual world results in triviality because not all objects have the same properties, viz. the world is not (fully) uniform. The $\mathbf{IL}^m$-consequences of our observational data contain the generalizations that would hold in the world if it were as uniform as is compatible with our observational data.

The set referred to in the last sentence is not as easily obtained as one might be tempted to think. If the data comprise $Pa$, $Qa$, $Rb$, $\sim Qb$, $Pc$ and $Rc$, neither $\forall x(Px \supset Qx)$ nor $\forall x(Rx \supset \sim Qx)$ are derivable because they are not jointly compatible with $Pc$ and $Rc$. I refer to [9] and [6] for the remarkable properties of the logic of inductive generalization and for the way in which it may be combined with adaptive logics handling background-knowledge.

As a final example, consider the (ampliative) adaptive logic of plausibility $\mathbf{T}^m$:

(1) *lower limit logic*: **T** (a specific predicative version of this logic—see for example [15])

(2) *set of abnormalities*: $\Omega = \{\Diamond A \wedge \sim A \mid A \in \mathcal{F}^p\}$, in which $\mathcal{F}^p$ is the set of formulas that contain no logical symbols other than identity

(3) *adaptive strategy*: Minimal Abnormality

The upper limit logic is the well-known system **Triv**, obtained by adding the axiom $\Diamond A \supset A$ to **S5**. Intuitively, the premises of the form $\Diamond A$ may be read as stating that $A$ is plausible.[5] The adaptive logic $\mathbf{T}^m$ interprets the premises in such a way that plausible formulas are true 'in as far as the premises permit'.

Of course, plausibilities come in degrees. But this is not a difficult problem. It is solved by superimposing a sequence of adaptive logics $\mathbf{T}^m_i$ that are exactly as $\mathbf{T}^m$, except that $\Omega_1 = \{\Diamond A \wedge \sim A \mid A \in \mathcal{F}^p\}$, $\Omega_2 = \{\Diamond\Diamond A \wedge \sim A \mid A \in \mathcal{F}^p\}$, etc. and, where $\Diamond^i$ is a sequence of $i$ times $\Diamond$, $\Diamond^0 A_0$, which is $A_0$, expresses that $A_0$ is certain, $\Diamond^1 A_1$ that $A_1$ is very plausible, $\Diamond^2 A_2$ that $A_2$ is somewhat less plausible, etc.

# 3 Proofs

The dynamics of the proofs is controlled by attaching *conditions* (finite subsets of $\Omega$) to derived formulas and by introducing a marking definition. While lines are added to a proof by applying the rules of inference, the marking definition determines for every stage of the proof[6] which lines are 'in' and which are 'out'. The rules of inference are determined by the lower limit logic **LLL** and the set of abnormalities $\Omega$, whereas the marking definition is determined by $\Omega$ and by the strategy. So the lines that occur (marked or unmarked) in a proof are independent of the strategy.

A line of an annotated proof consists of a line number, a formula, a justification, and a condition. The presence of the latter distinguishes dynamic proofs from usual proofs. The justification consists of a (possibly empty) list of line numbers (from which the formula is derived) and of the name of a rule.

As remarked before, the rules determine which lines (consisting of the four aforementioned elements) may be added to a given proof. The only effect of the marking definition is that, at every stage of the proof, certain lines are marked whereas others are unmarked. For all marking definitions, whether a line is marked depends only on the condition of the line and on the minimal *Dab*-formulas—see below—that have been derived in the proof. Whether the marks are considered as parts of the annotation is obviously a conventional matter.

[5] On the present approach, all **CL**-consequences of $A$ are also plausible, but this can be avoided, for example as in [9].

[6] A stage of a proof can be seen as a sequence of lines and a proof can be seen as a chain of stages. Every proof starts off with stage 1. Adding a line to a proof by applying one of the rules of inference brings the proof to its next stage, which is the sequence of all lines written so far.

I shall discuss the notion of an adaptive proof below, but first introduce the rules of inference and the marking definitions. The rules of inference reduce to three generic rules. Where

$$A \qquad \Delta$$

abbreviates that $A$ occurs in the proof on the condition $\Delta$, the generic rules are:

PREM If $A \in \Gamma$:

$$\begin{array}{ll} \ldots & \ldots \\ \hline A & \emptyset \end{array}$$

RU If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B$:

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \end{array}$$

RC If $A_1, \ldots, A_n \vdash_{\mathbf{LLL}} B \sqcup Dab(\Theta)$

$$\begin{array}{ll} A_1 & \Delta_1 \\ \ldots & \ldots \\ A_n & \Delta_n \\ \hline B & \Delta_1 \cup \ldots \cup \Delta_n \cup \Theta \end{array}$$

Consider, by way of example, the lower limit logic **CLuN**. In view of $p,\ p \supset q \vdash_{\mathbf{CLuN}} q$, RU can be applied if $p$ and $p \supset q$ occur in the proof. In view of $\sim p,\ p \vee q \vdash_{\mathbf{CLuN}} q \vee (p \wedge \sim p)$, RC can be applied if $\sim p$ and $p \vee q$ occur in the proof.

Before we move on to the marking definitions, let me point out the important relation between adaptive proofs and **LLL**-proofs. An **AL**-proof from $\Gamma$ can be seen as a **LLL**-proof in disguise. In the latter, the members of the condition are joined to the formula by classical conjunctions.

**Lemma 1** *There is an* **AL***-proof from* $\Gamma$ *that contains a line on which* $A$ *is derived on the condition* $\Delta$ *iff* $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$.

*Proof.* $\Rightarrow$ By an obvious induction on the length of the **AL**-proof from $\Gamma$, the proof can be transformed to a **LLL**-proof from $\Gamma$, every line in which some $A_i$ is derived on a condition $\Delta_i$ being replaced by a line in which $A_i \sqcup Dab(\Delta_i)$ is derived.

$\Leftarrow$ In view of the compactness of **LLL**, there is a **LLL**-proof of $A \sqcup Dab(\Delta)$ from $\Gamma$. So there is a **AL**-proof from $\Gamma$, obtained by applications of PREM and RU, in which $A \sqcup Dab(\Delta)$ is derived on the condition $\emptyset$. By applying RC to the last step, one obtains a proof from $\Gamma$ in which $A$ is derived on the condition $\Delta$. ■

There is also a direct relation between an **AL**-proof and an **ULL**-proof. As the members of $\Omega$ lead to triviality on **ULL**, deleting the conditions in an **AL**-proof results in an **ULL**-proof.

The marking definitions require some preparation. $Dab(\Delta)$ is a *minimal Dab-formula* at stage $s$ of the proof iff it is the formula of a line with condition $\emptyset$ and no $Dab(\Delta')$ with $\Delta' \subset \Delta$ is the formula of a line with condition $\emptyset$. A *choice set* of $\Sigma = \{\Delta_1, \Delta_2, \ldots\}$ is a set that contains one element out of each member of $\Sigma$. A *minimal choice set* of $\Sigma$ is a choice set of $\Sigma$ of which no proper subset is a choice set of $\Sigma$. Where $Dab(\Delta_1)$, ..., $Dab(\Delta_n)$ are the minimal $Dab$-formulas that are derived on condition $\emptyset$ at stage $s$, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$ and $\Phi_s(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \ldots, \Delta_n\}$.[7]

**Definition 3** *Marking for Reliability: Line $i$ is marked at stage $s$ iff, where $\Delta$ is its condition, $\Delta \cap U_s(\Gamma) \neq \emptyset$.*

Intuitively, $U_s(\Gamma) = \Delta_1 \cup \ldots \cup \Delta_n$ is the set of all abnormalities that are unreliable. Indeed, they are disjuncts of a minimal $Dab$-formula; the premises state that one of the disjuncts is true, but fail to specify which disjunct is true.

**Definition 4** *Marking for Minimal Abnormality: Line $i$ is marked at stage $s$ iff, where $A$ derived on the condition $\Delta$ at line $i$, (i) there is no $\varphi \in \Phi_s(\Gamma)$ such that $\varphi \cap \Delta = \emptyset$, or (ii) for some $\varphi \in \Phi_s(\Gamma)$, there is no line at which $A$ is derived on a condition $\Theta$ for which $\varphi \cap \Theta = \emptyset$.*

The idea behind this definition is derived from the semantics—see Section 4. *If* the minimal $Dab$-formulas at stage $s$ are indeed the minimal $Dab$-consequences if $\Gamma$, then a $A$ *is* derivable iff it is true in every model of $\Gamma$ that verifies one of the members of $\Phi_s(\Gamma)$.

Marks may come and go. So the rules of inference combined with the marking definitions determine an unstable notion of derivability, viz. derivability at a stage: $A$ is *derived* from $\Gamma$ *at stage s* of the proof iff $A$ is the formula of a line that is unmarked at stage $s$. However, we also want a different, stable, kind of derivability: *final derivability.* Intuitively, $A$ line is finally derived at line $i$ in an **AL**-proof from $\Gamma$ iff $A$ is the formula of line $i$, line $i$ is unmarked, and the proof is stable with respect to line $i$. The latter phrase means that line $i$ will not be marked in any extension of the proof. For some **AL**, $\Gamma$, and $A$, only an infinite proof from $\Gamma$ in which $A$ is the formula of a line $i$ is stable with respect to line $i$. A simple example is the proof of $p$ from $\{p \vee q, {\sim}q, (q \wedge {\sim}q) \vee (r_i \wedge {\sim}r_i), (q \wedge {\sim}q) \supset (r_i \wedge {\sim}r_i)\}_{(i \in \{0,1,\ldots\})}$. Only after $r_i \wedge {\sim}r_i$ was derived for all $i \in \mathbb{N}$ does the proof become stable.

Needless to say, the existence of an infinite proof is not established by producing the proof but by reasoning in the metalanguage. So it seems more attractive to define final derivability as follows (as it was defined from the very beginning).

**Definition 5** *$A$ is* finally derived *from $\Gamma$ on line $i$ of a proof at stage $s$ iff (i) $A$ is the second element of line $i$, (ii) line $i$ is not marked at stage $s$, and (iii) every extension of the proof in which line $i$ is marked may be further extended in such a way that line $i$ is unmarked.*

**Definition 6** $\Gamma \vdash_{\mathbf{AL}} A$ *($A$ is* finally **AL**-derivable *from $\Gamma$) iff $A$ is finally derived on a line of a proof from $\Gamma$.*

Definition 5 has an attractive game-theoretic interpretation. The proponent has shown that $A$ is finally derived at line $i$ iff, whenever the opponent extends the proof in such a way that line $i$ is marked, the proponent is able to extend the extension further in such a way that line $i$ is unmarked.

[7] The proofs can be made more effective by slightly modifying the definition, for example by defining $\Phi_s(\Gamma)$ as the minimal $\Omega$-closed choice sets, where the a $\Omega$-closure of a choice set $\varphi$ is $Cn_{\mathbf{LLL}}(\varphi) \cap \Omega$.

From now on $\mathbf{AL}^r$, respectively $\mathbf{AL}^m$, will refer to an adaptive logic the third element of which is the Reliability strategy, respectively the Minimal Abnormality strategy. So $\Gamma \vdash_{\mathbf{AL}^r} A$ ($A$ is *finally* $\mathbf{AL}^r$*-derivable* from $\Gamma$) iff $A$ is finally derived on a line of a proof from $\Gamma$ in which lines are marked according to the Reliability strategy. Similarly, $\Gamma \vdash_{\mathbf{AL}^m} A$ ($A$ is *finally* $\mathbf{AL}^m$*-derivable* from $\Gamma$) iff $A$ is finally derived on a line of a proof from $\Gamma$ in which lines are marked according to the Minimal Abnormality strategy.

Remark that while **ULL** extends **LLL** by validating some further rules, **AL** extends **LLL** by validating some *applications* of those rules.

# 4 Semantics

The adaptive semantics selects some **LLL**-models of $\Gamma$ as **AL**-models *of* $\Gamma$. The selection depends on $\Omega$ and on the strategy. First we need some technicalities. Let $Dab(\Delta)$ be a *minimal Dab-consequence of* $\Gamma$ iff $\Gamma \vDash_{\mathbf{LLL}} Dab(\Delta)$ and, for all $\Delta' \subset \Delta$, $\Gamma \nvDash_{\mathbf{LLL}} Dab(\Delta')$. Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal $Dab$-consequences of $\Gamma$,

$$U(\Gamma) = \Delta_1 \cup \Delta_2 \cup \ldots .$$

Where $M$ is a **LLL**-model, $Ab(M) = \{A \in \Omega \mid M \models A\}$.

**Definition 7** *A* **LLL***-model* $M$ *of* $\Gamma$ *is* reliable *iff* $Ab(M) \subseteq U(\Gamma)$.

**Definition 8** $\Gamma \vDash_{\mathbf{AL}^r} A$ *iff* $A$ *is verified by all reliable models of* $\Gamma$.

**Definition 9** *A* **LLL***-model* $M$ *of* $\Gamma$ *is* minimally abnormal *iff there is no* **LLL***-model* $M'$ *of* $\Gamma$ *such that* $Ab(M') \subset Ab(M)$.

**Definition 10** $\Gamma \vDash_{\mathbf{AL}^m} A$ *iff* $A$ *is verified by all minimally abnormal models of* $\Gamma$.

Let $\mathcal{M}_\Gamma^{\mathbf{LLL}}$ be the set of **LLL**-models of $\Gamma$, $\mathcal{M}_\Gamma^{\mathbf{ULL}}$ be the set of **ULL**-models of $\Gamma$, $\mathcal{M}_\Gamma^m$ the set of $\mathbf{AL}^m$-models (minimal abnormal models) of $\Gamma$, and $\mathcal{M}_\Gamma^r$ the set of $\mathbf{AL}^r$-models (reliable models) of $\Gamma$. If $\mathcal{M}_\Gamma^{\mathbf{LLL}}$, the definitions warrant that $M \in \mathcal{M}_\Gamma^r$ iff $M$ verifies no other abnormalities than those that are unreliable with respect to $\Gamma$ and that $M \in \mathcal{M}_\Gamma^m$ iff no other **LLL**-model of $\Gamma$ is (set theoretically) less abnormal than $M$.

**Lemma 2** $\mathcal{M}_\Gamma^{\mathbf{ULL}} \subseteq \mathcal{M}_\Gamma^m \subseteq \mathcal{M}_\Gamma^r \subseteq \mathcal{M}_\Gamma^{\mathbf{LLL}}$.

*Proof.* Immediate in view of Definitions 2, 7 and 9. ■

# 5 The Upper Limit Logic

Definitions 1 and 2 give us at once:

**Theorem 1** $\Gamma \vdash_{\mathbf{ULL}} A$ *iff* $\Gamma \vDash_{\mathbf{ULL}} A$.

The following theorem is extremely important. It is the 'motor' for the adaptive logic. By applying **AL**, we try to get as close to **ULL** as possible. Theorem 2 informs us that this can be done by considering 'as many' abnormalities false as $\Gamma$ permits.

**Theorem 2** $\Gamma \vdash_{\mathbf{ULL}} A$ *iff there is a finite* $\Delta \subseteq \Omega$ *such that* $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$. *(Derivability Adjustment Theorem)*

*Proof.* The following six statements are equivalent:

$$\Gamma \vdash_{\mathbf{ULL}} A$$

By the Definition 1:

$$\Gamma \cup \Omega^{\neg} \vdash_{\mathbf{LLL}} A$$

As **LLL** is compact:

$$\Gamma' \cup \Delta^{\neg} \vdash_{\mathbf{LLL}} A \text{ for a finite } \Gamma' \subseteq \Gamma \text{ and a finite } \Delta \subseteq \Omega$$

As **LLL** contains **CL**:

$$\Gamma' \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta) \text{ for those } \Gamma' \text{ and } \Delta$$

As **LLL** is monotonic:

$$\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta) \text{ for a finite } \Delta \subseteq \Omega\,.$$

■

**Theorem 3** **ULL** *is reflexive, transitive, monotonic, and compact and contains* **CL**.

*Proof.* Immediate in view of Definition 1. ■

The upper limit logic is axiomatized by adding to **LLL** the axiom schema $\neg\mathsf{F}$ with the restriction that pertains to the logical form characterizing $\Omega$. This may be seen as not very elegant. However, most upper limit logics can be axiomatized very simply, viz. as the lower limit logic extended with the axiom schema $\neg\mathsf{F}$. This works whenever there is no restriction on $\mathsf{F}$, or whenever for every $A$ of the form $\mathsf{F}$ there is a finite $\Delta \in \Omega$ for which $A \vdash_{\mathbf{LLL}} Dab(\Delta)$.[8] But even if this is not the case although $\Omega$ is characterized by a restricted logical form, it often is the case that no **ULL**-models verifies any formula of the unrestricted form $\mathsf{F}$. The adaptive logic $\mathbf{IL}^m$ is a simple example.

# 6 Strong Reassurance

Graham Priest's $\mathbf{LP}^m$ from [23] is an adaptive logic which is not in standard format because it selects models in terms of properties of the assignment (or interpretation) not in terms of the formulas verified by the model—see [2] for a discussion. $\mathbf{LP}^m$ has the odd property that some models are not selected because there are less abnormal models, but that none of the latter are selected either because there are still less abnormal models. So there is an infinite sequence of less and less abnormal models. This is often seen as a disadvantage—see also [3]. That a model is not selected should be justified by the presence of a selected model. This property was labelled Strong Reassurance, Smoothness, or Stopperedness. I now prove that it holds for adaptive logics in standard format. It will play a role in proofs of subsequent theorems.

[8]Not restricting the form $\mathsf{F}$ that characterizes $\Omega$ sometimes leads to flip-flop logics—see [5] or [13].

**Theorem 4** *If* $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^m$*, then there is a* $M' \in \mathcal{M}_\Gamma^m$ *such that* $Ab(M') \subset Ab(M)$*. (Strong Reassurance for Minimal Abnormality.)*

*Proof.* The theorem holds vacuously if $\mathcal{M}_\Gamma^m = \mathcal{M}_\Gamma^{\mathbf{LLL}}$. So consider a $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^m$. Let $D_1, D_2, \ldots$ be a list of all members of $\Omega$ and define:

$$\Delta_0 = \emptyset\,;$$

if $Ab(M') \subseteq Ab(M)$ for some **LLL**-model $M'$ of $\Gamma \cup \Delta_i \cup \{\neg D_{i+1}\}$, then

$$\Delta_{i+1} = \Delta_i \cup \{\neg D_{i+1}\}\,,$$

otherwise

$$\Delta_{i+1} = \Delta_i\,;$$

finally

$$\Delta = \Delta_0 \cup \Delta_1 \cup \Delta_2 \cup \ldots$$

The theorem is established by the following three steps.

*Step 1*: $\Gamma \cup \Delta$ has **LLL**-models. This follows immediately from the construction of $\Delta$ and from the compactness of **LLL**.

*Step 2*: If $M'$ is a model of $\Gamma \cup \Delta$, then $Ab(M') \subseteq Ab(M)$.

Suppose there is a $D_j \in \Omega$ such that $D_j \in Ab(M') - Ab(M)$. Let $M''$ be a model of $\Gamma \cup \Delta_{j-1}$ for which $Ab(M'') \subseteq Ab(M)$. As $D_j \notin Ab(M)$, $D_j \notin Ab(M'')$. Hence $M''$ is a model of $\Gamma \cup \Delta_{j-1} \cup \{\neg D_j\}$ and $Ab(M'') \subseteq Ab(M)$. So $\neg D_j \in \Delta_j \subseteq \Delta$. As $M'$ is a model of $\Gamma \cup \Delta$, $D_j \notin Ab(M')$. But this contradicts the supposition.

*Step 3*: Every model of $\Gamma \cup \Delta$ is a minimal abnormal model of $\Gamma$.

Suppose that $M'$ is a model of $\Gamma \cup \Delta$, but is not a minimal abnormal model of $\Gamma$. Hence, by Definition 9, there is a model $M''$ of $\Gamma$ for which $Ab(M'') \subset Ab(M')$.

It follows that $M''$ is a model of $\Gamma \cup \Delta$. If it were not, then, as $M''$ is a model of $\Gamma$, there is a $\neg D_j \in \Delta$ such that $M'$ verifies $\neg D_j$ and $M''$ falsifies $\neg D_j$. But then $M'$ falsifies $D_j$ and $M''$ verifies $D_j$, which is impossible in view of $Ab(M'') \subset Ab(M')$.

Consider any $D_j \in Ab(M') - Ab(M'') \neq \emptyset$. As $M''$ is a model of $\Gamma \cup \Delta_{j-1}$ that falsifies $D_j$, it is a model of $\Gamma \cup \Delta_{j-1} \cup \{\neg D_j\}$. As $Ab(M'') \subset Ab(M')$ and $Ab(M') \subseteq Ab(M)$, $Ab(M'') \subset Ab(M)$. It follows that $\Delta_j = \Delta_{j-1} \cup \{\neg D_j\}$ and hence that $\neg D_j \in \Delta$. But then $D_j \notin Ab(M')$. Hence, $Ab(M'') = Ab(M')$. So the supposition leads to a contradiction. ■

**Theorem 5** *If* $M \in \mathcal{M}_\Gamma^{\mathbf{LLL}} - \mathcal{M}_\Gamma^r$*, then there is a* $M' \in \mathcal{M}_\Gamma^r$ *such that* $Ab(M') \subset Ab(M)$*. (Strong Reassurance for Reliability.)*

*Proof.* Immediate in view of Theorem 4 and Lemma 2. ■

**Corollary 1** *If* $\Gamma$ *has* **LLL***-models,* $\Gamma$ *has* $\mathbf{AL}^m$*-models as well as* $\mathbf{AL}^m$*-models. (Reassurance.)*

# 7 Soundness and Completeness

The proofs in this section rely on the standard format, including the fact that the lower limit logic is supposed to have a characteristic semantics.

**Lemma 3** *If $A$ is finally derived at line $i$ of an $\mathbf{AL}^r$-proof from $\Gamma$, and $\Delta$ is the condition of line $i$, then $\Delta \cap U(\Gamma) = \emptyset$.*

*Proof.* Suppose that the antecedent is true but that $\Delta \cap U(\Gamma) \neq \emptyset$. Then there is a minimal *Dab*-consequence of $\Gamma$, say $Dab(\Delta')$, for which $\Delta \cap \Delta' \neq \emptyset$. So the $\mathbf{AL}^r$-proof from $\Gamma$ has an extension in which $Dab(\Delta')$ is derived (on the condition $\emptyset$). But then, where $s$ is the last stage of the extension, $\Delta' \subseteq U_s(\Gamma)$ and $\Delta \cap U_s(\Gamma) \neq \emptyset$, whence line $i$ is marked at stage $s$ in view of Definition 3. As $Dab(\Delta')$ is a minimal *Dab*-consequence of $\Gamma$, $\Delta' \subseteq U_{s'}(\Gamma)$ for all stages following $s$. So the extension has no further extension in which line $i$ is unmarked. In view of Definition 6, this contradicts that $A$ is finally derived at line $i$ of the $\mathbf{AL}^r$-proof from $\Gamma$. ■

**Theorem 6** $\Gamma \vdash_{\mathbf{AL}^r} A$ *iff, for some finite $\Delta \subset \Omega$, $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

*Proof.* $\Rightarrow$ Suppose that $\Gamma \vdash_{\mathbf{AL}^r} A$. So $A$ is finally derived on line $i$ of an $\mathbf{AL}^r$-proof from $\Gamma$. Let $\Delta$ be the condition of line $i$. But then $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$ by Lemma 1 and $\Delta \cap U(\Gamma) = \emptyset$ by Lemma 3.

$\Leftarrow$ Suppose that, for some finite $\Delta \subset \Omega$, $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$ for a finite $\Delta \subset \Omega$. So there is a $\mathbf{AL}^r$-proof from $\Gamma$ (containing only applications of PREM and RU) in which $A \sqcup Dab(\Delta)$ is derived on the condition $\emptyset$. By an application of RU, a line $i$ can be added that has $A$ as its formula and $\Delta$ as its condition and this line is unmarked. In any extension of this proof in which line $i$ is marked $Dab(\Theta)$ is derived on the condition $\emptyset$ for some $\Theta \subset \Omega$ such that $\Theta \cap \Delta \neq \emptyset$. As $\Delta \cap U(\Gamma) = \emptyset$, there is a $\Theta' \subset \Theta$ for which $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta')$. So the extension can be further extended in such a way that $Dab(\Theta)$ occurs on the condition $\emptyset$. But then $A$ is finally derived at line $i$ in view of Definition 6. ■

**Theorem 7** $\Gamma \vDash_{\mathbf{AL}^r} A$ *iff there is a (finite) $\Delta \subset \Omega$ for which $\Gamma \vDash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$.*

*Proof.* $\Rightarrow$ Suppose that $\Gamma \vDash_{\mathbf{AL}^r} A$, whence all members of $\mathcal{M}^r_\Gamma$ verify $A$. So $\Gamma \cup (\Omega - U(\Gamma))^\neg \vDash_{\mathbf{LLL}} A$. As **LLL** is compact, $\Gamma' \cup \Delta^\neg \vDash_{\mathbf{LLL}} A$ for a finite $\Gamma' \subset \Gamma$ and a finite $\Delta \subset \Omega$. But then, by **CL**, $\Gamma' \vDash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$. So, as **LLL** is monotonic, $\Gamma \vDash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$.

$\Leftarrow$ Suppose there is a $\Delta \subset \Omega$ for which $\Gamma \vDash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$ and $\Delta \cap U(\Gamma) = \emptyset$. $\Gamma \vDash_{\mathbf{AL}^r} A$ holds vacuously if $\mathcal{M}^{\mathbf{LLL}}_\Gamma = \emptyset$. So suppose that $\mathcal{M}^{\mathbf{LLL}}_\Gamma \neq \emptyset$ and that all members of $\mathcal{M}^{\mathbf{LLL}}_\Gamma \neq \emptyset$ verify $A \sqcup Dab(\Delta)$. By Theorem 5, $\mathcal{M}^r_\Gamma \neq \emptyset$. As $\Delta \cap U(\Gamma) = \emptyset$, all $\mathbf{AL}^r$-models of $\Gamma$ falsify $Dab(\Delta)$. So all $\mathbf{AL}^r$-models of $\Gamma$ verify $A$. ■

As **LLL** was supposed to be sound and complete with respect to its semantics:

**Corollary 2** $\Gamma \vdash_{\mathbf{AL}^r} A$ *iff* $\Gamma \vDash_{\mathbf{AL}^r} A$. *(Soundness and Completeness for $\mathbf{AL}^r$.)*

For the Minimal Abnormality strategy we need one more bit of terminology. Where $Dab(\Delta_1)$, $Dab(\Delta_2)$, ... are the minimal *Dab*-consequences of $\Gamma$, $\Phi(\Gamma)$ is the set of minimal choice sets of $\{\Delta_1, \Delta_2, \ldots\}$.

The proofs of Lemma 4 and Theorem 14 require some facts about minimal choice sets. I merely list them because proving them would require too much space (even if the proofs are simple).

**Fact 1** *If $\Sigma$ is a set of sets and $\varphi$ is a minimal choice set of $\Sigma$, then, for every $A \in \varphi$, there is a $\Delta \in \Sigma$ for which $\varphi \cap \Delta = \{A\}$.*

**Fact 2** *If $\Sigma$ and $\Sigma'$ are sets of sets, then, for every minimal choice set $\varphi$ of $\Sigma \cup \Sigma'$, there is a minimal choice set $\psi$ of $\Sigma$ for which $\varphi \supseteq \psi$.*

**Fact 3** *If $\Sigma$ and $\Sigma'$ are sets of sets and for every $\Theta \in \Sigma'$ there is a $\Delta \in \Sigma$ for which $\Delta \subseteq \Theta$, then $\varphi$ is a minimal choice set of $\Sigma \cup \Sigma'$ iff $\varphi$ is a minimal choice set of $\Sigma$.*

**Fact 4** *If $\Sigma$ and $\Sigma'$ are sets of sets, $\psi$ is a minimal choice set of $\Sigma$, and there is no minimal choice set $\varphi$ of $\Sigma \cup \Sigma'$ for which $\varphi \supseteq \psi$, then there is a $\Delta \in \Sigma'$ such that $\Delta \cap \psi = \emptyset$ and, for every $B \in \Delta$, there is a minimal choice set $\psi'$ of $\Sigma$, for which $B \in \Delta \cap \psi'$, and $\psi \supseteq \psi' - \{B\}$.*

If $\Gamma$ has no **LLL**-models, it obviously has no $\mathbf{AL}^m$-models.

**Lemma 4** *If $\Gamma$ has* **LLL***-models, then $\varphi \in \Phi(\Gamma)$ iff $\varphi = Ab(M)$ for some $M \in \mathcal{M}^m_\Gamma$.*

*Proof.* Suppose that $\Gamma$ has **LLL**-models. As every **LLL**-model $M$ of $\Gamma$ verifies all minimal *Dab*-consequences of $\Gamma$, Fact 1 gives us:

(†) Every **LLL**-model $M$ of $\Gamma$ verifies the members of a $\varphi \in \Phi(\Gamma)$.

Suppose that, for some $\varphi \in \Phi(\Gamma)$, $\Gamma \cup (\Omega - \varphi)^\neg$ has no **LLL**-model. By the compactness of **LLL**, there is a finite $\Gamma' \subseteq \Gamma$ and a finite $\Delta \subseteq (\Omega - \varphi)$ such that $\Gamma' \cup \Delta^\neg$ has no **LLL**-model. But then, by **CL**-properties, $\Gamma' \vDash_{\mathbf{LLL}} Dab(\Delta)$ and, by the monotonicity of **LLL**, $\Gamma \vDash_{\mathbf{LLL}} Dab(\Delta)$, which contradicts $\Delta \subseteq (\Omega - \varphi)$. So, for every $\varphi \in \Phi(\Gamma)$, $\Gamma \cup (\Omega - \varphi)^\neg$ has a **LLL**-model $M$ and, as $M$ verifies $\varphi$ in view of (†), $Ab(M) = \varphi$.

We have established that, for every $\varphi \in \Phi(\Gamma)$, there is a **LLL**-model $M$ of $\Gamma$ for which $Ab(M) = \varphi$. But then, in view of (†), every **LLL**-model $M$ of $\Gamma$ for which $Ab(M) \in \Phi(\Gamma)$ is a minimal abnormal model of $\Gamma$ and no other **LLL**-model of $\Gamma$ is a minimal abnormal model of $\Gamma$. ■

**Theorem 8** $\Gamma \vdash_{\mathbf{AL}^m} A$ *iff, for every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$.*

*Proof.* $\Rightarrow$ Suppose that $\Gamma \vdash_{\mathbf{AL}^m} A$. By Definitions 6 and 5 an $\mathbf{AL}^m$-proof from $\Gamma$ contains a line $i$ that has $A$ as its formula and some $\Delta \subset \Omega$ as its condition, line $i$ is unmarked, and every extension of the proof in which line $i$ is marked may be further extended in such a way that line $i$ is unmarked.

Suppose we extend the proof by deriving every minimal *Dab*-consequences of $\Gamma$ on the condition $\emptyset$, whence $\Phi_{s'}(\Gamma) = \Phi(\Gamma)$. In view of Definition 4, line $i$ is unmarked iff the extended proof has a further extension such that (i) $\Delta \cap \varphi = \emptyset$

for some $\varphi \in \Phi(\Gamma)$ and (ii) for every $\varphi \in \Phi(\Gamma)$, there is a line that has $A$ as its formula and some $\Delta'$ as its condition such that $\Delta' \cap \varphi = \emptyset$. There is such an extension iff for every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$.

$\Leftarrow$ Suppose that, for every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$. Then there is an $\mathbf{AL}^m$-proof from $\Gamma$ in which (i) every minimal *Dab*-consequences of $\Gamma$ is derived on the condition $\emptyset$ and (ii) for every $\varphi \in \Phi(\Gamma)$, $A$ is derived on a condition $\Delta \subset \Omega$ for which $\Delta \cap \varphi = \emptyset$. Clearly $A$ is finally derived in this proof. ■

**Theorem 9** $\Gamma \vdash_{\mathbf{AL}^m} A$ *iff* $\Gamma \vDash_{\mathbf{AL}^m} A$. *(Soundness and Completeness for* $\mathbf{AL}^m$*.)*

*Proof.* Each of the following are equivalent:
(1) $\Gamma \vdash_{\mathbf{AL}^m} A$.
By Theorem 8:
(2) For every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vdash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$.
By the Soundness and Completeness of **LLL**:
(3) For every $\varphi \in \Phi(\Gamma)$, there is a $\Delta \subset \Omega$ such that $\Delta \cap \varphi = \emptyset$ and $\Gamma \vDash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$.
By Lemma 4:
(4) For every $M \in \mathcal{M}^m_\Gamma$, there is a $\Delta \subset \Omega$ such that $\Delta \cap Ab(M) = \emptyset$ and $\Gamma \vDash_{\mathbf{LLL}} A \sqcup Dab(\Delta)$.
By **CL**:
(4) Every $M \in \mathcal{M}^m_\Gamma$ verifies $A$.
By Definition 10:
(4) $\Gamma \vDash_{\mathbf{AL}^m} A$.
■

In view of Corollary 2 and Theorem 9, Corollary 1 gives us:

**Corollary 3** *If* $Cn_{\mathbf{LLL}}(\Gamma)$ *is non-trivial, then* $Cn_{\mathbf{AL}^m}(\Gamma)$ *and* $Cn_{\mathbf{AL}^r}(\Gamma)$ *are non-trivial. (Syntactic Reassurance)*

# 8 Some Further Properties

**Theorem 10** $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$ *iff* $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$. *(***AL** *is Dab-conservative with respect to* **LLL***/Immunity .)*

*Proof.* If all **LLL**-models of $\Gamma$ verify $Dab(\Delta)$, then so do all Reliable models of $\Gamma$ and all Minimal abnormal models of $\Gamma$ (Definitions 7 and 9). So the right–left direction is obvious in view of the soundness and completeness of **LLL**-with respect to its semantics and in view of Corollary 2 and Theorem 9.

For the right–left direction suppose that every adaptive model of $\Gamma$ verifies $Dab(\Delta)$. Let $M$ be a **LLL**-model of $\Gamma$. If $M$ is an adaptive model of $\Gamma$, it verifies $Dab(\Delta)$ by the supposition. If $M$ is not an adaptive model of $\Gamma$, then, by Theorems 4 and 5, there is an adaptive model $M'$ of $\Gamma$ such that $Ab(M') \subset Ab(M)$. But then, as $M'$ verifies $Dab(\Delta)$ by the supposition, so does $M$. ■
*Proof.* If $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$, then $Dab(\Delta)$ is derivable on the condition $\emptyset$ from

$\Gamma$ in an **AL**-proof from $\Gamma$ and hence $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$. If $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$, there are two cases.

*Case 1*: $Dab(\Delta)$ is derivable on the condition $\emptyset$ in an **AL**-proof from $\Gamma$. Then $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$ in view of Lemma 1.

*Case 2*: $Dab(\Delta)$ is derivable in an **AL**-proof from $\Gamma$ but only on non-empty conditions.

*Case 2.1*: the strategy is Reliability. Let $\Theta$ be a minimal such condition. In view of Lemma 1, (i) $Dab(\Delta \cup \Theta)$ is derivable on the condition $\emptyset$ in the **AL**-proof from $\Gamma$ and (ii) $Dab(\Delta' \cup \Theta)$ is a minimal *Dab*-consequence of $\Gamma$ for some $\Delta' \subset \Delta$. So $\Theta \subseteq U(\Gamma)$ and every line at which $Dab(\Delta)$ is derived on a condition $\Theta' \supseteq \Theta$ is marked.

*Case 2.1*: the strategy is Minimal Abnormality. Suppose that $Dab(\Delta)$ is finally derived on a condition $\Theta_0$ at line $i$ of an $\mathbf{AL}^m$-proof from $\Gamma$ and that $\Theta_1, \Theta_2, \ldots$ are the minimal conditions on which $Dab(\Delta)$ is derivable in the proof. So there are $\Delta_i \subseteq \Delta$ such that $Dab(\Delta_1 \cup \Theta_1)$, $Dab(\Delta_2 \cup \Theta_2)$, ... are minimal *Dab*-consequences of $\Gamma$. It is easily seen that some minimal choice set of these contains a member of every $\Theta_i$, which contradicts the supposition. ∎

**Theorem 11**

1. $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.
2. $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma)$. *(Reflexivity.)*
3. *If* $A \in \Omega - U(\Gamma)$, *then* $\neg A \in Cn_{\mathbf{AL}^r}(\Gamma)$.
4. *If* $Dab(\Delta)$ *is a minimal Dab-consequence of* $\Gamma$ *and* $A \in \Delta$, *then some* $M \in \mathcal{M}^m_\Gamma$ *verifies* $A$ *and falsifies all members (if any) of* $\Delta - \{A\}$.
5. $U(\Gamma) = \bigcup \Phi(\Gamma)$.
6. $\mathcal{M}^m_\Gamma = \mathcal{M}^m_{Cn_{\mathbf{AL}^m}(\Gamma)}$ *whence* $Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{AL}^m}(Cn_{\mathbf{AL}^m}(\Gamma))$. *(Fixed Point/Idempotence for Reliability.)*
7. $\mathcal{M}^r_\Gamma = \mathcal{M}^r_{Cn_{\mathbf{AL}^r}(\Gamma)}$ *whence* $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^r}(Cn_{\mathbf{AL}^r}(\Gamma))$. *(Fixed Point/Idempotence for Minimal Abnormality.)*
8. $Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$. *(Redundance of* **LLL** *with respect to* **AL***.)*
9. *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, *then* $U(\Gamma \cup \Gamma') = U(\Gamma)$ *and* $\Phi(\Gamma \cup \Gamma') = \Phi(\Gamma)$.
10. *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, *then* $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') = Cn_{\mathbf{AL}}(\Gamma)$.
11. *If* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ *then* $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') \subseteq Cn_{\mathbf{AL}}(\Gamma)$. *(Cautious Cut/Cumulative Transitivity.)*
12. *If* $\Gamma \vDash_{\mathbf{AL}} A$ *for every* $A \in \Gamma'$, *and* $\Gamma \vDash_{\mathbf{AL}} B$, *then* $\Gamma \cup \Gamma' \vDash_{\mathbf{AL}} B$.
    *Viz. if* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ *then* $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Gamma')$. *(Cautious Monotonicity/Cumulative Monotonicity.)*
13. *If* $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma')$ *and* $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$, *then* $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma')$. *(Reciprocity.)*

*Proof. Ad* 1. We know from Lemma 2 that $\mathcal{M}^{\mathbf{ULL}}_{\Gamma} \subseteq \mathcal{M}^{m}_{\Gamma} \subseteq \mathcal{M}^{r}_{\Gamma} \subseteq \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$. So, the formulas verified by all **LLL**-models of Γ are verified by all $\mathbf{AL}^r$-models of Γ, etc.[9] The Soundness and Completeness of **LLL** with respect to its semantics was presupposed for and it was proved for the three other logics: Theorems 1 and 9 and Corollary 2.

*Ad* 2. As **LLL** is reflexive, $\Gamma \subseteq Cn_{\mathbf{LLL}}(\Gamma)$. So $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma)$ in view of 1.

*Ad* 3. If $A \in \Omega - U(\Gamma)$, all reliable models of Γ falsify $A$ in view of Definition 7. So all reliable models of Γ verify $A$. So Corollary 2 warrants that $\neg A \in Cn_{\mathbf{AL}^r}(\Gamma)$.

*Ad* 4. Suppose that every $M \in \mathcal{M}^{m}_{\Gamma}$ verifies a member of $\Delta - \{A\}$. So $\Gamma \vDash_{\mathbf{AL}^m} Dab(\Delta - \{A\})$, and hence, by Theorem 9, $\Gamma \vdash_{\mathbf{AL}^m} Dab(\Delta - \{A\})$. By Theorem 10, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta - \{A\})$, whence $Dab(\Delta)$ is not a minimal *Dab*-consequence of Γ.

It follows that, if $Dab(\Delta)$ is a minimal *Dab*-consequence of Γ and $A \in \Delta$, then some $M \in \mathcal{M}^{m}_{\Gamma}$ falsifies all members (if any) of $\Delta - \{A\}$ and hence, as $\Gamma \vDash_{\mathbf{AL}^m} Dab(\Delta)$, $M$ verifies $A$.

*Ad* 5. Immediate in view of 4.

*Ad* 6. This is vacuously true if $\mathcal{M}^{\mathbf{LLL}}_{\Gamma} = \emptyset$. So suppose that $\mathcal{M}^{\mathbf{LLL}}_{\Gamma} \neq \emptyset$. As **LLL** is reflexive, transitive and monotonic, $Cn_{\mathbf{LLL}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{LLL}}(\Gamma)$. As **LLL** is sound and complete with respect to its semantics, $\mathcal{M}^{\mathbf{LLL}}_{Cn_{\mathbf{LLL}}(\Gamma)} = \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$. In view of Theorem 10, $\Phi(Cn_{\mathbf{LLL}}(\Gamma)) = \Phi(\Gamma)$. But then $\mathcal{M}^{m}_{\Gamma} = \mathcal{M}^{m}_{Cn_{\mathbf{AL}^r}(\Gamma)}$ by Lemma 4 and $Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{AL}^m}(Cn_{\mathbf{AL}^r}(\Gamma))$ by Theorem 9.

*Ad* 7. The reasoning is identical to the one for 6, up to $\mathcal{M}^{\mathbf{LLL}}_{Cn_{\mathbf{LLL}}(\Gamma)} = \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$. In view of Theorem 10, $U(Cn_{\mathbf{LLL}}(\Gamma)) = U(\Gamma)$. It follows that $\mathcal{M}^{r}_{\Gamma} = \mathcal{M}^{r}_{Cn_{\mathbf{AL}^r}(\Gamma)}$ by Definition 7 and $Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^r}(Cn_{\mathbf{AL}^r}(\Gamma))$ by Corollary 2.

*Ad* 8. As **LLL** is Reflexive, $Cn_{\mathbf{AL}}(\Gamma) \subseteq Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$. For the other direction, suppose that $A \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$. By the Soundness of **LLL** with respect to its semantics, all members of $\mathcal{M}^{\mathbf{LLL}}_{Cn_{\mathbf{AL}}(\Gamma)}$ verify $A$. By Definitions 9 and 7, $\mathcal{M}^{\mathbf{AL}}_{Cn_{\mathbf{AL}}(\Gamma)} \subseteq \mathcal{M}^{\mathbf{LLL}}_{Cn_{\mathbf{AL}}(\Gamma)}$, whence all members of $\mathcal{M}^{\mathbf{AL}}_{Cn_{\mathbf{AL}}(\Gamma)}$ verify $A$. By 6 and 7, $\mathcal{M}^{\mathbf{AL}}_{Cn_{\mathbf{AL}}(\Gamma)} = \mathcal{M}^{\mathbf{AL}}_{\Gamma}$, whence all members of $\mathcal{M}^{\mathbf{AL}}_{\Gamma}$ verify $A$. By Theorem 9 and Corollary 2, $A \in Cn_{\mathbf{AL}}(\Gamma)$.

*Ad* 9. Suppose that $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$. As **LLL** is monotonic, $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma')$ if $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$.

Suppose that $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma')$. By the monotonicity of **LLL**, $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma \cup Cn_{\mathbf{AL}}(\Gamma))$. So $Dab(\Delta) \in Cn_{\mathbf{LLL}}(Cn_{\mathbf{AL}}(\Gamma))$ by 2. It follows that $Dab(\Delta) \in Cn_{\mathbf{AL}}(\Gamma)$ by 8, and hence that $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$ by Theorem 10.

So we have established that $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma \cup \Gamma')$ iff $Dab(\Delta) \in Cn_{\mathbf{LLL}}(\Gamma)$.

*Ad* 10. Suppose that $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$. As **LLL** is monotonic, $\mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \Gamma'} \subseteq \mathcal{M}^{\mathbf{LLL}}_{\Gamma}$. $\mathcal{M}^{\mathbf{AL}}_{\Gamma}$ comprises the members of $\mathcal{M}^{\mathbf{LLL}}_{\Gamma}$ that have a certain property, viz. either $Ab(M) \in \Phi(\Gamma)$ or $Ab(M) \subseteq U(\Gamma)$, depending on the strategy. $\mathcal{M}^{\mathbf{AL}}_{\Gamma \cup \Gamma'}$ comprises the members of $\mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \Gamma'}$ that have the same property (in view of 9). So, as all members of $\mathcal{M}^{\mathbf{AL}}_{\Gamma}$ are members of $\mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \Gamma'}$, $\mathcal{M}^{\mathbf{AL}}_{\Gamma} = \mathcal{M}^{\mathbf{AL}}_{\Gamma \cup \Gamma'}$. By Theorem 9 and Corollary 2, $Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') = Cn_{\mathbf{AL}}(\Gamma)$.

*Ad* 11 and 12. Immediate in view of 10.

[9] Inexperienced readers should remember that, if there are no $\mathbf{AL}^r$-models of Γ, then $\Gamma \vDash_{\mathbf{AL}^r} A$ for all (closed) formulas $A$. Indeed, every $\mathbf{AL}^r$-model is either not a model of Γ or it verifies $A$.

*Ad* 13 Suppose that $\Gamma' \subseteq Cn_{\mathbf{AL}}(\Gamma)$ and $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma')$. In view of 10, $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{AL}}(\Gamma \cup \Gamma') = Cn_{\mathbf{AL}}(\Gamma')$. ■

**Theorem 12**

1. *If* $\Gamma$ *is normal, then* $\mathcal{M}^{\mathbf{ULL}}_\Gamma = \mathcal{M}^m_\Gamma = \mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$ *and hence* $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^r}(\Gamma) = Cn_{\mathbf{AL}^m}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$.

2. *If* $\Gamma$ *is abnormal and* $\mathcal{M}^{\mathbf{LLL}}_\Gamma \neq \emptyset$*, then* $\mathcal{M}^{\mathbf{ULL}}_\Gamma \subset \mathcal{M}^m_\Gamma$ *and hence* $Cn_{\mathbf{AL}^m}(\Gamma) \subset Cn_{\mathbf{ULL}}(\Gamma)$.

3. $\mathcal{M}^{\mathbf{ULL}}_\Gamma \subseteq \mathcal{M}^m_\Gamma \subseteq \mathcal{M}^r_\Gamma \subseteq \mathcal{M}^{\mathbf{LLL}}_\Gamma$ *whence* $Cn_{\mathbf{LLL}}(\Gamma) \subseteq Cn_{\mathbf{AL}^r}(\Gamma) \subseteq Cn_{\mathbf{AL}^m}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(\Gamma)$.

4. $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$ *iff* $\Gamma \cup \{A\}$ *is* **LLL***-satisfiable for some* $A \in \Omega - U(\Gamma)$.

5. $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^r}(\Gamma)$ *iff* $\mathcal{M}^r_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$.

6. $\mathcal{M}^m_\Gamma \subset \mathcal{M}^{\mathbf{LLL}}_\Gamma$ *iff there is a (possibly infinite)* $\Delta \subseteq \Omega$ *such that* $\Gamma \cup \Delta$ *is* **LLL***-satisfiable and there is no* $\varphi \in \Phi_\Gamma$ *for which* $\Delta \subseteq \varphi$.

7. *If there are* $A_1, \ldots, A_n \in \Omega$ $(n \geq 1)$ *such that* $\Gamma \cup \{A_1, \ldots, A_n\}$ *is* **LLL***-satisfiable and, for every* $\varphi \in \Phi_\Gamma$, $\{A_1, \ldots, A_n\} \not\subseteq \varphi$*, then* $Cn_{\mathbf{LLL}}(\Gamma) \subset Cn_{\mathbf{AL}^m}(\Gamma)$.

8. $Cn_{\mathbf{AL}^m}(\Gamma)$ *and* $Cn_{\mathbf{AL}^r}(\Gamma)$ *are non-trivial iff* $\mathcal{M}^{\mathbf{LLL}}_\Gamma \neq \emptyset$.

*Proof. Ad* 1. If $\Gamma$ is normal, $U(\Gamma) = \emptyset$ and only **ULL**-models of $\Gamma$ are minimal abnormal.

*Ad* 2. If $\Gamma$ is abnormal, then $\mathcal{M}^{\mathbf{ULL}}_\Gamma = \emptyset$ and $Cn_{\mathbf{ULL}}(\Gamma)$ is trivial. If $\Gamma$ has **LLL**-models, then it has $\mathbf{AL}^m$-models by Corollary 1, whence $Cn_{\mathbf{AL}^m}(\Gamma)$ is non-trivial (there are no trivial **LLL**-models and all $\mathbf{AL}^m$-models of $\Gamma$ are **LLL**-models of $\Gamma$).

*Ad* 3. by 1 and 2, $\mathcal{M}^{\mathbf{ULL}}_\Gamma \subseteq \mathcal{M}^m_\Gamma$. $\mathcal{M}^r_\Gamma \subseteq \mathcal{M}^{\mathbf{LLL}}_\Gamma$ is immediate in view of Definition 7. $\mathcal{M}^m_\Gamma \subseteq \mathcal{M}^r_\Gamma$ follows from Definitions 7 and 9.

*Ad* 4. From Definitions 7 and 8.

*Ad* 5. $\Rightarrow$ Suppose $A \in Cn_{\mathbf{LLL}}(\Gamma) - Cn_{\mathbf{AL}^r}(\Gamma)$. So, for some $A \in \Omega - U(\Gamma)$, all $M \in \mathcal{M}^r_\Gamma$ falsify $A$ whereas some $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma - \mathcal{M}^r_\Gamma$ verifies $A$. $\Leftarrow$ obvious.

*Ad* 6. From Definitions 9 and 10.

*Ad* 7. Suppose the antecedent is true. Every $M \in \mathcal{M}^m_\Gamma$ falsifies some $A_i$ whereas some $M \in \mathcal{M}^{\mathbf{LLL}}_\Gamma$ (viz. an $M \in \mathcal{M}^{\mathbf{LLL}}_{\Gamma \cup \{A_1, \ldots, A_n\}}$) verifies $A_1 \sqcap \ldots \sqcap A_n$, in which $\sqcap$ is classical conjunction.

*Ad* 8. Immediate from Corollary 1 and the fact that no **LLL**-model is trivial. ■

**Theorem 13**

1. *For some* $\Gamma$ *and* $\Delta$, $Cn_{\mathbf{AL}}(\Gamma) \not\subseteq Cn_{\mathbf{AL}}(\Gamma \cup \Delta)$. *(***AL** *is non-monotonic.)*

2. *For some* $\Gamma$ *and* $\Delta$, $\Delta \subseteq Cn_{\mathbf{AL}}(\Gamma)$ *but* $Cn_{\mathbf{AL}}(\Delta) \not\subseteq Cn_{\mathbf{AL}}(\Gamma)$. *(Cut/Transitivity does not hold for* **AL***.)*

3. *There are* $\Gamma$, $A$ *and* $B$ *such that* $\Gamma \cup \{A\} \vdash_{\mathbf{AL}^r} B$ *but* $\Gamma \nvdash_{\mathbf{AL}^r} \neg A \sqcup B$. *(The Deduction Theorem does not hold for* $\mathbf{AL}^r$*).*

*Proof.* For each of the properties, I rely on the following two facts. As the form that characterizes $\Omega$ is **LLL**-contingent, $\Omega \neq \emptyset$. As the form contains at least one logical symbol, $p \notin \Omega$.

*Ad* 1. Where $A \in \Omega$, $p \sqcup A \vdash_{\mathbf{AL}} p$ and $p \sqcup A, A \nvdash_{\mathbf{AL}} p$.

*Ad* 2. Where $A \in \Omega$, $A \vdash_{\mathbf{AL}} p \sqcup A$ and $p \sqcup A \vdash_{\mathbf{AL}} p$ but $A \nvdash_{\mathbf{AL}} p$.

*Ad* 3. Where $A, B \in \Omega$, $A \sqcup B, B \sqcup p, A \vdash_{\mathbf{AL}^r} p$ but $A \sqcup B, B \sqcup p \nvdash_{\mathbf{AL}^r} \neg A \sqcup p$. ■

The fact that every adaptive logic in standard format is non-monotonic, does not rule out that a monotonic consequence relation is characterized by an adaptive logic under a translation. Thus Rescher's Weak consequence relation which is monotonic, is characterized by an adaptive logic—see footnote 4.

In preparation of Theorem 14, remark that the Deduction Theorem holds for **LLL**, viz. that $\Gamma \vdash_{\mathbf{LLL}} \neg A \sqcup B$ if $\Gamma \cup \{A\} \vdash_{\mathbf{LLL}} B$—remember that $\neg$ is classical negation, $\sqcup$ is classical disjunction, and **LLL** contains **CL** and is compact.

**Theorem 14** *If* $\Gamma \cup \{A\} \vdash_{\mathbf{AL}^m} B$ *then* $\Gamma \vdash_{\mathbf{AL}^m} \neg A \sqcup B$. *(Deduction Theorem for* $\mathbf{AL}^m$*.)*

*Proof.* Suppose that the antecedent is true. So, for every $\varphi \in \Phi(\Gamma \cup \{A\})$, there is a $\Delta \subset \Omega$ for which $\Gamma \cup \{A\} \vdash_{\mathbf{LLL}} B \sqcup Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$. It follows (by **CL**-properties) that $\Gamma \vdash_{\mathbf{LLL}} (\neg A \sqcup B) \sqcup Dab(\Delta)$ for all these $\Delta$.

By Fact 2, for every $\psi \in \Phi(\Gamma)$, there is a $\varphi \in \Phi(\Gamma \cup \{A\})$ for which $\varphi \supseteq \psi$. Let $\Sigma$ be the set of minimal *Dab*-consequences of $\Gamma$ and let $\Sigma'$ be the set of minimal *Dab*-consequences of $\Gamma \cup \{A\}$ that are not minimal *Dab*-consequences of $\Gamma$. $\Phi(\Gamma)$ is the set of minimal choice sets of $\Sigma$ and, in view of Fact 3, $\Phi(\Gamma \cup \{A\})$ is the set of minimal choice sets of $\Sigma \cup \Sigma'$. Consider a $\psi \in \Phi(\Gamma)$.

*Case 1*: There is a $\varphi \in \Phi(\Gamma \cup \{A\})$ for which $\varphi \supseteq \psi$. Then there is a $\Delta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} (\neg A \sqcup B) \sqcup Dab(\Delta)$ and $\Delta \cap \varphi = \emptyset$, and hence $\Delta \cap \psi = \emptyset$.

*Case 2*: There is no $\varphi \in \Phi(\Gamma \cup \{A\})$ for which $\varphi \supseteq \psi$. By Fact 4, there is a $\Delta \in \Sigma'$ such that $\Delta \cap \psi = \emptyset$ and, for every $B \in \Delta$, there is a $\psi' \in \Phi(\Gamma)$ for which $B \in \Delta \cap \psi'$ and $\psi \supseteq \psi' - \{B\}$. But then, as $\Gamma \cup \{A\} \vdash_{\mathbf{LLL}} Dab(\Delta)$ and the Deduction Theorem holds for **LLL**, $\Gamma \vdash_{\mathbf{LLL}} \neg A \sqcup Dab(\Delta)$. It follows that there is a $\Delta \subset \Omega$ for which $\Gamma \vdash_{\mathbf{LLL}} (\neg A \sqcup B) \sqcup Dab(\Delta)$ and $\Delta \cap \psi = \emptyset$.

So in both cases, there is a $\Delta$ such that $\Gamma \vdash_{\mathbf{LLL}} (\neg A \sqcup B) \sqcup Dab(\Delta)$ and $\Delta \cap \psi = \emptyset$. It follows that $\Gamma \vdash_{\mathbf{AL}^m} \neg A \sqcup B$. ■

**Theorem 15**

1. *For all* $\Gamma$, $\Gamma \subseteq Cn_{\mathbf{AL}}(\Gamma)$. *(***AL** *is reflexive.)*
2. *For all* $\Gamma$, $Cn_{\mathbf{ULL}}(Cn_{\mathbf{AL}}(\Gamma)) = Cn_{\mathbf{ULL}}(\Gamma)$. *(***AL** *is conservative with respect to* **ULL***.)*
3. *For all* $\Gamma$, $Cn_{\mathbf{AL}}(Cn_{\mathbf{LLL}}(\Gamma)) = Cn_{\mathbf{AL}}(\Gamma)$. *(***LLL** *is conservative with respect to* **AL***.)*

*Proof. Ad* 1. All members of $\Gamma$ can be derived on the condition $\emptyset$ in an **AL**-proof by PREM.

*Ad* 2. As **ULL** is monotonic, $Cn_{\mathbf{ULL}}(\Gamma) \subseteq Cn_{\mathbf{ULL}}(Cn_{\mathbf{AL}}(\Gamma))$ in view of 1. If $\Gamma$ is normal, then $Cn_{\mathbf{AL}}(\Gamma) = Cn_{\mathbf{ULL}}(\Gamma)$ (in view of 1 of Theorem 12), whence $Cn_{\mathbf{ULL}}(\Gamma) = Cn_{\mathbf{ULL}}(Cn_{\mathbf{AL}}(\Gamma))$ (because $Cn_{\mathbf{ULL}}(\Gamma) = Cn_{\mathbf{ULL}}(Cn_{\mathbf{ULL}}(\Gamma))$).

If $\Gamma$ is abnormal, then $Cn_{\mathbf{ULL}}(\Gamma) = Cn_{\mathbf{ULL}}(Cn_{\mathbf{AL}}(\Gamma))$ because both sets are trivial.

*Ad* 3. By applications of PREM and RU, every member of $Cn_{\mathbf{LLL}}(\Gamma)$ can be derived on the condition $\emptyset$ in an **AL**-proof from $\Gamma$. So every **AL**-proof from $Cn_{\mathbf{LLL}}(\Gamma)$ can be transformed into an **AL**-proof from $\Gamma$ by deriving the members of $Cn_{\mathbf{LLL}}(\Gamma) - \Gamma$ on the condition $\emptyset$, whence $Cn_{\mathbf{AL}}(Cn_{\mathbf{LLL}}(\Gamma)) \subseteq Cn_{\mathbf{AL}}(\Gamma)$. Moreover, any **AL**-proof from $\Gamma$ can be transformed into an **AL**-proof from $Cn_{\mathbf{LLL}}(\Gamma)$. ■

**Theorem 16** *If $\Gamma \vdash_{\mathbf{AL}} A$, then every* **AL***-proof from $\Gamma$ can be extended in such a way that $A$ is finally derived in it. (Proof Invariance)*

*Proof.* Let $\mathcal{P}_1 = \langle l_1, l_2, \ldots \rangle$ be the (stage of the) proof in which $A$ is finally derived from $\Gamma$ at line $l_k$ and let $\mathcal{P}_2 = \langle l'_1, l'_2, \ldots \rangle$ be a (stage of an) arbitrary proof from $\Gamma$. (If $\mathcal{P}_1$ is finite, there is a last element in the sequence; similarly for $\mathcal{P}_2$.)

In view of Definitions 4 and 3, the following is obvious. Whether $B$ is derived at a stage $s$ in a proof from $\Gamma$ depends on the lines that occur in the proof, not on the order in which these lines occur. So the sequence $\mathcal{P}_3 = \langle l_1, l'_1, l_2, l'_2, \ldots \rangle$ (if there are more $l_i$ than $l'_j$, the sequence will contain only $l_i$ from some point on, etc.) may be seen as an extension of $\mathcal{P}_1$ and also as an extension of $\mathcal{P}_2$. So, as $A$ is finally derived in $\mathcal{P}_1$, it follows by Definitions 5 and 6 that $\mathcal{P}_3$ as well as every extension of $\mathcal{P}_3$ in which line $l_k$ is marked has a further extension in which line $l_k$ is unmarked. ■

# 9 Criteria for Final Derivability

In view of the reasoning processes explicated by $\vdash_{\mathbf{AL}}$, this relation is not decidable (in general) and there is no positive test for it. This leads to two questions. Does the dynamics of the proofs go anywhere? And are there criteria for final derivability?

In view of the block analysis of proofs (and the block semantics) from [1], (i) a stage of a proof provides a certain insight in the premises, (ii) every step of the proof is either informative, in which case insight in the premises is gained, or non-informative, in which case no insight lost, and (iii) sensible proofs converge toward maximal insight—sensible proofs are obtained, for example, by the procedure described below.

There have been several attempts to devise criteria for final derivability. The first ones originated from the block analysis of [1]. These are very complicated and rather confusing for people that are not fluent in adaptive logics. More elegant criteria were found in the context of tableau methods—see [11] and [12]. The disadvantage of these criteria is that tableaus require writing a lot of formulas and entering roads that lead nowhere and would be skipped in sensible proofs. Tableaux are not goal directed and it is difficult to make them goal directed. The most interesting criteria are *procedural criteria*, which I now explain.

The procedural criterion for final derivability on the Reliability strategy is based on a a special kind of goal-directed proofs, called prospective proofs. Typical for these proofs is that most of the proof heuristics—Hintikka calls this the strategy, for example in [22]—is pushed into the proofs themselves. While constructing a proof, the heuristic reasoning instructs one, for example, to add a certain line and not another one. So, in a sense, a usual proof contains a part of the reasoning required to obtain the proof, and not other parts. Which part of the reasoning is actually written down in the proof is largely a conventional matter. The prospective proofs for the propositional fragment of **CL** is described for example in [16], that for the propositional fragments of **CLuN** and $\mathbf{CLuN}^r$ for example in [8].[10] Predicative results are forthcoming. The procedure, of which the structure is described below, presupposes that prospective proofs are available for the lower limit logic. Given this, the structure of the procedure is the same for all adaptive logics that have Reliability as their strategy.

The procedure for testing whether $\Gamma \vdash_{\mathbf{AL}^r} A$ consists of three phases. If the procedure stops, an answer is obtained: either yes or no. Of course the procedure cannot always stop, because there is no positive test, but it can be shown (where developed) to be at least as good as tableau methods.

*Phase 1* Here the procedure is directed at deriving derive the conclusion or goal, henceforth called $G$, on a condition. If the procedure stops without deriving $G$, $\Gamma \nvdash_{\mathbf{ACLuN1}} G$. If the procedure stops because $G$ is derived on a condition $\Delta$ at a line $i$, then (i) if $\Delta = \emptyset$, $\Gamma \vdash_{\mathbf{ACLuN1}} G$ and (ii) if $\Delta \neq \emptyset$n the procedure moves to phase 2—see below. After returning from phase 2 to phase 1, (i) if line $i$ is not marked, $\Gamma \vdash_{\mathbf{ACLuN1}} G$ and (ii) if line $i$ is marked, the procedure tries to derive $G$ on a (different) condition. And so on.

*Phase 2* If the procedure arrives here, $G$ was derived on condition $\Delta$ ($\neq \emptyset$) at line $i$. The procedure tries to derive $Dab(\Delta)$ on some condition. If the procedure is unsuccessful, $\Delta \cap U(\Gamma) = \emptyset$, it returns to phase 1, line $i$ being unmarked. If the procedure is successful, $Dab(\Delta)$ is derived on a condition $\Theta$ at line $j$. (i) If $\Theta = \emptyset$, line $i$ is marked and the procedure returns to phase 1. (ii) If $\Theta \neq \emptyset$, the procedure moves to phase 3. If, after returning from phase 3, line $j$ is not marked, $\Theta \cap U(\Gamma) = \emptyset$ whence $\Delta \cap U(\Gamma) \neq \emptyset$; in this case line $i$ is marked and the procedure returns to phase 1. If, after returning from phase 3, line $j$ is marked, $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta)$, so *possibly* $\Delta \cap U(\Gamma) = \emptyset$. So the procedure tries to derive $Dab(\Delta)$ on a (different) condition. And so on.

*Phase 3* If the procedure arrives here, $G$ is derived on condition $\Delta$ ($\neq \emptyset$) at line $i$ and $Dab(\Delta)$ is derived on condition $\Theta$ at line $j$, whence $\Gamma \vdash_{\mathbf{LLL}} Dab(\Delta \cup \Theta)$. The procedure tries to derive $Dab(\Theta)$ on a the condition $\emptyset$. (i) If the procedure is unsuccessful, it returns to phase 2, line $j$ being unmarked. In this case so $\Gamma \nvdash_{\mathbf{LLL}} Dab(\Theta)$, whence $\Delta \cap U(\Gamma) \neq \emptyset$. (ii) If the procedure is successful, line $j$ is marked and the procedure returns to phase 2. In this case $\Gamma \vdash_{\mathbf{LLL}} Dab(\Theta)$, so *possibly* $\Delta \cap U(\Gamma) = \emptyset$.

Of course, the previous paragraphs do not clarify the way in which the procedure actually works—see for example [8] for that—but it gives a clear idea of the steps taken by the procedure to arrive at a conclusion about the question whether $G$ is or is not derivable from $\Gamma$ by any adaptive logic that uses the Reliability strategy. If the procedure stops, we have an answer. If it does not, *we* shall have to stop anyway, and act on present insights. That's life.

[10]The program `pdp2.exe` implements the procedure for propositional $\mathbf{ACLuN}^r$. it is available at `http://logica.ugent.be/centrum/programs/`.

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