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An explicit, totally analytic approximate solution for Blasius' viscous flow problems

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Abstract

By means of using an operator $\mathscr{A}$ to denote non-linear differential equations in general, we first give a systematic description of a new kind of analytic technique for non-linear problems, namely the homotopy analysis method (HAM). Secondly, we generally discuss the convergence of the related approximate solution sequences and show that, as long as the approximate solution sequence given by the HAM is convergent, it must converge to one solution of the non-linear problem under consideration. Besides, we illustrate that even though a non-linear problem has one and only one solution, the sole solution might have an infinite number of expressions. Finally, to show the validity of the HAM, we apply it to give an explicit, purely analytic solution of the 2D laminar viscous flow over a semi-infinite flat plate. This explicit analytic solution is valid in the whole region $\eta = [0, +\infty)$ and can give, the first time in history (to our knowledge), an analytic value $f''(0) = 0.33206$, which agrees very well with Howarth's numerical result. This verifies the validity and great potential of the proposed homotopy analysis method as a new kind of powerful analytic tool. © 1999 Elsevier Science Ltd. All rights reserved.

Keywords: 2D Blasius' viscous flow; Explicit analytic solution; Non-linear differential equation; The homotopy analysis method; Independent upon small parameters

1. Introduction

The two-dimensional (2D) laminar viscous flow over a semi-infinite flat plate is governed by a non-linear differential equation (see Refs. 1–4)

$$f'''(\eta) + \tfrac{1}{2} f(\eta) f''(\eta) = 0, \quad \eta \in [0, +\infty), \tag{1.1}$$

with boundary conditions

$$f(0) = f'(0) = 0, \quad f'(+\infty) = 1, \tag{1.2}$$

where the prime denotes derivative w. r. t. η.

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The non-linear differential Eq (1.1) appears to be simple. However, it is difficult to solve it analytically. In 1908, Blasius [5] gave a solution in power series

$$f(\eta) = \sum_{k=0}^{+\infty} \left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2}, \tag{1.3}$$

where

$$A_0 = A_1 = 1,$$

$$A_k = \sum_{r=0}^{k-1} \binom{3k-1}{3r} A_r A_{k-r-1} \quad (k \geqslant 2). \tag{1.4}$$

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Here $\sigma = f''(0)$ must be numerically given. Howarth [6, 7] obtained a numerical result $f''(0) = 0.33206$. So, rigorously speaking, Blasius' solution (1.3) is a semi-analytic and semi-numerical one. As pointed out by Liao [8], the power series (1.3) can be given by perturbation techniques (see Ref. [9]), too. However, Blasius' power solution (1.3) converges in a rather restricted region $|\eta| \leqslant \rho_0$, where $\rho_0 \approx 5.690$.

In tradition, perturbation techniques are widely applied to give analytic approximations of non-linear problems. In essence, perturbation techniques use one or more "small parameters" to convert a non-linear problem into an infinite sequence of auxiliary linear sub-problems. However, it is just the so-called small parameter which greatly restricts application areas of perturbation techniques. First, a lot of non-linear problems do not contain such a kind of "small parameters". Secondly, the validity of perturbation approximations is in general strongly dependent upon the value of the so-called "small parameters". Finally, we have nearly no freedom to select the related initial approximations and governing equations of the related auxiliary sub-problems. Therefore, it is worthwhile developing a kind of new analytic technique which can provide us with greater freedom and larger flexibility to apply it and besides has nothing to do with whether considered non-linear problems contain small parameters or not.

The author has made some attempts in this direction. Based on the homotopy method in topology (see Ref. [10]), Liao [8, 11, 12] proposed such a kind of analytic technique, namely the homotopy analysis method (HAM). The HAM has such an advantage that its validity is in general independent of whether non-linear problems under consideration contain "small parameters" or not. Thus, the HAM is valid for most of the non-linear problems, especially those with strong non-linearity. Furthermore, the HAM provides us with great freedom and large flexibility to select related initial approximations, governing equations of auxiliary linear sub-problems. It is this kind of freedom and flexibility which provide us with a larger possibility to ensure that the corresponding approximation sequences of the HAM is convergent.

In many cases, the HAM can give better analytic approximations than perturbation ones. For example, Liao [8] applied the HAM to solve the foregoing Blasius' flow and obtained such a family of power series

$$f(\eta) = \lim_{m \to +\infty} \sum_{k=0}^{m} \left[\left(-\frac{1}{2}\right)^k \frac{A_k \sigma^{k+1}}{(3k+2)!} \eta^{3k+2} \right] \Phi_{m,k}(\hbar),$$

$$\eta \in [0, +\infty), \ -2 < \hbar < 0, \tag{1.5}$$

where $\sigma = f''(0)$, $A_k (k \geqslant 0)$ is defined by Eq. (1.4), and the real function $\Phi_{m,n}(\hbar)$, called the approaching function, is defined by

$$\Phi_{m,n}(\hbar) = 0, \quad (n > m),$$

$$\Phi_{m,n}(\hbar) = (-\hbar)^n \sum_{k=0}^{m-n} \binom{m}{m-n-k} \binom{n+k-1}{k} \hbar^k,$$

$$(1 \leqslant n \leqslant m),$$

$$\Phi_{m,n}(\hbar) = 1, \quad (n \leqslant 0). \tag{1.6}$$

As pointed out by Liao [12], the power series (1.5) is convergent in the region

$$-\rho_0 < \eta < \rho_0 \left[\frac{2}{|\hbar|} - 1 \right]^{1/3}, \quad (-2 < \hbar < 0), \tag{1.7}$$

which becomes larger and larger as $|\hbar|$ $(-2 < \hbar < 0)$ gets smaller and smaller, where $\rho_0 \approx 5.690$ is the convergence radius of the Blasius' power series (1.3). Therefore, the series (1.5) may be valid in the *whole* domain $\eta = [0, +\infty)$ as $|\hbar|$ $(-2 < \hbar < 0)$ tends to zero! Moreover, as pointed out by Liao [8], the Blasius' power series (1.3) is only a special case of Eq. (1.5) when $\hbar = -1$, because the real function $\Phi_{m,n}(\hbar)$ has such an interesting property that $\Phi_{m,n}(-1) = 1$ for $0 \leqslant n \leqslant m$. Thus, the power series (1.5) is more general than Eq. (1.3).

However, rigorously speaking, even the power series (1.5) is still not a *purely* analytic but a semi-analytic and semi-numerical solution, because the value of $\sigma = f''(0)$ had to be given by *numerical* techniques. To our knowledge, up to now, no one has given, in rigorous meaning, a purely analytic solution of Blasius' viscous flow and an analytic value of $f''(0)$. Of course, we might neglect this kind of imperfection. However, it still remains a challenge for us to give a *purely*, or *totally*, analytic solution of Eqs. (1.1) and (1.2).

In this paper, by using an operator $\mathscr{A}$ to denote a non-linear differential equation in general, we first give a systematic description of the HAM. Secondly, we generally discuss the convergence of the related infinite sequences of approximations given by the HAM. Finally, to show the validity of the HAM, we apply a new auxiliary linear operator, which is better and more general than that used by Liao [8], to give an explicit, purely analytic solution of the foregoing Blasius' flow. This explicit analytic solution is valid in the whole region $\eta = [0, +\infty)$ and can give an analytic value $f''(0) = 0.33206$, which agrees very well with Howarth's numerical result.

2. The systematic description of the homotopy analysis method

Let $\hbar \neq 0$, p be complex numbers, and $A(p)$, $B(p)$ be complex functions analytic in the region $|p| \leqslant 1$, which satisfy

$$A(0) = B(0) = 0, \quad A(1) = B(1) = 1, \tag{2.1}$$

respectively. Besides, let

$$A(p) = \sum_{k=1}^{+\infty} \alpha_{1,k} p^k, \quad B(p) = \sum_{k=1}^{+\infty} \beta_{1,k} p^k \tag{2.2}$$

denote the Maclaurin series of $A(p)$ and $B(p)$, respectively. Because $A(p)$ and $B(p)$ are analytic in the region $|p| \leqslant 1$, therefore we have

$$\sum_{k=1}^{+\infty} \alpha_{1,k} = A(1) = 1, \quad \sum_{k=1}^{+\infty} \beta_{1,k} = B(1) = 1. \tag{2.3}$$

For simplicity, the above-defined complex functions $A(p)$ and $B(p)$ are called the *embedding functions*, and p is called the *embedding parameter*.

Consider a non-linear equation in a general form

$$\mathscr{A}[u(\mathbf{r})] = 0, \quad \mathbf{r} \in \Omega, \tag{2.4}$$

where $\mathscr{A}$ is a differential operator, $u(\mathbf{r})$ is a solution defined in the region $\mathbf{r} \in \Omega$. Applying the homotopy analysis method to solve it, we first of all need to construct such a family of equations

$$[1 - B(p)]\{\mathscr{L}[\theta(\mathbf{r}, p)] - \mathscr{L}[u_0(\mathbf{r})]\} = \hbar A(p)\mathscr{A}[\theta(\mathbf{r}, p)], \tag{2.5}$$

where $\mathscr{L}$ is a properly selected auxiliary linear operator satisfying

$$\mathscr{L}(0) = 0, \tag{2.6}$$

$\hbar \neq 0$ is an auxiliary parameter, $u_0(\mathbf{r})$ is an initial approximation, $A(p)$ and $B(p)$ are the above-defined embedding functions, p is the embedding parameter. According to the definition of the embedding functions $A(p)$ and $B(p)$, Eq. (2.5) gives when $p = 0$ that

$$\theta(\mathbf{r}, 0) = u_0(\mathbf{r}). \tag{2.7}$$

Similarly, when $p = 1$, Eq. (2.5) is the same as Eq. (2.4) so that we have

$$\theta(\mathbf{r}, 1) = u(\mathbf{r}). \tag{2.8}$$

Assume that $\hbar$, $A(p)$, $B(p)$ are properly selected so that Eq. (2.5) has solution $\theta(\mathbf{r}, p)$ for any $p \in [0, 1]$, and besides, at $p = 0$ the solution $\theta(\mathbf{r}, p)$ has derivatives of the order of up-to infinity, say,

$$\theta_0^{[k]}(\mathbf{r}) = \left.\frac{\partial^k \theta(\mathbf{r}, p)}{\partial p^k}\right|_{p=0}, \quad k = 1, 2, 3, \ldots, \tag{2.9}$$

Thus, as p increases from 0 to 1, the solution $\theta(\mathbf{r}, p)$ of Eq. (2.5) varies continuously from the initial approximation $u_0(\mathbf{r})$ to the solution $u(\mathbf{r})$ of the original Eq. (2.4). In topology, this kind of continuous variation is called deformation. So, we call Eq. (2.5) the *zeroth-order deformation equations*, and $\theta_0^{[k]}(\mathbf{r})$ the *kth-order deformation derivatives*.

Clearly, Eqs. (2.7) and (2.8) give an indirect relationship between the initial approximation $u_0(\mathbf{r}_0)$ and the solution $u(\mathbf{r})$ of the original equation (2.4)—the numerical technique, namely the continuous method, is just based on this kind of relationship. Here, we deduce a direct relationship between them, which is the cornerstone of our analytic technique. Note that the Maclaurin series of $\theta(\mathbf{r}, p)$ about p is

$$\theta(\mathbf{r}, 0) + \sum_{k=1}^{+\infty} \left[\frac{\theta_0^{[k]}(\mathbf{r})}{k!}\right] p^k. \tag{2.10}$$

Assuming that $\hbar$, $A(p)$, $B(p)$, the initial approximation $u_0(\mathbf{r})$ and the auxiliary linear operator $\mathscr{L}$ are properly selected so that the above Maclaurin series converges at $p = 1$, we have by Eqs. (2.7) and (2.8) the relationship

$$u(\mathbf{r}) = u_0(\mathbf{r}) + \sum_{m=1}^{+\infty} \varphi_m(\mathbf{r}), \tag{2.11}$$

where

$$\varphi_m(\mathbf{r}) = \frac{\theta_0^{[m]}(\mathbf{r})}{m!}, \quad m \geqslant 1. \tag{2.12}$$

Obviously, it is necessary to give the governing equations determining $\varphi_m(\mathbf{r})$. Differentiating the zeroth-order deformation equation (2.5) m times with respect to p, we get

$$\sum_{k=0}^{m} \binom{m}{k} \frac{\mathrm{d}^k[1 - B(p)]}{\mathrm{d}p^k} \frac{\mathrm{d}^{m-k}}{\mathrm{d}p^{m-k}} \{\mathscr{L}[\theta(\mathbf{r}, p)] - \mathscr{L}[u_0(\mathbf{r})]\} = \hbar \sum_{k=0}^{m} \binom{m}{k} \frac{\mathrm{d}^k A(p)}{\mathrm{d}p^k} \frac{\mathrm{d}^{m-k}\mathscr{A}[\theta(\mathbf{r}, p)]}{\mathrm{d}p^{m-k}}. \tag{2.13}$$

Further dividing it by $m!$ and then setting $p = 0$, we have the so-called *mth-order deformation equations*

$$\mathscr{L}\left[\varphi_m(\mathbf{r}) - \sum_{k=1}^{m-1} \beta_{1,k}\varphi_{m-k}(\mathbf{r})\right] = R_m(\mathbf{r}), \tag{2.14}$$

where

$$R_m(\mathbf{r}) = \hbar \sum_{k=1}^{m} \alpha_{1,k}\delta_{m-k}(\mathbf{r}) \tag{2.15}$$

and

$$\delta_k(\mathbf{r}) = \frac{1}{k!} \frac{\mathrm{d}^k\mathscr{A}[\theta(\mathbf{r}, p)]}{\mathrm{d}p^k}\bigg|_{p=0}. \tag{2.16}$$

We emphasize that the mth-order ($m \geqslant 1$) deformation equation (2.14) is *linear*. Moreover, its right-hand side $R_m(\mathbf{r})$ ($m = 1, 2, 3, \ldots$) is known when the first $(m-1)$th-order approximations have been obtained. So, using the selected initial approximation $u_0(\mathbf{r})$, we can obtain $\varphi_1(\mathbf{r}), \varphi_2(\mathbf{r}), \varphi_3(\mathbf{r}), \ldots$, one after the other in order. Therefore, according to Eq. (2.11), we in fact convert the original *non-linear* problem (2.4) into an *infinite* sequence of *linear* sub-problems governed by Eq. (2.14). We emphasize that unlike perturbation techniques, this kind of transformation has nothing to do with whether Eq. (2.4) contains the so-called "small parameters" or not. This is the essential difference between the HAM and perturbation techniques. Besides, we emphasize that the HAM provides us with great freedom and large flexibility to select the non-zero auxiliary parameter $\hbar$, the embedding functions $A(p)$ and $B(p)$, the initial approximation $u_0(\mathbf{r})$ and the auxiliary linear operators $\mathscr{L}$. This kind of freedom and flexibility may greatly increase the possibility for us to ensure that the approximation solution series (2.11) is convergent. We will show this point in Section 4 by means of a simple example.

Note that most of the non-linear problems can be expressed by some governing equations (either ODEs or PDEs) and related boundary conditions. However, for the sake of simplicity, we consider here only one non-linear equation (2.4) which may be either a governing equation or a boundary condition. Clearly, all governing equations and boundary conditions of a non-linear problem can be treated similarly. Moreover, for different governing equations or boundary conditions, we might select different auxiliary linear operators $\mathscr{L}$, different values of non-zero parameter $\hbar$, and different types of embedding functions $A(p)$ and $B(p)$. All of these provide us with great freedom and large flexibility to apply the HAM to get satisfactory analytic approximations of strongly non-linear problems. Finally, we point out that the operator $\mathscr{A}$ is rather general so that Eq. (2.4) may express either an ordinary differential equation (ODE) or a partial differential equation (PDE) or the related boundary conditions. Therefore, the HAM is valid for both ODEs and PDEs.

3. The convergence of the approximate solution sequence

In this section, we prove that, as long as the sequence of approximations given by the above-mentioned approach (HAM) is convergent, it must be a solution of the non-linear problem under considerations.

Theorem 1. *If the series*

$$u_0(\mathbf{r}) + \sum_{m=1}^{+\infty} \varphi_m(\mathbf{r})$$

is convergent, it must be a solution of Eq. (2.4).

Proof. By Eq. (2.14), we have

$$\begin{aligned}
\sum_{m=1}^{+\infty} R_m(\mathbf{r}) &= \sum_{m=1}^{+\infty} \mathscr{L}\left[\varphi_m(\mathbf{r}) - \sum_{k=1}^{m-1} \beta_{1,k}\varphi_{m-k}(\mathbf{r})\right] \\
&= \mathscr{L}\left[\sum_{m=1}^{+\infty} \varphi_m(\mathbf{r}) - \sum_{m=1}^{+\infty}\sum_{k=1}^{m-1} \beta_{1,k}\varphi_{m-k}(\mathbf{r})\right] \\
&= \mathscr{L}\left[\sum_{m=1}^{+\infty} \varphi_m(\mathbf{r}) - \sum_{k=1}^{+\infty}\sum_{m=k+1}^{+\infty} \beta_{1,k}\varphi_{m-k}(\mathbf{r})\right] \\
&= \mathscr{L}\left[\sum_{m=1}^{+\infty} \varphi_m(\mathbf{r}) - \sum_{k=1}^{+\infty} \beta_{1,k}\sum_{m=1}^{+\infty} \varphi_m(\mathbf{r})\right] \\
&= \mathscr{L}\left[\left(1 - \sum_{k=1}^{+\infty} \beta_{1,k}\right)\sum_{m=1}^{+\infty} \varphi_m(\mathbf{r})\right],
\end{aligned} \tag{3.1}$$

which gives by Eqs. (2.3) and (2.6) that

$$\sum_{m=1}^{+\infty} R_m(\mathbf{r}) = 0. \tag{3.2}$$

On the other hand, we have by Eqs. (2.15) and (2.16) that

$$\begin{aligned}
\sum_{m=1}^{+\infty} R_m(\mathbf{r}) &= \sum_{m=1}^{+\infty} \hbar \sum_{k=1}^{m} \alpha_{1,k}\delta_{m-k}(\mathbf{r}) \\
&= \hbar \sum_{k=1}^{+\infty} \alpha_{1,k} \sum_{m=k}^{+\infty} \delta_{m-k}(\mathbf{r}) \\
&= \hbar \sum_{k=1}^{+\infty} \alpha_{1,k} \sum_{m=0}^{+\infty} \delta_m(\mathbf{r}) \\
&= \hbar \sum_{k=1}^{+\infty} \alpha_{1,k} \sum_{m=0}^{+\infty} \frac{1}{m!} \left.\frac{\mathrm{d}^m \mathscr{A}[\theta(\mathbf{r}, p)]}{\mathrm{d}p^m}\right|_{p=0}.
\end{aligned} \tag{3.3}$$

According to Eq. (2.3), we have $\sum_{k=1}^{+\infty} \alpha_{1,k} = 1$. Thus, the above expression becomes

$$\sum_{m=1}^{+\infty} R_m(\mathbf{r}) = \hbar \sum_{m=0}^{+\infty} \delta_m(\mathbf{r}) = \hbar \sum_{m=0}^{+\infty} \frac{1}{m!} \left.\frac{\mathrm{d}^m \mathscr{A}[\theta(\mathbf{r}, p)]}{\mathrm{d}p^m}\right|_{p=0}. \tag{3.4}$$

Note that $\hbar \neq 0$. Therefore, by Eqs. (3.2) and (3.4), we have

$$\sum_{m=0}^{+\infty} \frac{1}{m!} \left.\frac{\mathrm{d}^m \mathscr{A}[\theta(\mathbf{r}, p)]}{\mathrm{d}p^m}\right|_{p=0} = 0. \tag{3.5}$$

Noting that, in general, $\theta(\mathbf{r}, p)$ is not the solution of Eq. (2.4) when $p \neq 1$. Write $\Delta(\mathbf{r}, p) = \mathscr{A}[\theta(\mathbf{r}, p)]$. Clearly, $\Delta(\mathbf{r}, p)$ denotes the residual error of Eq. (2.4). The Maclaurin series of this residual error about p is

$$\begin{aligned}
&\sum_{m=0}^{+\infty} \frac{1}{m!} \left.\frac{\mathrm{d}^m \Delta[\theta(\mathbf{r}, p)]}{\mathrm{d}p^m}\right|_{p=0} p^m \\
&\quad = \sum_{m=0}^{+\infty} \frac{1}{m!} \left.\frac{\mathrm{d}^m \mathscr{A}[\theta(\mathbf{r}, p)]}{\mathrm{d}p^m}\right|_{p=0} p^m.
\end{aligned} \tag{3.6}$$

According to Eq. (3.5), the above Maclaurin series is convergent at $p = 1$, say

$$\begin{aligned}
\Delta(\mathbf{r}, 1) = \mathscr{A}[\theta(\mathbf{r}, 1)] &= \sum_{m=0}^{+\infty} \frac{1}{m!} \left.\frac{\mathrm{d}^m \mathscr{A}[\theta(\mathbf{r}, p)]}{\mathrm{d}p^m}\right|_{p=0} \\
&= 0.
\end{aligned} \tag{3.7}$$

It means that

$$\theta(\mathbf{r}, 1) = u_0(\mathbf{r}) + \sum_{m=1}^{+\infty} \varphi_m(\mathbf{r})$$

must be a solution of Eq. (2.4). This completes the proof. □

Theorem 2. *Write*

$$\kappa_m = \sum_{k=1}^{m} R_k(\mathbf{r}) \quad \text{and} \quad \upsilon_m = \sum_{k=1}^{m} \delta_k(\mathbf{r}),$$

where $R_k(\mathbf{r})$, $\delta_k(\mathbf{r})$ are defined by Eqs. (2.15) and (2.16), respectively. If the series $u_0(\mathbf{r}) + \sum_{m=1}^{+\infty} \varphi_m(\mathbf{r})$ converges, both of the sequences $\kappa_0, \kappa_1, \kappa_2, \ldots$ and $\upsilon_0, \upsilon_1, \upsilon_2, \ldots$ converge to zero.

Proof. According to Eqs. (3.4) and (3.5), this theorem is obviously true. This completes the proof. □

Due to Theorem 1, we just need to focus on selecting proper initial approximations $u_0(\mathbf{r})$, non-zero auxiliary parameter $\hbar$, the embedding functions $A(p)$, $B(p)$, and auxiliary linear operators $\mathscr{L}$ to ensure that the related approximation sequences is convergent. Theorem 2 provides us with a simple way to estimate whether the related approximation solution sequence is divergent: when the sequences $\kappa_0, \kappa_1, \kappa_2, \ldots$ and/or $\upsilon_0, \upsilon_1, \upsilon_2, \ldots$ are divergent, the related series (3.1) must be divergent, too. Theorem 2 is especially useful for some numerical techniques based on the HAM, such as the general boundary element method ([13, 14]) and so on [15].

4. The explicit analytic solution of Blasius' viscous flow

4.1. The explicit expression of the solution

For 2D Blasius' viscous flow problem, Liao [8] used an auxiliary linear operator

$$\mathscr{L}_0 = \frac{\partial^3}{\partial \eta^3}$$

and an initial approximation $\tilde{f}_0(\eta) = \sigma\eta^2/2$ to construct the corresponding zeroth-order deformation equation. Note that the above auxiliary linear operator directly comes from the linear term of Eq. (1.1). However, by means of the HAM, this is *not* necessary because the HAM provides us with large freedom and flexibility to select other auxiliary linear operators and initial approximations in different forms. In this paper, we illustrate that, using a more general auxiliary linear operator

$$\mathscr{L} = \left(\frac{\partial}{\partial \eta} + \beta\right)\frac{\partial^2}{\partial \eta^2} = \frac{\partial^3}{\partial \eta^3} + \beta\frac{\partial^2}{\partial \eta^2}, \quad \beta > 0, \tag{4.1}$$

we can obtain a new family of explicit, totally analytic solutions better than Eq. (1.5). This explicit analytic solution can converge to the solution of Eqs. (1.1) and (1.2) in the whole region $\eta = [0, +\infty)$, and moreover, it can independently give such an *analytic* value $f''(0) = 0.32206$ that agrees very well with Howarth's numerical one.

First of all, using $\mathscr{L}$ defined by Eq. (4.1) as an auxiliary linear operator and $A(p) = p$ and $B(p) = p$ as the embedding functions, we construct the following zeroth-order deformation equation:

$$(1-p)\mathscr{L}[F(\eta,\hbar,\beta,p) - f_0(\eta)] = p\hbar\left[\frac{\partial^3 F(\eta,\hbar,\beta,p)}{\partial \eta^3} + \frac{1}{2}F(\eta,\hbar,\beta,p)\frac{\partial^2 F(\eta,\hbar,\beta,p)}{\partial \eta^2}\right]$$

$$\eta \in [0, +\infty), \hbar \neq 0, \beta > 0, p \in [0, 1], \tag{4.2}$$

with boundary conditions

$$F(0,\hbar,\beta,p) = F'(0,\hbar,\beta,p) = 0, \quad F'(+\infty,\hbar,\beta,p) = 1,$$

$$p \in [0,1], \quad \hbar \neq 0, \quad \beta > 0. \tag{4.3}$$

Note that we have at $p = 0$ the result

$$F(\eta,\hbar,\beta,0) = f_0(\eta), \quad \eta \in [0, +\infty), \quad \hbar \neq 0, \quad \beta > 0 \tag{4.4}$$

and at $p = 1$ the relationship

$$F(\eta,\hbar,\beta,1) = f(\eta), \quad \eta \in [0, +\infty), \quad \hbar \neq 0, \quad \beta > 0, \tag{4.5}$$

respectively, where the prime denotes the partial derivative w. r. t. η, and we select

$$f_0(\eta) = \eta - \frac{1 - \exp(-\beta\eta)}{\beta}, \quad \beta > 0 \tag{4.6}$$

as the initial approximation which satisfies the boundary conditions (1.2). Therefore, the process of p varying from 0 to 1 is just the continuous variation (or deformation) of the function $F(\eta,\hbar,\beta,p)$ from the known initial approximation $f_0(\eta)$ to the unknown solution $f(\eta)$ of Eqs. (1.1) and (1.2).

Assume that the deformation $F(\eta,\hbar,\beta,p)$, governed by Eqs. (4.2) and (4.3), is smooth enough about p, so that the kth-order deformation derivative

$$f_0^{[k]}(\eta,\hbar,\beta) = \left.\frac{\partial^k F(\eta,\hbar,\beta,p)}{\partial p^k}\right|_{p=0} \quad (k \geqslant 1) \tag{4.7}$$

exists. Then, according to Eq. (4.4) and the Taylor formula, we have

$$F(\eta,\hbar,\beta,p) = f_0(\eta) + \sum_{k=1}^{+\infty}\left[\frac{f_0^{[k]}(\eta,\hbar,\beta)}{k!}\right]p^k. \tag{4.8}$$

Clearly, the convergence region of above infinite series is dependent upon $\hbar$ ($\hbar \neq 0$) and β ($\beta > 0$). Assume that both $\hbar$ and β are so properly selected that the series (4.8) is convergent at $p = 1$. Then, due to Eqs. (4.5) and (4.8), we get at $p = 1$ the relationship

$$f(\eta) = f_0(\eta) + \sum_{k=1}^{+\infty} \frac{f_0^{[k]}(\eta,\hbar,\beta)}{k!} = \sum_{k=0}^{+\infty} \varphi_k(\eta,\hbar,\beta) \quad (4.9)$$

between the initial approximation $f_0(\eta)$ and the unknown solution $f(\eta)$, where we define

$$\varphi_0(\eta,\hbar,\beta) = f_0(\eta), \varphi_k(\eta,\hbar,\beta) = \frac{f_0^{[k]}(\eta,\hbar,\beta)}{k!}, \quad (k \geqslant 1). \quad (4.10)$$

The governing equations of the unknown function $\varphi_m(\eta,\hbar,\beta)$ ($m \geqslant 1$) are obtained by first differentiating Eqs. (4.2) and (4.3) m times w. r. t. p and then setting $p = 0$ and finally dividing it by $m!$, i.e.

$$(\varphi_m)''' + \beta(\varphi_m)'' = G_m(\eta,\hbar,\beta),$$

$$\eta \in [0, +\infty), \quad \beta > 0, \quad \hbar \neq 0, \quad m \geqslant 1 \quad (4.11)$$

with the related boundary conditions

$$\varphi_m(0,\hbar,\beta) = \varphi'_m(0,\hbar,\beta) = \varphi'_m(+\infty,\hbar,\beta) = 0,$$

$$\beta > 0, \quad \hbar \neq 0, \quad m \geqslant 1, \quad (4.12)$$

where the prime denotes the partial derivative w.r.t. η and

$$G_1(\eta,\hbar,\beta) = \hbar\left[\frac{\partial^3\varphi_0(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{1}{2}\varphi_0(\eta,\hbar,\beta)\frac{\partial^2\varphi_0(\eta,\hbar,\beta)}{\partial\eta^2}\right], \quad (4.13)$$

$$G_m(\eta,\hbar,\beta) = \frac{\partial^3\varphi_{m-1}(\eta,\hbar,\beta)}{\partial\eta^3} + \beta\frac{\partial^2\varphi_{m-1}(\eta,\hbar,\beta)}{\partial\eta^2} + \hbar\left[\frac{\partial^3\varphi_{m-1}(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{1}{2}\sum_{k=0}^{m-1}\varphi_k(\eta,\hbar,\beta)\frac{\partial^2\varphi_{m-1-k}(\eta,\hbar,\beta)}{\partial\eta^2}\right] \quad (m \geqslant 2). \quad (4.14)$$

Note that by Eqs. (4.6) and (4.13) we can first calculate the term $G_1(\eta,\hbar,\beta)$ and then obtain $\varphi_1(\eta,\hbar,\beta)$ by solving the *linear* differential equation (4.11) with linear boundary conditions (4.12). Similarly, we can further calculate the term $G_2(\eta,\hbar,\beta)$ by Eq. (4.14) and then get $\varphi_2(\eta,\hbar,\beta)$, and so on. In this way, the linear mth-order ($m \geqslant 1$) deformation equation (4.11) and (4.12) can be solved one after the other in order. We use the widely applied symbolic computation software *MATHEMATICA* (see Ref. [16]) to solve the first several equations (4.11) and (4.12) and find, as a little surprise, that $\varphi_m(\eta,\hbar,\beta)$ has the following structure:

$$\varphi_m(\eta,\hbar,\beta) = \sum_{k=0}^{m+1} \Psi_{m,k}(\eta,\hbar,\beta)\exp(-k\beta\eta), \quad m \geqslant 0, \quad (4.15)$$

where the function $\Psi_{m,k}(\eta,\hbar,\beta)$ is defined by

$$\Psi_{0,0}(\eta,\hbar,\beta) = b_{0,0}^0 + b_{0,0}^1\eta, \quad (4.16)$$

$$\Psi_{0,1}(\eta,\hbar,\beta) = b_{0,1}^0, \quad (4.17)$$

$$\Psi_{m,0}(\eta,\hbar,\beta) = b_{m,0}^0, \quad m \geqslant 1, \quad (4.18)$$

$$\Psi_{m,k}(\eta,\hbar,\beta) = \sum_{i=0}^{2(m+1-k)} b_{m,k}^i\eta^i, m \geqslant 1, 1 \leqslant k \leqslant m+1. \quad (4.19)$$

and the related coefficients are

$$b_{0,0}^0 = -\frac{1}{\beta}, \quad b_{0,0}^1 = 1, \quad b_{0,1}^0 = \frac{1}{\beta}, \quad (4.20)$$

$$b_{1,0}^0 = \hbar\left[\frac{1}{\beta} - \frac{5}{8}\frac{1}{\beta^3}\right], \quad (4.21)$$

$$b_{1,1}^0 = -\hbar\left(\frac{1}{\beta} - \frac{3}{4}\frac{1}{\beta^3}\right),$$

$$b_{1,1}^1 = -\hbar\left(1 - \frac{1}{2\beta^2}\right), \quad b_{1,1}^2 = \frac{\hbar}{4\beta} \quad (4.22)$$

$$b_{1,2}^0 = -\frac{\hbar}{8\beta^3}, \quad (4.23)$$

$$b_{2,0}^0 = \frac{\hbar}{\beta} + \frac{5}{4}\frac{\hbar^2}{\beta^3} - \frac{5}{8}\frac{\hbar}{\beta^3} - \frac{359}{288}\frac{\hbar^2}{\beta^5}, \quad (4.24)$$

$$b^0_{2,1} = \frac{3}{4}\frac{\hbar}{\beta^3} - \frac{\hbar}{\beta} + \frac{157}{96}\frac{\hbar^2}{\beta^5} - \frac{3}{2}\frac{\hbar^2}{\beta^3},$$

$$b^1_{2,1} = -\hbar + \frac{1}{2}\frac{\hbar}{\beta^2} + \frac{17}{16}\frac{\hbar^2}{\beta^4} - \frac{5}{4}\frac{\hbar^2}{\beta^2}, \tag{4.25}$$

$$b^2_{2,1} = \frac{1}{4}\frac{\hbar}{\beta} + \frac{9}{16}\frac{\hbar^2}{\beta^3} - \frac{1}{2}\frac{\hbar^2}{\beta} + \frac{1}{2}\beta\hbar^2,$$

$$b^3_{2,1} = \frac{1}{8}\frac{\hbar^2}{\beta^2} - \frac{1}{4}\hbar^2, \quad b^4_{2,1} = \frac{1}{32}\frac{\hbar^2}{\beta}, \tag{4.26}$$

$$b^0_{2,2} = -\frac{1}{8}\frac{\hbar}{\beta^3} - \frac{13}{32}\frac{\hbar^2}{\beta^5} + \frac{1}{4}\frac{\hbar^2}{\beta^3},$$

$$b^1_{2,2} = -\frac{3}{16}\frac{\hbar^2}{\beta^4} + \frac{1}{4}\frac{\hbar^2}{\beta^2}, \quad b^2_{2,2} = -\frac{1}{16}\frac{\hbar^2}{\beta^3}, \tag{4.27}$$

$$b^0_{2,3} = \frac{5}{288}\frac{\hbar^2}{\beta^5}, \ldots \tag{4.28}$$

and so on. Knowing the structure (4.15) of $\varphi_m(\eta,\hbar,\beta)$, we can rigorously deduce a recurrence formula about the coefficients $b^k_{m,n}$ of $\varphi_m(\eta,\hbar,\beta)$, where $m \geqslant 1$, $0 \leqslant n \leqslant m+1$ and $0 \leqslant k \leqslant 2(m-n+1)$, as follows:

$$b^0_{m,0} = \chi_m b^0_{m-1,0} - \beta^{-1}\sum_{q=0}^{2m-1}\Gamma^q_{m,1}\mu^q_{1,1} - \sum_{n=2}^{m+1}\left[(n-1)\Gamma^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2(m-n+1)}\Gamma^q_{m,n}(n\mu^q_{n,0} - \mu^q_{n,0} - \beta^{-1}\mu^q_{n,1})\right], \tag{4.29}$$

$$b^1_{m,0} = 0, \tag{4.30}$$

$$b^0_{m,1} = \chi_m b^0_{m-1,1} + \beta^{-1}\sum_{q=0}^{2m-1}\Gamma^q_{m,1}\mu^q_{1,1} + \sum_{n=2}^{m+1}\left[n\Gamma^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2(m-n+1)}\Gamma^q_{m,n}(n\mu^q_{n,0} - \beta^{-1}\mu^q_{n,1})\right], \tag{4.31}$$

$$b^k_{m,1} = \chi_m b^k_{m-1,1} + \sum_{q=k-1}^{2m-1}\Gamma^q_{m,1}\mu^q_{1,k}, \quad 1 \leqslant k \leqslant 2m-2, \tag{4.32}$$

$$b^k_{m,1} = \sum_{q=k-1}^{2m-1}\Gamma^q_{m,1}\mu^q_{1,k}, \quad 2m-1 \leqslant k \leqslant 2m, \tag{4.33}$$

$$b^k_{m,n} = \chi_m b^k_{m-1,n} - \sum_{q=k}^{2(m-n+1)}\Gamma^q_{m,n}\mu^q_{n,k}, \quad 0 \leqslant k \leqslant 2(m-n), \quad 2 \leqslant n \leqslant m, \tag{4.34}$$

$$b^k_{m,n} = -\sum_{q=k}^{2(m-n+1)}\Gamma^q_{m,n}\mu^q_{n,k}, \quad 2(m-n)+1 \leqslant k \leqslant 2(m-n)+2, \ 2 \leqslant n \leqslant m, \tag{4.35}$$

$$b^0_{m,m+1} = -\Gamma^0_{m,m+1}\mu^0_{m+1,0}, \tag{4.36}$$

where

$$\chi_m = \begin{cases} 0, & \text{when } m = 1, \\ 1, & \text{otherwise.} \end{cases} \tag{4.37}$$

$$\mu^q_{1,k} = \frac{q!}{k!}\frac{(q-k+2)}{\beta^{q-k+3}}, \quad 0 \leqslant k \leqslant q+1, \quad q \geqslant 0, \tag{4.38}$$

$$\mu^q_{n,k} = \frac{q!}{k!}\frac{1}{(n-1)^{q-k+1}\beta^{q-k+3}}\left\{1 - \left(1-\frac{1}{n}\right)^{q-k+1} \times\left[(q-k+2) - (q-k+1)\left(1-\frac{1}{n}\right)\right]\right\}, \quad 0 \leqslant k \leqslant q, \ n \geqslant 2, \ q \geqslant 0 \tag{4.39}$$

and

$$\Gamma^q_{m,1} = \hbar(d^q_{m-1,1} + \delta^q_{m,1}), \quad 0 \leqslant q \leqslant 2m-2, \tag{4.40}$$

$$\Gamma^{2m-1}_{m,1} = \hbar\delta^{2m-1}_{m,1}, \tag{4.41}$$

$$\Gamma^0_{m,m+1} = \hbar\delta^0_{m,m+1} \tag{4.42}$$

$$\Gamma^q_{m,n} = \hbar(d^q_{m-1,n} + \delta^q_{m,n}), \ 0 \leqslant q \leqslant 2(m-n), \ 2 \leqslant n \leqslant m,$$

$$\Gamma^q_{m,n} = \hbar\delta^q_{m,n}, \quad 2(m-n)+1 \leqslant q \leqslant 2(m-n)+2, \quad 2 \leqslant n \leqslant m,$$

$$\Gamma^q_{m,n} = 0 \quad \text{otherwise.} \tag{4.43}$$

The related coefficient $\delta^q_{m,n}(m \geqslant 1,\ 0 \leqslant n \leqslant m+1,\ 0 \leqslant q \leqslant 2(m-n+1))$ is defined as follows:

$$\delta^q_{m,n} = \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=J_0}^{J_1}\sum_{i=I_0}^{I_1} c^i_{k,j}\, b^{q-i}_{m-1-k,n-j}\,\lambda^{q-i}_{m-1-k,n-j},$$

$$J_0 = \max\{1, n+k-m\}, \ J_1 = \min\{n, k+1\},$$

$$I_0 = \max\{0, q-2(m-k-n+j)\},$$

$$I_1 = \min\{q, 2(k-j+1)\},, \tag{4.44}$$

where

$$\lambda_{i,j}^k = 0, \quad i = j = 0,\ k \geqslant 2,$$

$$\lambda_{i,j}^k = 0, \quad i > 0,\ j = 0,\ k \geqslant 1,$$

$$\lambda_{i,j}^k = 0, \quad j > i + 1,$$

$$\lambda_{i,j}^k = 0, \quad k > 2(i + 1 - j),$$

$$\lambda_{i,j}^k = 1, \quad \text{otherwise}, \tag{4.45}$$

$$c_{m,m+1}^0 = (m+1)^2 \beta^2 b_{m,m+1}^0, \tag{4.46}$$

$$d_{m,m+1}^0 = -(m+1)^3 \beta^3 b_{m,m+1}^0 \tag{4.47}$$

and for $1 \leqslant k \leqslant m$,

$$c_{m,k}^i = (i+1)(i+2) b_{m,k}^{i+2} - 2(k\beta)(i+1) b_{m,k}^{i+1} + (k\beta)^2 b_{m,k}^i, \quad 0 \leqslant i \leqslant 2(m-k), \tag{4.48}$$

$$c_{m,k}^i = -2(k\beta)(i+1) b_{m,k}^{i+1} + (k\beta)^2 b_{m,k}^i, \quad i = 2(m-k)+1, \tag{4.49}$$

$$c_{m,k}^i = (k\beta)^2 b_{m,k}^i, \quad i = 2(m-k)+2, \tag{4.50}$$

$$d_{m,k}^i = (i+1) c_{m,k}^{i+1} - (k\beta) c_{m,k}^i, \quad 0 \leqslant i \leqslant 2(m-k)+1, \tag{4.51}$$

$$d_{m,k}^{2(m-k+1)} = -(k\beta) c_{m,k}^{2(m-k+1)}. \tag{4.52}$$

For details, please refer to Appendix A. Using the above recurrence formulas, we can calculate all coefficients $b_{m,n}^k$ by using only the first three coefficients $b_{0,0}^0 = -\beta^{-1}$, $b_{0,0}^1 = 1$, $b_{0,1}^0 = \beta^{-1}$. Clearly, owing to Eqs. (4.9) and (4.15), the Mth-order approximation is

$$\begin{aligned} f_0(\eta) + \sum_{k=1}^{M} \varphi_k(\eta, \hbar, \beta) &= \sum_{m=0}^{M} \sum_{k=0}^{m+1} \Psi_{m,k}(\eta, \hbar, \beta) \exp(-k\beta\eta) \\ &= \sum_{m=0}^{M} \Psi_{m,0}(\eta, \hbar, \beta) + \sum_{k=1}^{M+1} \exp(-k\beta\eta) \left(\sum_{m=k-1}^{M} \Psi_{m,k}(\eta, \hbar, \beta) \right) \\ &= t + \left(\sum_{m=0}^{M} b_{m.0}^0 \right) + \sum_{n=1}^{M+1} \exp(-n\beta\eta) \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m-n+1)} b_{m,n}^{\mathrm{k}} \eta^k \right). \end{aligned}$$

Therefore, we obtain in fact an *explicit*, *totally* analytic solution of the 2D Blasius' viscous flow problems

$$\begin{aligned} f(\eta) &= \lim_{M \to +\infty} \sum_{k=0}^{M} \varphi_k(\eta, \hbar, \beta) \\ &= t + \lim_{M \to +\infty} \left[\left(\sum_{m=0}^{M} b_{m,0}^0 \right) + \sum_{n=1}^{M+1} \exp(-n\beta\eta) \times \left(\sum_{m=n-1}^{M} \sum_{k=0}^{2(m-n+1)} b_{m,n}^k \eta^k \right) \right]. \end{aligned} \tag{4.53}$$

4.2. *The convergence of the explicit analytic solution (4.53)*

In Section 3, we point out in general terms that, if the sequence of the approximation solutions given by the HAM is convergent, it must converge to one of the solutions of the non-linear problem under consideration. Here, using the 2D Blasius' flow problems as an example, we can show this point more clearly.

Assume that $\hbar$ and β are properly selected so that the related series (4.9)

$$\sum_{k=0}^{+\infty} \varphi_k(\eta, \hbar, \beta) \tag{4.54}$$

converges. Then,

$$\lim_{k \to +\infty} \varphi_k(\eta, \hbar, \beta) = 0 \tag{4.55}$$

must hold. Thus, we further have by Eq. (4.11) that

$$\lim_{k \to +\infty} G_k(\eta, \hbar, \beta) = \lim_{k \to +\infty} \left[\frac{\partial^3 \varphi_k(\eta, \hbar, \beta)}{\partial \eta^3} + \beta \frac{\partial^2 \varphi_k(\eta, \hbar, \beta)}{\partial \eta^2} \right] = 0, \quad \eta \in [0, +\infty), \tag{4.56}$$

say, the infinite sequence

$$G_1(\eta, \hbar, \beta), \quad G_2(\eta, \hbar, \beta), \quad G_3(\eta, \hbar, \beta), \ldots,$$

converges to zero. On the other hand, owing to Eq. (3.12), or Eqs. (4.13) and (4.14), we have by straightforward calculations that

$$G_m(\eta,\hbar,\beta) = G_{m-1}(\eta,\hbar,\beta) + \hbar\left[\frac{\partial^3\varphi_{m-1}(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{1}{2}\sum_{k=0}^{m-1}\varphi_k(\eta,\hbar,\beta)\frac{\partial^2\varphi_{m-1-k}(\eta,\hbar,\beta)}{\partial\eta^2}\right] = \hbar\sum_{i=1}^{m}\left[\frac{\partial^3\varphi_{i-1}(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{1}{2}\sum_{k=0}^{i-1}\varphi_k(\eta,\hbar,\beta)\frac{\partial^2\varphi_{i-1-k}(\eta,\hbar,\beta)}{\partial\eta^2}\right]. \quad (4.57)$$

Therefore, owing to Eqs. (4.56) and (4.57), one may deduce

$$\begin{aligned}\lim_{m\to+\infty} G_m(\eta,\hbar,\beta) &= \hbar\sum_{i=1}^{+\infty}\left[\frac{\partial^3\varphi_{i-1}(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{1}{2}\sum_{k=0}^{i-1}\varphi_k(\eta,\hbar,\beta)\frac{\partial^2\varphi_{i-1-k}(\eta,\hbar,\beta)}{\partial\eta^2}\right]\\
&= \hbar\sum_{s=0}^{+\infty}\frac{\partial^3\varphi_s(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{\hbar}{2}\sum_{s=0}^{+\infty}\sum_{k=0}^{s}\varphi_k(\eta,\hbar,\beta)\frac{\partial^2\varphi_{s-k}(\eta,\hbar,\beta)}{\partial\eta^2}\\
&= \hbar\sum_{s=0}^{+\infty}\frac{\partial^3\varphi_s(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{\hbar}{2}\sum_{k=0}^{+\infty}\sum_{s=k}^{+\infty}\varphi_k(\eta,\hbar,\beta)\frac{\partial^2\varphi_{s-k}(\eta,\hbar,\beta)}{\partial\eta^2}\\
&= \hbar\sum_{s=0}^{+\infty}\frac{\partial^3\varphi_s(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{\hbar}{2}\sum_{k=0}^{+\infty}\varphi_k(\eta,\hbar,\beta)\sum_{s=k}^{+\infty}\frac{\partial^2\varphi_{s-k}(\eta,\hbar,\beta)}{\partial\eta^2}\\
&= \hbar\sum_{s=0}^{+\infty}\frac{\partial^3\varphi_s(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{\hbar}{2}\sum_{k=0}^{+\infty}\varphi_k(\eta,\hbar,\beta)\sum_{s=0}^{+\infty}\frac{\partial^2\varphi_s(\eta,\hbar,\beta)}{\partial\eta^2}\\
&= \hbar\left\{\frac{\partial^3}{\partial\eta^3}\left[\sum_{k=0}^{+\infty}\varphi_k(\eta,\hbar,\beta)\right] + \frac{1}{2}\left[\sum_{k=0}^{+\infty}\varphi_k(\eta,\hbar,\beta)\right]\frac{\partial^2}{\partial\eta^2}\left[\sum_{k=0}^{+\infty}\varphi_k(\eta,\hbar,\beta)\right]\right\}\\
&= 0,\end{aligned} \quad (4.58)$$

which gives, due to $\hbar \neq 0$,

$$\frac{\partial^3}{\partial\eta^3}\left[\sum_{k=0}^{+\infty}\varphi_k(\eta,\hbar,\beta)\right] + \frac{1}{2}\left[\sum_{k=0}^{+\infty}\varphi_k(\eta,\hbar,\beta)\right]\frac{\partial^2}{\partial\eta^2}\left[\sum_{k=0}^{+\infty}\varphi_k(\eta,\hbar,\beta)\right] = 0. \quad (4.59)$$

Furthermore, by Eqs. (4.6) and (4.12), we have

$$\sum_{k=0}^{+\infty}\varphi_k(0,\hbar,\beta) = 0, \quad \sum_{k=0}^{+\infty}\varphi_k'(0,\hbar,\beta) = 0,$$

$$\sum_{k=0}^{+\infty}\varphi_k'(+\infty,\hbar,\beta) = 1.$$

Therefore, as long as the infinite series (4.53) converges, it must be the solution of the Blasius' viscous flow problems governed by Eqs. (1.1) and (1.2).

Note that the infinite series (4.53) gives a *family* of explicit analytic solutions in two parameters β ($\beta > 0$) and $\hbar$ ($\hbar \neq 0$). Some among them converge to $f(\eta)$ but some do not, dependent upon the values of $\hbar$ and β. Besides, some of them might be "better" than others. Note that the HAM provides us with great freedom and large flexibility to select "better" values of β and $\hbar$ so as to ensure that the related series (4.53) converges to $f(\eta)$. Certainly, if Eq. (4.53) converges, its second-order derivative with respect to η at $\eta = 0$, say,

$$\sum_{k=0}^{+\infty}\varphi_k''(0,\hbar,\beta) \quad (4.60)$$

must converge, too. By Eq. (A.11), we have its corresponding mth-order approximation

$$\sigma_m = \sum_{k=0}^{m}\varphi_k''(0,\hbar,\beta) = \sum_{k=0}^{m}\sum_{n=1}^{k+1} c_{k,n}^0. \quad (4.61)$$

By the foregoing recurrence formulas, we have

$$\sigma_1 = \beta(1+\hbar) - \frac{\hbar}{4\beta}, \quad (4.62)$$

$$\sigma_2 = \beta(1+\hbar)^2 - \frac{1}{2}\frac{\hbar}{\beta} - \frac{5}{24}\frac{\hbar^2}{\beta^3}, \quad (4.63)$$

$$\sigma_3 = \beta(1+\hbar)^3 - \frac{3}{4}\frac{\hbar}{\beta} - \frac{5}{8}\frac{\hbar^2}{\beta^3} - \frac{275}{576}\frac{\hbar^3}{\beta^5} + \frac{5}{24}\frac{\hbar^3}{\beta^3}, \tag{4.64}$$

$$\sigma_4 = \beta(1+\hbar)^4 - \frac{\hbar}{\beta} - \frac{5}{4}\frac{\hbar^2}{\beta^3} - \frac{275}{144}\frac{\hbar^3}{\beta^5} + \frac{5}{6}\frac{\hbar^3}{\beta^3} + \frac{275}{288}\frac{\hbar^4}{\beta^5} - \frac{4879}{2880}\frac{\hbar^4}{\beta^7}, \tag{4.65}$$

$$\sigma_5 = \beta(1+\hbar)^5 - \frac{5}{4}\frac{\hbar}{\beta} - \frac{25}{12}\frac{\hbar^2}{\beta^3} - \frac{1375}{288}\frac{\hbar^3}{\beta^5} + \frac{25}{12}\frac{\hbar^3}{\beta^3} + \frac{1375}{288}\frac{\hbar^4}{\beta^5} - \frac{4879}{576}\frac{\hbar^4}{\beta^7} - \frac{275}{576}\frac{\hbar^5}{\beta^5} + \frac{4879}{960}\frac{\hbar^5}{\beta^7} - \frac{2740789}{345600}\frac{\hbar^5}{\beta^9}, \ldots \tag{4.66}$$

Note that σ_m contains the term $\beta(1+\hbar)^m$. Thus, $\hbar$ must belong to a subset of the region $|1+\hbar| \leqslant 1$, i.e.

$$-2 \leqslant \hbar \leqslant 0. \tag{4.67}$$

In other words, we are ascertained that the infinite series (4.60) and (4.53) diverge when $\hbar > 0$ or $\hbar < -2$. Note that in Eqs. (4.2) and (4.3) we have defined $\hbar \neq 0$. Our calculations indicate that the series (4.60) converges if

$$-2 < \hbar < 0, \quad \beta > \beta_c, \tag{4.68}$$

where $\beta_c \approx 2.5$. However, the convergence rate is dependent upon the values of $\hbar$ and β. Our calculations indicate that the series (4.60) converges rather slowly when $\hbar$ is in the neighborhoods of -2 or zero. In the region $-1.25 \leqslant \hbar \leqslant -0.75$, the series (4.60) converges sufficiently fast. Furthermore, for each given value of $\hbar \in (-2, 0)$, there exists a best value of β ($\beta > \beta_c$) which corresponds to the smallest convergence rate, say, the corresponding series (4.60) converges fastest. For example, when $\hbar = -9/10$, $\beta = 3$, the series (4.60) converges fast enough to the value 0.33206, as shown in Table 1, which agrees well with Howarth's numerical one. We emphasize that such a totally *analytic* result is given the first time in history (to our knowledge). This verifies the validity of the HAM.

Furthermore, we examine the convergence of the series (4.53). Our calculations indicate that, Eq. (4.53) is convergent in the whole region $\eta \in [0, +\infty)$ to the solution $f(\eta)$ of Eqs. (1.1) and (1.2), as long as the series (4.60) converges, say, the series (4.53) converges when

$$-2 < \hbar < 0, \quad \beta > \beta_c$$

where $\beta_c \approx 2.5$. Moreover, when the series (4.60) converges fast enough, the corresponding series (4.53) also converges sufficiently fast. For example, when $\hbar = -9/10$, $\beta = 3$, the series (4.53) converges to the solution of Eqs. (1.1) and (1.2) with a satisfactory convergence rate, so that the corresponding 35th-order analytic approximation agrees very well with Howarth's numerical results, as shown in Tables 2–4. Clearly, the higher the order of approximation, the better the approximation is. Note that

Table 1
Analytic approximations $f_m''(0, \beta, \hbar)$ when $\beta = 3$ and $\hbar = -0.90$

Order of Approx.	$f''(0)$
5th order	0.28098
10th order	0.32992
15th order	0.33164
20th order	0.33198
25th order	0.33204
30th order	0.33205
35th order	0.33206

Table 2
Residual errors $|f_{35}'''(\eta) + f_{35}''(\eta) f_{35}(\eta)/2|$ of the 35th-order analytic approximation when $\beta = 3$ and $\hbar = -9/10$

η_i	10th approx.	20th approx.	30th approx.	35th approx.
0.4	2.4×10^{-2}	1.9×10^{-4}	8.3×10^{-6}	7.2×10^{-6}
0.8	7.2×10^{-3}	4.2×10^{-4}	1.4×10^{-5}	3.7×10^{-6}
1.2	8.0×10^{-2}	2.0×10^{-3}	2.3×10^{-5}	4.6×10^{-6}
1.6	0.102	2.1×10^{-3}	1.4×10^{-4}	1.4×10^{-5}
2.0	2.9×10^{-2}	5.3×10^{-3}	3.9×10^{-4}	9.5×10^{-5}
2.4	4.8×10^{-2}	1.2×10^{-2}	4.2×10^{-5}	1.4×10^{-4}
2.8	7.6×10^{-2}	8.3×10^{-3}	1.2×10^{-3}	1.7×10^{-4}
3.2	6.5×10^{-2}	6.7×10^{-4}	1.6×10^{-3}	5.5×10^{-4}
3.6	4.2×10^{-2}	7.2×10^{-3}	7.3×10^{-4}	5.1×10^{-4}
4.0	2.3×10^{-2}	8.7×10^{-3}	4.2×10^{-4}	1.2×10^{-4}
4.4	1.1×10^{-2}	1.9×10^{-3}	1.1×10^{-3}	2.6×10^{-4}
5	2.7×10^{-3}	3.2×10^{-3}	9.6×10^{-4}	4.1×10^{-4}
6	1.8×10^{-4}	4.3×10^{-4}	2.4×10^{-4}	1.4×10^{-4}
7	7.3×10^{-6}	2.8×10^{-5}	2.4×10^{-5}	1.7×10^{-5}
8	1.7×10^{-7}	1.0×10^{-6}	1.2×10^{-6}	9.5×10^{-7}
Averaged	3.4×10^{-2}	3.5×10^{-3}	4.6×10^{-4}	1.6×10^{-4}

Table 3
Comparison of the analytic approximations $f_m(\eta, \beta, \hbar)$ in case of $\beta = 3$ and $\hbar = -9/10$ with Howarth's (Refs. [6, 7]) numerical result

η	5th order	10th order	15th order	20th order	25th order	30th order	35th order	Numerical results
0.4	0.0218	0.0263	0.0266	0.0266	0.0266	0.0266	0.0266	0.0266
0.8	0.0920	0.1042	0.1059	0.1061	0.1061	0.1061	0.1061	0.1061
1.2	0.2319	0.2321	0.2373	0.2379	0.2379	0.2379	0.2380	0.2380
1.6	0.4544	0.4105	0.4185	0.4200	0.4203	0.4203	0.4203	0.4203
2.0	0.7489	0.6421	0.6467	0.6493	0.6499	0.6500	0.6500	0.6500
2.4	1.0932	0.9254	0.9188	0.9211	0.9220	0.9222	0.9223	0.9223
2.8	1.4665	1.2524	1.2300	1.2297	1.2305	1.2309	1.2310	1.2310
3.2	1.8547	1.6110	1.5733	1.5686	1.5686	1.5689	1.5690	1.5690
3.6	2.2498	1.9897	1.9402	1.9307	1.9294	1.9294	1.9295	1.9295
4.0	2.6479	2.3795	2.3224	2.3091	2.3063	2.3058	2.3058	2.3058
4.4	3.0472	2.7751	2.7137	2.6977	2.6936	2.6926	2.6924	2.6924
5.0	3.6468	3.3728	3.3088	3.2907	3.2854	3.2839	3.2835	3.2833
6.0	4.6468	4.3722	4.3072	4.2883	4.2825	4.2806	4.2800	4.2797
7.0	5.6468	5.3722	5.3071	5.2881	5.2822	5.2803	5.2796	5.2793
8.0	6.6468	6.3722	6.3071	6.2881	6.2822	6.2802	6.2796	6.2793
10	8.6468	8.3722	8.3071	8.2881	8.2822	8.2802	8.2796	8.2793
15	13.6468	13.3722	13.3071	13.2881	13.2822	13.2902	13.2796	13.2793
20	18.6468	18.3722	18.3071	18.2881	18.2822	18.2802	18.2796	18.2793
50	48.6468	48.3722	48.3071	48.2881	48.2822	48.2802	48.2796	48.2793
100	98.6468	98.3722	98.3071	98.2881	98.2822	98.2802	98.2796	98.2793

Table 4
Comparison of the analytic approximations $f'_m(\eta, \beta, \hbar)$ in case of $\beta = 3$ and $\hbar = -9/10$ with Howarth's (Refs. [6, 7]) numerical result

η_i	5th order	10th order	15th order	20th order	25th order	30th order	35th order	Numerical results
0.4	0.1090	0.13099	0.1326	0.1327	0.1328	0.1328	0.1328	0.1328
0.8	0.2529	0.2578	0.2640	0.2646	0.2647	0.2647	0.2647	0.2647
1.2	0.4527	0.3819	0.3918	0.3935	0.3937	0.3938	0.3938	0.3938
1.6	0.6544	0.5118	0.5130	0.5160	0.5166	0.5167	0.5168	0.5168
2.0	0.8083	0.6456	0.6269	0.6284	0.6295	0.6297	0.6298	0.6298
2.4	0.9044	0.7675	0.7316	0.7279	0.7285	0.7289	0.7290	0.7290
2.8	0.9563	0.8622	0.8216	0.8123	0.8112	0.8113	0.8115	0.8115
3.2	0.9814	0.9259	0.8914	0.8794	0.8765	0.8760	0.8760	0.8761
3.6	0.9926	0.9636	0.9395	0.9284	0.9246	0.9236	0.9234	0.9233
4.0	0.9972	0.9835	0.9692	0.9609	0.9573	0.9560	0.9557	0.9555
4.4	0.9990	0.9931	0.9855	0.9804	0.9777	0.9765	0.9761	0.9759
5.0	0.9998	0.9983	0.9960	0.9940	0.9927	0.9921	0.9918	0.9916
6.0	1.0000	1.0000	0.9997	0.9994	0.9993	0.9991	0.9991	0.9990
7.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
8.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
10	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
15	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
20	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
50	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
100	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

the first-order derivative of the series (4.53) also converges to the corresponding Howarth's numerical solution, as shown in Table 4. In fact by Eq. (4.53), we can easily get the high-order derivatives of $f(\eta)$. This is one of the advantages of analytic solutions over numerical ones.

5. Conclusions and discussions

In this paper, we first of all use a non-linear differential operator $\mathscr{A}$ to systematically describe the basic ideas of a new kind of analytic technique for non-linear problems, namely the homotopy analysis method (HAM). Then, we prove that, as long as the related approximation sequence is convergent, it must converge to one of the solutions of the non-linear problem under consideration. Besides, we provide a simple way to judge if the related approximation sequence is divergent. What we would especially emphasize is that the HAM can provide us with great freedom and large flexibility to select better initial approximations and auxiliary linear operators and nonzero auxiliary parameter $\hbar$ and embedding functions $A(p)$, $B(p)$ so as to ensure the HAM valid and to get better approximations. To illustrate this point, we apply the HAM to solve the 2D Blasius' viscous flow problems and give an explicit, purely *analytic* solution which is valid in the whole region $0 \leqslant \eta < +\infty$. This solution gives us an *analytic* value $f''(0) = 0.33206$, which agrees very well with Howarth's numerical one. We emphasize that it is the first time in history (to our knowledge) that such an explicit, totally analytic solution of Blasius' flow problems and such an analytic value of $f''(0)$ are given. This well verify the validity and the great potential of the HAM as a new kind of analytic tool.

Note that the explicit analytic solution (4.53) contains two parameters β and $\hbar$. It is very interesting that this analytic solution converges to the *sole* solution of Blasius' flow problems when

$$-2 < \hbar < 0, \quad \beta > \beta_c,$$

where $\beta_c \approx 2.5$. Note that different values of β correspond to different initial approximations $f_0(\eta)$ and different "auxiliary" linear operators $\mathscr{L}$, and moreover, different values of $\hbar$ give different "deformations" governed by the zeroth-order deformation equations (4.2) and (4.3). Thus, an *infinite* number of pairs of β and $\hbar$, or in other words, an *infinite* number of initial approximations and auxiliary linear operators and also many kinds of deformations, can make the series (4.53) convergent to the *sole* solution of the Blasius' flow problems. It means that, although the solution of Blasius flow problems is *sole*, it has however an *infinite* number of different expressions. How should we understand this *fact*? We emphasize that the solution (4.53) is a kind of *limit* and the values of β and $\hbar$ determine the way and speed of *approach* to the *sole* solution of Blasius' flow problems. In essence, this is similar to such a limit of a real function having two variables

$$\Pi = \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{|x|}. \tag{5.1}$$

It is well known that the result Π of the above limit is strongly dependent upon the *way* or the *approach* how the point (x, y) tends to $(0, 0)$. Assume that the point (x, y) tends to $(0, 0)$ along the path defined by

$$y = \alpha x^{\gamma}, \quad \gamma > 0, \tag{5.2}$$

then we have

$$\Pi = \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{|x|} = 1, \quad \gamma > 1,$$

$$\Pi = \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{|x|} = \sqrt{1+\alpha^2}, \quad \gamma = 1,$$

$$\Pi = \lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2+y^2}}{|x|} = +\infty, \quad 0 < \gamma < 1. \tag{5.3}$$

Here, we emphasize two points. First, the result Π of the limit (5.1) is strongly dependent upon the path tending to the *sole* point $(0, 0)$. Secondly, there exists an *infinite* number of paths corresponding to $\gamma > 1$, which give the *same* result $\Pi = 1$. In other words, there exists an *infinite* number of paths along which we get the *sole* result $\Pi = 1$. Now, let us come back to the explicit analytic solution (4.53) of the Blasius' flow problems. Note that solution (4.53) is defined by a kind of limit. Similarly, although Eqs. (1.1) and (1.2) have one and only one solution, there exist however an *infinite* number of different *approaches* to tend to this *sole* solution. Some among these approaches are better than others, but some of them (for example, those when $|1+\hbar|>1, \beta<\beta_c$) are so bad that the corresponding series is divergent. And certainly, there should exist some better values of β and $\hbar$. In other words, there should exist some better auxiliary linear operators, better initial approximations and better zeroth-order deformation equations which make the related approximation series to converge sufficiently fast. Note that it is the homotopy analysis method (HAM) which provides us with the possibility, flexibility and freedom to select operator (4.1), which is more general than that used in Ref. [8], as our auxiliary linear operator, so that, we can get such an explicit, totally analytic solution (4.53) of the 2D Blasius' flow problems.

Note that we get solution (4.53) by using the simplest embedding functions $A(p)=p$ and $B(p)=p$. Obviously, if we use other types of embedding functions such as

$$\tan\left(\frac{\pi}{4}p\right),\quad \sin\left(\frac{\pi}{2}p\right),\quad \frac{\exp(p)-1}{(e-1)}$$

and so on, we can get some new types of analytic approximations of Blasius' flow problems. Thus, it might be interesting to study whether or not there exist the best embedding functions and furthermore how to find them when the answer is positive.

Although the example, i.e. the 2D Blasius' viscous flow, is simple and the related governing Eq. (1.1) is an ordinary differential equation, the basic ideas of the HAM can be applied to solve complicated non-linear problems governed by ordinary differential equations or partial differential equations, as mention in Section 2.

Using an auxiliary linear operator different from Eq. (4.1), Liao [12] gave the power series (1.5) of Blasius' flow problems which contains the Blasius' power solution (1.3). However, compared with perturbation techniques, the HAM has the following advantages. First, it is based on homotopy and does not depend on small parameters. Therefore, the HAM can be applied to approximately solve a large class of strongly non-linear problems than with the straightforward perturbation method, even including those whose governing equations and boundary conditions do not contain any small parameters. Secondly, the HAM provides us with great freedom and large flexibility to properly select initial approximations, auxiliary linear operators, non-zero auxiliary parameter $\hbar$ and embedding functions $A(p)$, $B(p)$. This kind of freedom and flexibility not only implies great potential for us to further improve the HAM itself, but also provides us with a larger possibility to ensure that the related infinite series of approximations are convergent and, besides, to select "better" ones from the family of approximations in more general forms. In fact, based on the HAM, we have also developed some new numerical techniques such as the general boundary element method [13, 14] and so on (see Ref. [15]). Therefore, irrespective of whether small parameters or not, the HAM might become a new powerful analytic tool for non-linear problems in science and engineering, although it certainly needs further improvement and more applications.

Finally, we especially point out that we get the mathematical structure (4.15) of $\varphi_m(\eta,\hbar,\beta)$ by firstly applying the symbolic computation software *MATHEMATICA* to solve the first several (sufficiently many) deformation equations (4.11) and (4.12) and then rigorously proving (by logic) that this structure is indeed right (for details, please refer to Appendix A). Without a computer, it seems difficult for us to *guess* at first such a structure of $\varphi_m(\eta,\hbar,\beta)$. Thus, we agree with the view point of Dadfar *et al.* [17] that the importance of the role of symbolic computation should not be underestimated. Combined with high performance computers and symbolic computation software such as MATHEMATICA and so on, the Homotopy Analysis Method (HAM) might become a new, more powerful analytic tool to get satisfactory approximations of complicated non-linear problems in science and engineering.

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Appendix A

(I) First of all, we point out that the initial approximation (4.6) has the same structure as Eq. (4.15), where the real function $\Psi_{m,k}(\eta, \hbar, \beta)$ is defined by Eqs. (4.16)–(4.19).

(II) Secondly, if we assume that the first $(m-1)$ solutions $\varphi_k(\eta, \hbar, \beta)$ $(k = 0, 1, 2, 3, \ldots, m-1)$ have the same structure as Eq. (4.15), then, we can prove that $\varphi_m(\eta, \beta, \hbar)$ has the same structure as Eq. (4.15), too.

To prove this, we define for the sake of simplicity that

$$\begin{aligned}
&\lambda^k_{i,j} = 0, \quad i = j = 0, \quad k \geqslant 2,\\
&\lambda^k_{i,j} = 0, \quad i > 0, \quad j = 0, \quad k \geqslant 1,\\
&\lambda^k_{i,j} = 0, \quad j > i + 1,\\
&\lambda^k_{i,j} = 0, \quad k > 2(i + 1 - j),\\
&\lambda^k_{i,j} = 1, \quad \text{otherwise.}
\end{aligned} \tag{A.1}$$

Then, $\Psi_{m,k}(\eta, \beta, \hbar)$ can be simply rewritten as

$$\Psi_{m,k}(\eta, \beta, \hbar) = \sum_{i=0}^{2(m+1-k)} \lambda^i_{m,k} b^i_{m,k} \eta^i, \quad 0 \leqslant k \leqslant m+1 \tag{A.2}$$

for both $k = 0$ and $k \neq 0$. Thus, we have for $0 \leqslant k \leqslant m+1$ that

$$\begin{aligned}
\Psi'_{m,k}(\eta, \beta, \hbar) &= \sum_{i=0}^{2(m+1-k)} i \lambda^i_{m,k} b^i_{m,k} \eta^{i-1}\\
&= \sum_{i=0}^{2(m-k)+1} (i+1) \lambda^{i+1}_{m,k} b^{i+1}_{m,k} \eta^i\\
&= \sum_{i=0}^{2(m-k+1)} (i+1) \lambda^{i+1}_{m,k} b^{i+1}_{m,k} \eta^i,
\end{aligned} \tag{A.3}$$

$$\begin{aligned}
\Psi''_{m,k}(\eta, \beta, \hbar) &= \sum_{i=2}^{2(m+1-k)} i(i-1) \lambda^{i+1}_{m,k} b^i_{m,k} \eta^{i-2}\\
&= \sum_{i=0}^{2(m-k)} (i+2)(i+1) \lambda^{i+2}_{m,k} b^{i+2}_{m,k} \eta^i\\
&= \sum_{i=0}^{2(m-k+1)} (i+2)(i+1) \lambda^{i+2}_{m,k} b^{i+2}_{m,k} \eta^i,
\end{aligned} \tag{A.4}$$

where the prime denotes the derivative with respect to η. Differentiating Eq. (4.15) twice with respect to η, we have

$$\varphi''_m(\eta, \beta, \hbar) = \sum_{k=1}^{m+1} [\Psi''_{m,k} - 2k\beta \Psi'_{m,k} + (k\beta)^2 \Psi_{m,k}] \times \exp(-k\beta\eta), \tag{A.5}$$

Owing to Eqs. (A.3) and (A.4), we have when $1 \leqslant k \leqslant m+1$ that

$$\Psi''_{m,k} - 2k\beta \Psi'_{m,k} + (k\beta)^2 \Psi_{m,k} = \sum_{i=0}^{2(m-k+1)} c^i_{m,k} \eta^i, \tag{A.6}$$

where

$$c^0_{m,m+1} = (m+1)^2 \beta^2 b^0_{m,m+1}, \tag{A.7}$$

and when $1 \leqslant k \leqslant m$,

$$c^i_{m,k} = (i+1)(i+2) b^{i+2}_{m,k} - 2(k\beta)(i+1) b^{i+1}_{m,k} + (k\beta)^2 b^i_{m,k}, \quad 0 \leqslant i \leqslant 2(m-k), \tag{A.8}$$

$$c^i_{m,k} = -2(k\beta)(i+1) b^{i+1}_{m,k} + (k\beta)^2 b^i_{m,k}, \quad i = 2(m-k)+1, \tag{A.9}$$

$$c^i_{m,k} = (k\beta)^2 b^i_{m,k}, \quad i = 2(m-k)+2, \tag{A.10}$$

Thus, by Eqs. (A.5)–(A.6), we have

$$\varphi''_m(\eta, \beta, \hbar) = \sum_{k=1}^{m+1} \exp(-k\beta\eta) \left(\sum_{i=0}^{2(m-k+1)} c^i_{m,k} \eta^i \right), \quad m \geqslant 1. \tag{A.11}$$

Differentiating the above equation with respect to η, we have

$$\varphi'''_m(\eta, \beta, \hbar) = \sum_{k=1}^{m+1} \exp(-k\beta\eta) \left(\sum_{i=0}^{2(m-k+1)} d^i_{m,k} \eta^i \right), \quad m \geqslant 1, \tag{A.12}$$

where

$$d^0_{m,m+1} = -(m+1)^3 \beta^3 b^0_{m,m+1}, \tag{A.13}$$

and when $1 \leqslant k \leqslant m$,

$$d^i_{m,k} = (i+1) c^{i+1}_{m,k} - (k\beta) c^i_{m,k}, \; 0 \leqslant i \leqslant 2(m-k)+1, \tag{A.14}$$

$$d^{2(m-k+1)}_{m,k} = -(k\beta) c^{2(m-k+1)}_{m,k}. \tag{A.15}$$

Owing to Eqs. (4.13)–(4.14), we have by straightforward calculations that

$$G_m(\eta,\beta,\hbar) = G_{m-1}(\eta,\beta,\hbar) + \hbar\left[\frac{\partial^3\varphi_{m-1}(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{1}{2}\sum_{k=0}^{m-1}\varphi_{m-1-k}(\eta,\hbar,\beta)\frac{\partial^2\varphi_k(\eta,\hbar,\beta)}{\partial\eta^2}\right], \quad (A.16)$$

where $m > 1$. Note that

$$\varphi_1''' + \beta\varphi_1'' = \hbar\,[\varphi_0'''(\eta,\beta,\hbar) + \tfrac{1}{2}\varphi_0(\eta,\beta,\hbar)\varphi_0''(\eta,\beta,\hbar)], \quad (A.17)$$

and when $m > 1$ we have

$$\varphi_{m-1}''' + \beta\varphi_{m-1}'' = G_{m-1}(\eta,\beta,\hbar). \quad (A.18)$$

Thus, by Eqs. (A.16), (A.17) and (A.18), we obtain the following equation:

$$(\varphi_m - \chi_m\varphi_{m-1})''' + \beta(\varphi_m - \chi_m\varphi_{m-1})'' = \hbar\left[\frac{\partial^3\varphi_{m-1}(\eta,\hbar,\beta)}{\partial\eta^3} + \frac{1}{2}\sum_{k=0}^{m-1}\varphi_{m-1-k}(\eta,\hbar,\beta)\frac{\partial^2\varphi_k(\eta,\hbar,\beta)}{\partial\eta^2}\right] \quad (A.19)$$

with the boundary conditions

$$\varphi_m(0,\beta,\hbar) - \chi_m\varphi_{m-1}(0,\beta,\hbar) = 0, \quad (A.20)$$

$$\varphi_m'(0,\beta,\hbar) - \chi_m\varphi_{m-1}'(0,\beta,\hbar) = 0, \quad (A.21)$$

$$\varphi_m'(+\infty,\beta,\hbar) - \chi_m\varphi_{m-1}'(+\infty,\beta,\hbar) = 0. \quad (A.22)$$

where $m \geqslant 1$ and

$$\chi_m = \begin{cases} 0, & \text{when } m = 1, \\ 1, & \text{otherwise.} \end{cases} \quad (A.23)$$

When $0 \leqslant k \leqslant m-1$, we get by Eqs. (A.2) and (A.11) that

$$\varphi_{m-1-k}(\eta,\beta,\hbar) = \sum_{r=0}^{m-k}\Psi_{m-1-k,r}(\eta,\beta,\hbar)\exp(-r\beta\eta) = \sum_{r=0}^{m-k}\exp(-r\beta\eta)\left(\sum_{s=0}^{2(m-k-r)}\lambda_{m-1-k,r}^s b_{m-1-k,r}^s\eta^s\right) \quad (A.24)$$

and

$$\varphi_k''(\eta,\beta,\hbar) = \sum_{j=1}^{k+1}\exp(-j\beta\eta)\left(\sum_{i=0}^{2(k-j+1)}c_{k,j}^i\eta^i\right), \quad (A.25)$$

respectively. Therefore, when $0 \leqslant k \leqslant m-1$, we have by Eqs. (A.24) and (A.25) that

$$\begin{aligned}
&\frac{1}{2}\varphi_{m-1-k}(\eta,\hbar,\beta)\frac{\partial^2\varphi_k(\eta,\hbar,\beta)}{\partial\eta^2} \\
&= \frac{1}{2}\sum_{r=0}^{m-k}\exp(-r\,\beta\eta)\left(\sum_{s=0}^{2(m-k-r)}\lambda_{m-1,k,r}^s b_{m-1-k,r}^s\eta^s\right) \times \sum_{j=1}^{k+1}\exp(-j\,\beta\eta)\left(\sum_{i=0}^{2(k-j+1)}c_{k,j}^i\eta^i\right) \\
&= \frac{1}{2}\sum_{j=1}^{k+1}\sum_{r=0}^{m-k}\exp[-(j+r)\beta\eta] \times \left(\sum_{i=0}^{2(k-j+1)}\sum_{s=0}^{2(m-k-r)}c_{k,j}^i b_{m-1k,r}^s\lambda_{m-1-k,r}^s\eta^{s+i}\right) \\
&= \sum_{n=1}^{m+1}\exp(-n\,\beta\eta)\sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}} \times \left(\sum_{i=0}^{2(k-j+1)}\sum_{s=0}^{2(m-k-n+j)} \times \frac{1}{2}c_{k,j}^i b_{m-1-k,n-j}^s\lambda_{m-1-k,n-j}^s\eta^{s+i}\right) \\
&= \sum_{n=1}^{m+1}\exp(-n\,\beta\eta)\sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}} \times \left(\sum_{q=0}^{2(m-n+1)}\eta^q\sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k-j+1)\}} \times \frac{1}{2}c_{k,j}^i b_{m-1-k,n-j}^{q-i}\lambda_{m-1-k,n-j}^{q-i}\right) \\
&= \sum_{n=1}^{m+1}\exp(-n\,\beta\eta)\sum_{q=0}^{2(m-n+1)}\eta^q \times \left(\sum_{j=\max\{1,n+k-m\}}^{\min\{n,k+1\}}\sum_{i=\max\{0,q-2(m-k-n+j)\}}^{\min\{q,2(k-j+1)\}} \times \frac{1}{2}c_{k,j}^i b_{m-1-k,n-j}^{q-i}\lambda_{m-1-k,n-j}^{q-i}\right).
\end{aligned} \quad (A.26)$$

Therefore, we have

$$\frac{1}{2}\sum_{k=0}^{m-1} \varphi_{m-1-k}(\eta,\hbar,\beta)\frac{\partial^2\varphi_k(\eta,\hbar,\beta)}{\partial\eta^2}$$

$$= \exp(-\beta\eta)\sum_{q=0}^{2m-1}\delta_{m,1}^{q}\eta^q + \sum_{n=2}^{m+1}\exp(-n\,\beta\eta)\left(\sum_{q=0}^{2(m-n+1)}\delta_{m,n}^{q}\eta^q\right), \quad \text{(A.27)}$$

where for $1 \leqslant n \leqslant m+1$, $0 \leqslant q \leqslant 2(m-n+1)$,

$$\delta_{m,n}^{q} = \frac{1}{2}\sum_{k=0}^{m-1}\sum_{j=J_0}^{J_1}\sum_{i=I_0}^{I_1} c_{k,j}^{i} b_{m-1-k,n-j}^{q-i}\lambda_{m-1-k,n-j}^{q-i},$$

$$J_0 = \max\{1, n+k-m\}, \quad J_1 = \min\{n, k+1\},$$

$$I_0 = \max\{0, q-2(m-k-n+j)\},$$

$$I_1 = \min\{q, 2(k-j+1)\}. \quad \text{(A.28)}$$

Here, we especially point out that $\delta_{m,1}^{2m} = 0$ for $m \geqslant 1$.

Thus, by Eqs. (A.12) and (A.28), when $m \geqslant 1$,

$$\hbar\left[\varphi_{m-1}'''(\eta,\hbar,\beta) + \frac{1}{2}\sum_{k=0}^{m-1}\varphi_{m-1-k}(\eta,\hbar,\beta)\frac{\partial^2\varphi_k(\eta,\hbar,\beta)}{\partial\eta^2}\right]$$

$$= \sum_{n=1}^{m}\exp(-n\,\beta\eta)\left(\sum_{q=0}^{2(m-n)}\hbar d_{m-1,n}^{q}\eta^q\right) + \exp(-\beta\eta)\sum_{q=0}^{2m-1}\hbar\delta_{m,1}^{q}\eta^q + \sum_{n=2}^{m+1}\exp(-n\,\beta\eta)\left(\sum_{q=0}^{2(m-n+1)}\hbar\delta_{m,n}^{q}\eta^q\right)$$

$$= \exp(-\beta\eta)\sum_{q=0}^{2m-1}\Gamma_{m,1}^{q}\eta^q + \sum_{n=2}^{m+1}\exp(-n\,\beta\eta)\left(\sum_{q=0}^{2(m-n+1)}\Gamma_{m,n}^{q}\eta^q\right) \quad \text{(A.29)}$$

holds, where

$$\Gamma_{m,1}^{q} = \hbar(d_{m-1,1}^{q} + \delta_{m,1}^{q}), \quad 0 \leqslant q \leqslant 2m-2, \quad \text{(A.30)}$$

$$\Gamma_{m,1}^{2m-1} = \hbar\delta_{m,1}^{2m-1}, \quad \text{(A.31)}$$

$$\Gamma_{m,m+1}^{0} = \hbar\delta_{m,m+1}^{0} \quad \text{(A.32)}$$

and for $2 \leqslant n \leqslant m$,

$$\Gamma_{m,n}^{q} = \hbar(d_{m-1,n}^{q} + \delta_{m,n}^{q}), \quad 0 \leqslant q \leqslant 2(m-n),$$

$$\Gamma_{m,n}^{q} = \hbar\delta_{m,n}^{q}, \quad 2(m-n)+1 \leqslant q \leqslant 2(m-n)+2,$$

$$\Gamma_{m,n}^{q} = 0, \quad \text{otherwise}. \quad \text{(A.33)}$$

Thus, Eq. (A.19) can be rewritten in the following form:

$$(\varphi_m - \chi_m\varphi_{m-1})''' + \beta(\varphi_m - \chi_m\varphi_{m-1})''$$

$$= \exp(-\beta\eta)\sum_{q=0}^{2m-1}\Gamma_{m,1}^{q}\eta^q + \sum_{n=2}^{m+1}\exp(-n\beta\eta)\left(\sum_{q=0}^{2(m-n+1)}\Gamma_{m,n}^{q}\eta^q\right). \quad \text{(A.34)}$$

In order to solve the above equation, we should at first give solutions of the equation

$$Y'''(\eta) + \beta Y''(\eta) = \eta^q\exp(-n\beta\eta), \quad \text{(A.35)}$$

where $n \geqslant 1$ and $q \geqslant 0$ are integers. Here, we mention such a formula, say, for integers $q \geqslant 0$ and $n \geqslant 1$, the following holds

$$\int x^q e^{-n\beta x}\,dx = -e^{-n\beta x}\sum_{j=0}^{q}\left(\frac{q!}{j!}\right)\frac{x^j}{(n\beta)^{q-j+1}}, \quad \text{(A.36)}$$

i.e.

$$\int\left(\frac{x^q}{q!}\right)e^{-n\beta x}\,dx = -e^{-n\beta x}\sum_{j=0}^{q}\left(\frac{x^j}{j!}\right)\frac{1}{(n\beta)^{q-j+1}}.$$

We solve Eq. (A.35) in two different cases of $n = 1$ and $n \geqslant 2$, respectively.

(1) When $n = 1$, Eq. (A.35) becomes

$$Y''(\eta) = \exp(-\beta\eta)\int\exp(\beta\eta)\eta^q\exp(-\beta\eta)\,d\eta$$

$$= \frac{1}{q+1}\eta^{q+1}\exp(-\beta\eta), \quad \text{(A.37)}$$

which further by Eq. (A.36) gives that

$$Y'(\eta) = \int \frac{1}{q+1}\eta^{q+1}\exp(-\beta\eta)\,d\eta$$
$$= -\frac{1}{q+1}\exp(-\beta\eta)\sum_{j=0}^{q+1}\frac{(q+1)!}{j!}\frac{\eta^j}{\beta^{q-j+2}}$$
$$= -\exp(-\beta\eta)\sum_{j=0}^{q+1}\frac{q!}{j!}\frac{\eta^j}{\beta^{q-j+2}}. \quad (A.38)$$

Integrating the above equation, we get

$$Y(\eta) = -\int \exp(-\beta\eta)\sum_{j=0}^{q+1}\frac{q!}{j!}\frac{\eta^j}{\beta^{q-j+2}}\,d\eta$$
$$= -\sum_{j=0}^{q+1}\frac{q!}{j!}\frac{1}{\beta^{q-j+2}}\int \eta^j\exp(-\beta\eta)\,d\eta$$
$$=\sum_{j=0}^{q+1}\frac{q!}{j!}\frac{1}{\beta^{q-j+2}}\exp(-\beta\eta)\sum_{i=0}^{j}\frac{j!}{i!}\frac{\eta^j}{\beta^{j-i+1}}$$
$$=\exp(-\beta\eta)\sum_{j=0}^{q+1}\sum_{i=0}^{j}\frac{q!}{i!}\frac{\eta^i}{\beta^{q-i+3}}$$
$$=\exp(-\beta\eta)\sum_{i=0}^{q+1}\sum_{j=i}^{q+1}\frac{q!}{i!}\frac{\eta^i}{\beta^{q-i+3}}$$
$$=\exp(-\beta\eta)\sum_{i=0}^{q+1}\frac{q!}{i!}\frac{(q-i+2)}{\beta^{q-i+3}}\eta^i$$
$$=\exp(-\beta\eta)\sum_{k=0}^{q+1}\mu_{1,k}^q\eta^k, \quad (A.39)$$

where

$$\mu_{1,k}^q = \frac{q!}{k!}\frac{(q-k+2)}{\beta^{q-k+3}}, \quad 0 \leqslant k \leqslant q+1, \ q \geqslant 0. \quad (A.40)$$

(2) When $n \geqslant 2$ by Eq. (A.36), (A.35) becomes

$$Y''(\eta) = \exp(-\beta\eta)\int \exp(\beta\eta)\eta^q\exp(-n\beta\eta)\,d\eta$$
$$= \exp(-\beta\eta)\int \eta^q\exp[-(n-1)\beta\eta]\,d\eta$$
$$= -\exp(-n\beta\eta)\sum_{j=0}^{q}\frac{q!}{j!}\frac{\eta^j}{[(n-1)\beta]^{q-j+1}}, \quad (A.41)$$

which further gives

$$Y'(\eta) = -\sum_{j=0}^{q}\frac{q!}{j!}\frac{1}{[(n-1)\beta]^{q-j+1}} \times \int \eta^j\exp(-n\beta\eta)\,d\eta$$
$$= \sum_{j=0}^{q}\frac{q!}{j!}\frac{1}{[(n-1)\beta]^{q-j+1}}\exp(-n\beta\eta) \times \sum_{i=0}^{j}\frac{j!}{i!}\frac{\eta^i}{(n\beta)^{j-i+1}}$$
$$= \exp(-n\beta\eta) \times \sum_{j=0}^{q}\sum_{i=0}^{j}\frac{q!}{i!}\frac{1}{[(n-1)\beta]^{q-j+1}}\frac{\eta^i}{(n\beta)^{j-i+1}}. \quad (A.42)$$

Integrating the above equation, we get

$$Y(\eta) = \sum_{j=0}^{q}\sum_{i=0}^{J}\frac{q!}{i!}\frac{1}{[(n-1)\beta]^{q-j+1}}\frac{1}{(n\beta)^{j-i+1}} \times \int \eta^i\exp(-n\beta\eta)\,d\eta$$
$$= -\exp(-n\beta\eta)\sum_{j=0}^{q}\sum_{i=0}^{j}\frac{q!}{i!}\frac{1}{[(n-1)\beta]^{q-j+1}} \times \frac{1}{(n\beta)^{j-i+1}}\sum_{k=0}^{i}\frac{i!}{k!}\frac{\eta^k}{(n\beta)^{i-k+1}}$$
$$= -\exp(-n\beta\eta)\sum_{j=0}^{q}\sum_{i=0}^{j}\sum_{k=0}^{i}\frac{q!}{k!}\frac{1}{[(n-1)\beta]^{q-j+1}} \times \frac{\eta^k}{(n\beta)^{j-k+2}}$$
$$= -\exp(-n\beta\eta)\sum_{j=0}^{q}\sum_{k=0}^{j}\sum_{i=k}^{j}\frac{q!}{k!}\frac{1}{[(n-1)\beta]^{q-j+1}} \times \frac{\eta^k}{(n\beta)^{j-k+2}}$$
$$= -\exp(-n\beta\eta)\sum_{j=0}^{q}\sum_{k=0}^{j}\frac{q!}{k!}\frac{(j-k+1)}{[(n-1)\beta]^{q-j+1}} \times \frac{\eta^k}{(n\beta)^{j-k+2}}$$

$$= -\exp(-n\beta\eta)\sum_{k=0}^{q}\sum_{j=k}^{j}\frac{q!}{k!}\frac{(j-k+1)}{[(n-1)\beta]^{q-j+1}}\times\frac{\eta^k}{(n\beta)^{j-k+2}}$$

$$= -\exp(-n\beta\eta)\sum_{k=0}^{q}\mu_{n,k}^q\eta^k, \quad (A.43)$$

where

$$\mu_{n,k}^q = \frac{q!}{k!}\sum_{j=k}^{q}\frac{(j-k+1)}{[(n-1)\beta]^{q-j+1}}\frac{1}{(n\beta)^{j-k+2}}$$

$$= \frac{q!}{k!}\frac{1}{(n-1)^{q-k+1}\beta^{q-k+3}}\left\{1-\left(1-\frac{1}{n}\right)^{q-k+1}\times\left[(q-k+2)-(q-k+1)\left(1-\frac{1}{n}\right)\right]\right\}. \quad (A.44)$$

for $0 \leqslant k \leqslant q$, $n \geqslant 2$, $q \geqslant 0$.

Note that the solution of Eq. (A.34) is equal to the superposition of solutions of Eq. (A.35). Thus, by Eqs. (A.39) and (A.43), we obtain a special solution of Eq. (A.34) such that

$$(\varphi_m - \chi_m\varphi_{m-1})$$

$$= \exp(-\beta\eta)\sum_{q=0}^{2m-1}\Gamma_{m,1}^q\sum_{k=0}^{q+1}\mu_{1,k}^q\eta^k - \sum_{n=2}^{m+1}\exp(-n\beta\eta)\left(\sum_{q=0}^{2(m-n+1)}\Gamma_{m,n}^q\sum_{k=0}^{q}\mu_{n,k}^q\eta^k\right)$$

$$= \exp(-\beta\eta)\left[\sum_{q=0}^{2m-1}\Gamma_{m,1}^q\mu_{1,0}^q + \sum_{k=1}^{2m}\eta^k\left(\sum_{q=k-1}^{2m-1}\Gamma_{m,1}^q\mu_{1,k}^q\right)\right] - \sum_{n=2}^{m+1}\exp(-n\beta\eta)\times\left[\sum_{k=0}^{2(m-n+1)}\eta^k\left(\sum_{q=k}^{2(m-n+1)}\Gamma_{m,n}^q\mu_{n,k}^q\right)\right] \quad (A.45)$$

Therefore, the general solution of Eq. (A.19) is

$$(\varphi_m - \chi_m\varphi_{m-1})$$

$$= \exp(-\beta\eta)\left[\sum_{q=0}^{2m-1}\Gamma_{m,1}^q\mu_{1,0}^q + \sum_{k=1}^{2m}\eta^k\left(\sum_{q=k-1}^{2m-1}\Gamma_{m,1}^q\mu_{1,k}^q\right)\right] - \sum_{n=2}^{m+1}\exp(-n\beta\eta)\times\left[\sum_{k=0}^{2(m-n+1)}\eta^k\left(\sum_{q=k}^{2(m-n+1)}\Gamma_{m,n}^q\mu_{n,k}^q\right)\right] + C_1^m\exp(-\beta\eta) + C_2^m\eta + C_3^m. \quad (A.46)$$

Using the boundary conditions (A.20), (A.21) and (A.22), we have

$$C_1^m = \sum_{q=0}^{2m-1}\Gamma_{m,1}^q(\beta^{-1}\mu_{1,1}^q - \mu_{1,0}^q) + \sum_{n=2}^{m+1}\left[n\Gamma_{m,n}^0\mu_{n,0}^0 + \sum_{q=1}^{2(m-n+1)}\Gamma_{m,n}^q(n\mu_{n,0}^q - \beta^{-1}\mu_{n,1}^q)\right], \quad (A.47)$$

$$C_2^m = 0, \quad (A.48)$$

$$C_3^m = -C_1^m - \sum_{q=0}^{2m-1}\Gamma_{m,1}^q\mu_{1,0}^q + \sum_{n=2}^{m+1}\sum_{q=0}^{2(m-n+1)}\Gamma_{m,n}^q\mu_{n,0}^q$$

$$= -\sum_{q=0}^{2m-1}\beta^{-1}\Gamma_{m,1}^q\mu_{1,1}^q - \sum_{n=2}^{m+1}\left[(n-1)\Gamma_{m,n}^0\mu_{n,0}^0 + \sum_{q=1}^{2(m-n+1)}\Gamma_{m,n}^q(n\mu_{n,0}^q - \mu_{n,0}^q - \beta^{-1}\mu_{n,1}^q)\right]. \quad (A.49)$$

Therefore, $\varphi_m(\eta,\beta,\hbar)$ has the same structure as Eq. (4.15), and the related coefficients $b_{m,n}^k$ are as follows:

$$b_{m,0}^0 = \chi_m b_{m-1,0}^0 - \beta^{-1}\sum_{q=0}^{2m-1}\Gamma_{m,1}^q\mu_{1,1}^q - \sum_{n=2}^{m+1}\left[(n-1)\Gamma_{m,n}^0\mu_{n,0}^0 + \sum_{q=1}^{2(m-n+1)}\Gamma_{m,n}^q(n\mu_{n,0}^q - \mu_{n,0}^q - \beta^{-1}\mu_{n,1}^q)\right], \quad (A.50)$$

$$b^1_{m,0} = 0, \quad (A.51)$$

$$b^0_{m,1} = \chi_m b^0_{m-1,1} + \beta^{-1} \sum_{q=0}^{2m-1} \Gamma^q_{m,1}\mu^q_{1,1} + \sum_{n=2}^{m+1}\left[n\Gamma^0_{m,n}\mu^0_{n,0} + \sum_{q=1}^{2(m-n+1)} \Gamma^q_{m,n}(n\mu^q_{n,0} - \beta^{-1}\mu^q_{n,1})\right], \quad (A.52)$$

$$b^k_{m,1} = \chi_m b^k_{m-1,1} + \sum_{q=k-1}^{2m-1} \Gamma^q_{m,1}\mu^q_{1,k}, \quad 1 \leqslant k \leqslant 2m-2, \quad (A.53)$$

$$b^k_{m,1} = \sum_{q=k-1}^{2m-1} \Gamma^q_{m,1}\mu^q_{1,k}, \quad 2m-1 \leqslant k \leqslant 2m, \quad (A.54)$$

$$b^k_{m,n} = \chi_m b^k_{m-1,n} - \sum_{q=k}^{2(m-n+1)} \Gamma^q_{m,n}\mu^q_{n,k}, \quad 0 \leqslant k \leqslant 2(m-n), \quad 2 \leqslant n \leqslant m, \quad (A.55)$$

$$b^k_{m,n} = - \sum_{q=k}^{2(m-n+1)} \Gamma^q_{m,n}\mu^q_{n,k}, \quad 2(m-n)+1 \leqslant k \leqslant 2(m-n)+2, \quad 2 \leqslant n \leqslant m, \quad (A.56)$$

$$b^0_{m,m+1} = -\Gamma^0_{m,m+1}\mu^0_{m+1,0}, \quad (A.57)$$

where $m \geqslant 1$, $0 \leqslant n \leqslant m+1$ and $0 \leqslant k \leqslant 2(m-n+1)$.

(III) In (I), we point out that the initial approximation (4.6) has the same structure as Eq. (4.15). In (II), we not only deduce the recurrence formulas (A.50)–(A.57) but also rigorously prove that, if the first $(m-1)$ solutions $\varphi_k(\eta, \hbar, \beta)$ $(k = 0, 1, 2, 3, \ldots, m-1)$ have the structure (4.15), then the following mth solutions $\varphi_m(\eta, \hbar, \beta)$ $(m \geqslant 1)$ must have the same structure as Eq. (4.15), too. Therefore, owing to (I) and (II), all $\varphi_k(\eta, \hbar, \beta)$ $(k \geqslant 0)$ have the same structure as Eq. (4.15). Besides, using the foregoing recurrence formulas and only the known first three coefficients $b^0_{0,0} = -\beta^{-1}$, $b^1_{0,0} = 1$, $b^0_{0,1} = \beta^{-1}$, we can calculate all coefficients $b^k_{m,n}$ one after the other.

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