Applications of DFT to the Theory of Twentieth-Century Harmony

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Abstract. Music theorists have only recently, following groundbreaking work by Quinn, recognized the potential for the DFT on pcsets, initially proposed by Lewin, to serve as the foundation of a theory of harmony for the twentieth century. This paper investigates pcset "arithmetic" – subset structure, transpositional combination, and interval content – through the lens of the DFT. It discusses relationships between interval classes and DFT magnitudes, considers special properties of dyads, pcset products, and generated collections, and suggest methods of using the DFT in analysis, including interpreting DFT magnitudes, using phase spaces to understand subset structure, and interpreting the DFT of Lewin's interval function. Webern's op. 5/4 and Bartok's String Quartet 4, iv, are discussed.

Keywords: Discrete Fourier transform, pitch-class set theory, twentieth-century harmony, posttonal theory, Webern, Bartok, generated sets

1 Introduction

In American music theory of the 1960s and 1970s, the era of Allen Forte's ambitiously titled book *The Structure of Atonal Music* [12], a theory of harmony for the twentieth century seemed not only a possible but a natural goal of the discipline. In latter years, the idea of pursuing a general theory for such an eclectic century would come to seem increasingly audacious. But recent advances in mathematical music theory should reignite this enterprise: in particular the application of the Fourier transform to pcsets [3, 5, 9, 15, 18, 19].

Forte's project [11, 12] to develop a theory based on interval content and subset structure was propitious in that he identified general properities would be relevant to a wide range of music despite great disparities in compositional aesthetic and technique. The DFT makes it possible to establish a more solid mathematical foundation for such a theory than was available to Forte.

2 Preliminaries

Amiot and Sethares [5] define *scale vectors* as characteristic functions of pcsets:

Definition 1 *The* characteristic function *of a pcset is a vector with twelve places, one for each pc starting from C = 0, with a 1 indicating the presence of a pc and 0 indicating its absence.*

The characteristic function naturally generalizes to include pc-multisets (by allowing positive integers other than 1) and, more generally, *pc-distributions* (by allowing non-integers). I will refer to real-valued vectors corresponding to pc-distributions as *pc vectors*.

Pc-distributions are best identified with equivalence classes of pc vectors under addition of a constant. In other words, it is the differences between pc values that define a pc-distribution, not the values themselves. As we will see, these equivalence classes can be neatly described using the DFT. They also bypass the potential conundrum of assessing the meaning of negative-valued pc vectors: negative values can always be eliminated by addition of a constant.

The insight of Lewin [14, 15], Quinn [18], and others is that reparameterizing such characteristic functions by means of the DFT reveals a wealth of musically significant information:

Definition 2 *Let* $A = (a_0, a_1, a_2, \ldots, a_{11})$ *be the characteristic function of a pc-distribution. Let* $\hat{A} = (\hat{a}_0, \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_{11})$ *denote the* discrete Fourier transform (DFT) *of A. Then* $\hat{A}$ *is given by* $\forall(0 \leq k \leq 11)$,

$$\hat{a}_k = \sum_{j=0}^{11} a_j e^{-i2\pi kj/12} = \sum_{j=0}^{11} a_j(\cos(2\pi kj/12) + i\sin(2\pi kj/12)) \qquad (1)$$

The components of the Fourier transform, $\hat{A}$, as defined above, are complex numbers. They are most useful when viewed in polar form (magnitude and phase).

Definition 3 *Let* $\hat{a}_k = re^{i\theta}$*. Then the* magnitude *of* $\hat{a}_k$ *is* $r = \sqrt{Re(\hat{a}_k)^2 + Im(\hat{a}_k)^2}$ *and is denoted* $|\hat{a}_k|$*. The* phase *of* $\hat{a}_k$, φ_{a_k}, *is* $\arg(\hat{a}_k) = \theta = \arctan(Re(\hat{a}_k), Im(\hat{a}_k))$*. We will often normalize phases to a mod12 circle, denoted* ${}_{12}\varphi_{a_k} = 6\theta/\pi$*.*

Guerino Mazzola has pointed out (in informal response to [4]) that the DFT is one of many possible orthonormal bases for the space of pc-distributions. (See, e.g., [5].) Any of these would reflect the common-pc-content–based topology promoted by Yust [19] as a fundamental strength of this space. However, the DFT basis is of special music-theoretic value because it reflects evenness (i.e., periodicity) properties of fundamental musical importance. For instance, Amiot [3] and Yust [19] have shown that a space based on phases of the third and fifth components reflects many properties of tonal harmony by isolating evenness properties particular to triads and scales. Amiot [2] has also used the DFT to evaluate temperaments on the basis of the evenness of diatonic subsets. The analyses below consider the musical significance of other DFT components, especially the second and sixth.

Some basic properties of the DFT:

Remark 1. The components of the pc vector (pc magnitudes) are real valued. Therefore components 7–11 of the DFT have the same magnitude and opposite phase as their complementary components.

Remark 2. The zeroeth component of the DFT is always equal to the cardinality of a pcset or multiset.

Remark 3. Adding a constant to a pcset changes *only the zeroeth component* of the DFT. Therefore members of an equivalence class of pc vectors (as defined above) always have equivalent non-zero DFT components.

Remark 4. Negation preserves DFT magnitudes adds π to all well-defined phases. Adding a constant of 1 to all pcs of the negation produces the complement, meaning that these belong to an equivalence class, differing only in the zeroeth component of the DFT.

Remark 5. Transposition and inversion change the phases of DFT components but do not affect magnitudes.

3 Pcset arithmetic in Fourier coefficients

3.1 Sums of pcsets

A sum of pcsets is the componentwise sum of their characteristic functions. This differs from the set-theoretic concept of pcset *union* in that the latter eliminates doublings, whereas pcset sums preserve doublings by allowing for multisets.

Pcset sums also correspond to the componentwise sum of their DFTs. This is straightforward when the components of the DFT are expressed in real and imaginary parts, but less so in the more meaningful polar representation.

Proposition 1. *Let pcset B be a sum of pcsets $A, A', A'', \ldots$. Then for all $0 \leq k \leq 11$,*

$$\varphi_{b_k} = \arg(|\hat{a}_k| \cos(\varphi_{a_k}) + |\hat{a}'_k| \cos(\varphi_{a'_k}) + |\hat{a}''_k| \cos(\varphi_{a''_k}) + \ldots, \\ |\hat{a}_k| \sin(\varphi_{a_k}) + |\hat{a}'_k| \sin(\varphi_{a'_k}) + |\hat{a}''_k| \sin(\varphi_{a''_k}) + \ldots) \quad (2)$$

If φ_{b_k} is undefined, then $|\hat{b}_k| = 0$. Otherwise,

$$|\hat{b}_k| = |\hat{a}_k| \cos(\varphi_{b_k} - \varphi_{a_k}) + |\hat{a}'_k| \cos(\varphi_{b_k} - \varphi_{a'_k}) + |\hat{a}''_k| \cos(\varphi_{b_k} - \varphi_{a''_k}) + \ldots \quad (3)$$

Equation 2 is derived simply by converting to rectangular coordinates, summing, and converting back to polar. Equation 3 is most easily demonstrated geometrically, by projecting each summand, as a vector in the complex plane, onto the sum.

From (3) we see that the contribution of each pcset to the sum is determined by its magnitude and its difference in phase from the sum. It maximally reinforces the sum when its phase is the same, contributes nothing when its phase is oblique (a difference of $\pi/2$ or 3 mod 12) and maximally reduces the sum when its phase is opposite (π or 6 mod 12). The contribution of each pcset to the phase of the sum is also weighted by magnitude, as (2) shows. Two pcsets with equal magnitude and opposite phases cancel one another out in the sum.

3.2 Product of pcsets

"Multiplication" of pcsets was first defined by Pierre Boulez [7, 13] in reference to his own compositional technique. Cohn [10] demonstrates the applicability of the operation, which he calls "transpositional combination," in analysis of twentieth-century music. Mathematically, Boulez and Cohn's operation is a variant of *convolution*. For pc vectors A and B, the convolution $C = A * B$ is given by:

$$c_k = \sum_{j=0}^{11} a_j b_{(k-j) \bmod 12} \tag{4}$$

The difference between convolution and Boulez's multiplication or Cohn's transpositional combination is that it allows for pc-multisets, whereas Boulez and Cohn take the additional step of eliminating doublings (replacing all positive integers in the pc vector with 1s).

Boulez's term is fortuitous for present purposes, because according to one of the basic Fourier theorems, convolution of pc vectors corresponds to the termwise product of their DFTs.

$$\hat{c}_k = \hat{a}_k \hat{b}_k = |\hat{a}_k| e^{i\varphi_{a_k}} |\hat{b}_k| e^{i\varphi_{b_k}} = |\hat{a}_k||\hat{b}_k| e^{i(\varphi_{a_k} + \varphi_{b_k})} \tag{5}$$

As this shows, convolution is particularly straightforward when viewed from the polar form of the DFT: it corresponds to simply multiplying the magnitudes and adding the phases of each component. It is therefore appropriate to refer to the convolution as a *product of pcsets*.

Lewin [15] noted that the convolution of one pcset with the inverse of another (or the *cross-correlation*) gives his *interval function*, a vector that lists the number of occurences of each pc interval from the first pcset to the second. The interval function of a pcset to itself gives Forte's *interval vector* (as components 1–6 of the twelve-place interval function). The DFT of the interval vector is purely real-valued (all well-defined phases are zero), as can be seen from (5) and the fact that inversion (about 0) negates the phases and does not affect magnitudes (see Remark 5):

$$\hat{a}_k \left(\hat{Ia}\right)_k = |\hat{a}_k|^2 e^{i(\varphi_{a_k} - \varphi_{a_k})} = |\hat{a}_k|^2 \tag{6}$$

Singularities, zero-magnitude DFT components [5], are of special importance for pcset products in particular, because a singularity in one multiplicand leads to a singularity in the product. Note that phases are undefined when there is a singularity on a given component.

4 Fourier Components and Intervallic Content

4.1 Relating Fourier Components to Interval Classes

A motivating factor behind Forte's [11, 12] focus on interval vectors is their invariance with respect to transposition and inversion (implied by (6) and Remark 5). An advantage of the DFT is that while distilling essentially the same intervallic information as the interval vector in the magnitudes of its components, it also preserves essential information in their phases. These are important, for instance, in understanding subset structure, as equation 3 shows.

Quinn [18] has emphasized the association of individual Fourier components with specific interval classes (ic1 $\leftrightarrow \hat{a}_1$, ic2 $\leftrightarrow \hat{a}_6$, ic3 $\leftrightarrow \hat{a}_4$, ic4 $\leftrightarrow \hat{a}_3$, ic5 $\leftrightarrow \hat{a}_5$, ic6 $\leftrightarrow \hat{a}_2$). The primary grounds for such associations are that Quinn's *generic prototypes* (set classes maximal with respect to a given component, often generated sets – see Section 5) have maximal representation of the associated interval class. The associations can be misleading in other respects, however.

For example, let $A = \{\text{C, F, F}\sharp\}$ and let $B = \{\text{C, D, E}\}$. Although A contains an instance of ic5 and B does not, $|\hat{a}_5|^2 = 1$, while $|\hat{b}_5|^2 = 4$. Component 5 does not indicate the"fifthy-ness" of a pcset so much as its diatonicity, and B is a more characteristic diatonic subset than A. Or, for another example, consider the set $A = \{\text{C, D, E, F}\sharp\}$. It has a relatively large number of ic4s for a tetrachord, but $|\hat{a}_3|^2 = 0$, because the two ic4s cancel one another out ($_{12}\varphi_{\{C,E\}_3} = 0$ while $_{12}\varphi_{\{D,F\sharp\}_3} = 6$). (See also the discussion in [9].)

The relationship of interval classes to Fourier components is best summarized by their own DFTs, as shown in Table 1.

Table 1. Squared DFT magnitudes for all twelve-tone interval classes

	0	**1**	**2**	**3**	**4**	**5**	**6**	7	8	9	10	11
ic1	4	**3.73**	**3**	**2**	**1**	**0.27**	**0**	0.27	1	2	3	3.73
ic2	4	**3**	**1**	**0**	**1**	**3**	**4**	3	1	0	1	3
ic3	4	**2**	**0**	**2**	**4**	**2**	**0**	2	4	2	4	2
ic4	4	**1**	**1**	**4**	**1**	**1**	**4**	1	1	4	1	1
ic5	4	**0.27**	**3**	**2**	**1**	**3.73**	**0**	3.73	1	2	3	0.27
ic6	4	**0**	**4**	**0**	**4**	**0**	**4**	0	4	0	4	0

This information can be summarized by defining *delta values* as minimal phase distances between the two pcs in the dyad:

Definition 1. *Let h be an interval in a u-ET universe. The* delta value *of h for each component k is the shortest* mod u *distance represented by $h \cdot k$, $\delta = |((hk + u/2) \bmod u) - u/2|$.*

For 12-tET, $\delta = |((hk+6) \bmod 12) - 6|$, ranging from 0 to 6, and the squared DFT magnitude for any ic/component pair is $|\hat{a}_k| = 4cos^2(\delta\pi/12)$. (See also Section 5 below.)

Components are not neatly associated one-to-one with interval classes. Each has a maximum value, but the maximum could correspond to $\delta = 0$ (for components 2, 3, 4, and 6) or $\delta = 1$ (for 1 and 5). In the former situation a component might have maximum value for more than one ic, as is the case for components 4 and 6. Also, the maximum values do not tell the full story: at least as important are the *singularities* of each interval class, where $\delta = 6$ ($= u/2$).

4.2 Webern, op. 5, no. 4

Forte [11], in his classic analysis of Webern's *Satz für Streichquartett*, op. 5 no. 4, uses interval vectors and abstract subset structure to demonstrate how the piece is sectionalized by harmonic content. A similar conclusion can be reached using the DFT. Figure 1 shows mm. 1–10, the first two sections of the ternary form, and labels some significant pcsets. Table 2 lists the squared magnitudes of the DFT components for each of these. From this we can make the following generalizations: Universe A (sets A, A', A'' and combinations involving them) is characterized by a high component 2 and low odd components, Universe B (sets D, E, F) by a high component 3 and low component 2. Intermediate between these are sets B and C, which have a high component 2 and moderate presence of 3. As the values for dyads show (Table 1), the interval classes that Forte associates with universes A and B (ic6 and ic4 respectively) manifest these properties only in part: component 2 is one of three maximum values for ic6, and one of four relatively low values in ic4. The reason for this is evident from the fact that many of these are products of dyads: $A =$ ic1×ic6 or ic5×ic6, $A + A' =$ ic1×ic1×ic6 or ic5×ic5×ic6 or ic1×ic5×ic6, $A \cup A' \cup A'' =$ ic1×ic2×ic6 or ic5×ic2×ic6, $E =$ic3×ic4. Interval classes 1 and 5 have higher values of component 2 than 4 or 6, so they contribute this feature when multiplied by ic6. Similarly, ic3 has a singularity on component 2, making $E =$ ic3×ic4 a particularly appropriate antipode to A. Other sets from Universe B are more similar to ic4: D is generated by ic4 (see section 5), and $D + E$ and $E + F$ are, like ic4, weighted towards component 6 as well as 3. The accompaniment by itself, F, gives the weakest contrast from the harmony previous section. B and C are also factorable: $B =$ ic1×ic5 and $C =$ (012)×ic5.

Table 2. Squared DFT magnitudes for pcsets from Webern's Op. 5 No. 4

	1	**2**	**3**	**4**	**5**	**6**		**1**	**2**	**3**	**4**	**5**	**6**
A	0	**12**	0	4	0	0	D	0	0	**9**	0	0	**9**
$A \cup A'$	0	**16**	0	0	0	4	E	2	0	**8**	4	2	0
$A \cup A' \cup A''$	0	**12**	0	4	0	0	$D + E$	2	0	**5**	4	2	**9**
$A + A'$	0	**36**	0	4	0	0	$E + F$	3	3	**9**	3	3	**9**
$A + A' + A''$	0	**48**	0	0	0	0	F	0.27	3	**5**	1	3.73	**9**
$A \cup B$	1	**13**	1	1	1	1							
B	1	**9**	**4**	1	1	0	V	0.27	**7**	2	1	3.73	4
C	2	**12**	2	0	2	0	*Flyaway*	1	**7**	1	**7**	1	1

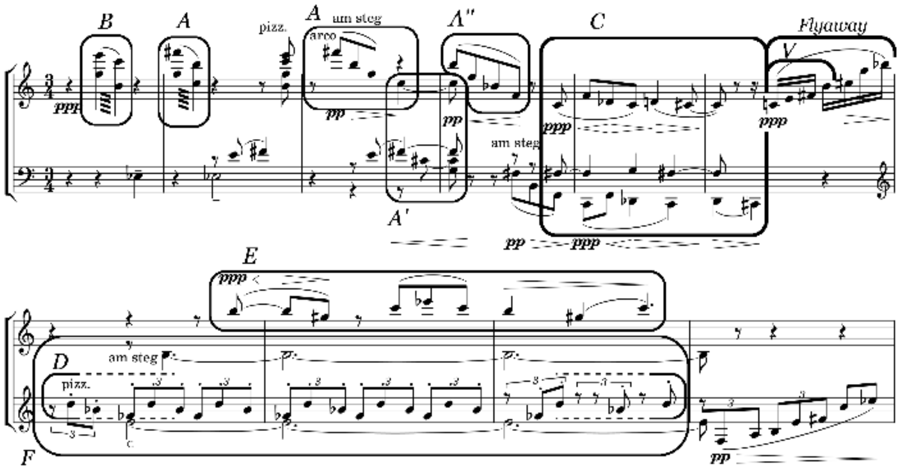

Fig. 1. Webern op. 5 no. 4, mm. 1–10: Some significant pcsets

Other authors (Perle [17] and Burkhart [8]) see more continuity in the piece by emphasizing V, which is a (literal) subset of $A \cup B$, $A \cup A' \cup A''$, and, most explicitly, *Flyaway* (borrowing Lewin's [16] nickname for this motive). Perle observes that the same set class is a subset of F ({EG♭B♭B}). As Table 2 shows, V and *Flyaway* are intermediate between the harmonic universes, like B and C. And the presence of component 2 in F suggests a link with Universe A.

We can clarify subset/superset relationships by using the kind of phase spaces proposed by Amiot [3] and Yust [19]. These authors construct toroidal spaces using phases of Fourier components 3 and 5 as axes to make *Tonnetz*-like maps for tonal harmony. For Webern's piece, a space based on phases of components 2 and 3, as seen in Figure 2, is appropriate. Yust [19] demonstrates how the third component represents the triadic aspect of tonality; Webern's Universe B might accordingly be heard as a reference to tonal harmony, made hazy by a lack of diatonicity. The second DFT component does not feature in Amiot and Yust's treatment of tonal harmony, but its use can be identified with the "quartal" sonorities emblematic of early twentieth-century modernism – i.e., the second component comes into play specifically in a harmonic palette that avoids thirds and sixths (ics 3 and 4), as is evident from Table 1.

According to (3), the more spread out pcsets are in phase, the weaker their sums are on a given component. The pcsets from the first section of the piece are concentrated in a small zone of φ_2, but spread out in φ_3. Pcsets connected by lines in Figure 2 have equal magnitude but opposite phase on φ_3, so their sums have component 3 singularities, except $F + E$ and $-F$ (the negation of F) which are opposite on φ_2. The position of F is opposite that of $-F$ (a difference of 6 in all dimensions, see Remark 4), so, like $F + E$, the phase of its second component is within the region defined by the pcsets of Universe A.

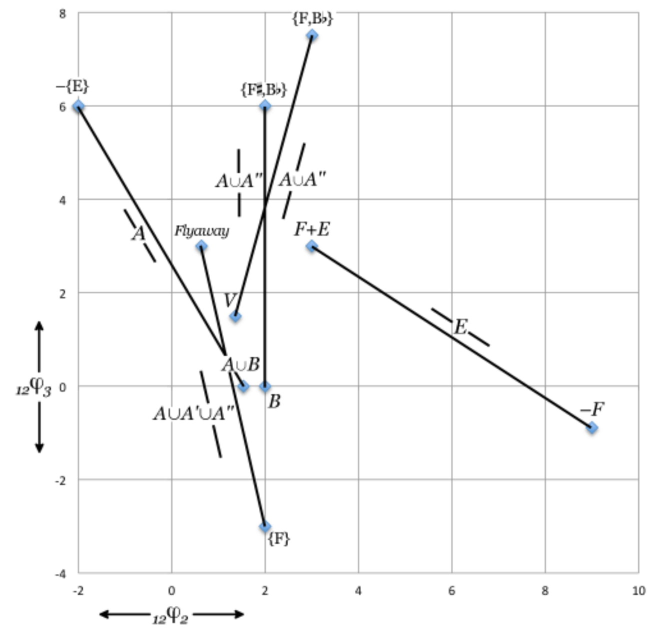

Fig. 2. Pcset sums from Webern op. 5 no. 4 shown in phase space

5 DFT of Generated Collections

As noted above, Quinn [18] and Amiot [1] place special emphasis on generated collections (and maximally even sets in particular) as pcsets most representative of a particular interval. The following formulas simplify the calculation of the DFT of generated collections and provide some insight into their properties.

Proposition 1 *Let A be a pcset of cardinality n generated by interval g/u, where u is the cardinality of the ET universe. Then,*

$$\varphi_{a_k} = -(n-1)gk\pi/u \tag{7}$$

$$|\hat{a}_k| = \begin{cases} n & \textit{if } gk = 0 \bmod 12, \\ \dfrac{\sin(ngk\pi/u)}{\sin(gk\pi/u)} & \textit{otherwise} \end{cases} \tag{8}$$

Proof. The first case of (8) is evident from the fact that when $gk = 0 \bmod 12$, the unit vectors all have the same phase, 0.

For $gk \neq 0$ the Fourier series (1) can be written out in the order of generation and simplified as a geometric series:

$$\hat{a}_k = \sum_{j=0}^{n-1} e^{-i2\pi kgj/u} = \frac{1 - e^{-i2ngk\pi/u}}{1 - e^{-i2gk\pi/u}}$$
$$= (\frac{e^{ingk\pi/u} - e^{-ingk\pi/u}}{e^{igk\pi/u} - e^{-igk\pi/u}})e^{-(n-1)igk\pi/u} = \frac{\sin(ngk\pi/u)}{\sin(gk\pi/u)}e^{-(n-1)igk\pi/u} \quad (9)$$

The second step factors out the phase of the component, and the final step applies Euler's formula. The resulting magnitude function is a Dirichlet kernel.[1]

This result complements those of Amiot [1] and generalize his formulas for maximum values, which represent the special cases $\delta = 0$ and $\delta = 1$, giving a more general picture of the special status of generated collections viewed through the DFT. Equation 8 can be viewed as a function of n, so that the denominator, $sin^{-1}(gk\pi/u)$, is a constant, indicating the maximum value of the given component for the given generator. Note that (8) gives the same result for $-gk$ as for gk, so δ (Def. 1) can substitute for gk, making the maximum value $sin^{-1}(\delta\pi/u)$. $|\hat{a}_k|$ is a sinusoidal function of n with period u/δ, minimum value (0) at $n = 0 \bmod u/\delta$, and maximum at $n = \frac{1}{2}u/\delta \bmod u/\delta$. Amiot and Quinn focus on the cases $\delta = 0$, where $|\hat{a}_k|$ is unbounded, and $\delta = 1$, which maximizes $sin^{-1}(\delta\pi/u)$ and gives a period of u (for $|\hat{a}_k|$ as a function of n), and hence a unique maximum. However, this is one extreme of a range of possibilities, the other being $\delta = \frac{1}{2}u$, which minimizes $sin^{-1}(\delta\pi/u) = 1$ and gives a period of 2. (In other words, this component alternates between magnitudes 0 for n even and 1 for n odd). For D in the Webern analysis above (the augmented triad, $g = 4$), $\delta = 4$ for components 1, 2, 4, and 5, reaching a minimum at $n = 3$, while $\delta = 0$ for components 3 and 6.

As another example, compare B and C from the Webern analysis, which both involve the product of an ic1-generated collection with ic5. From $n = 2$ to $n = 3$, components with large δ values and short periods (3 and 4) decrease, while component 5, with a long period, increases incrementally. The result in C intensifies the strength of component 2 relative to 3 and 4.

6 Example: Bartók, "Allegro Pizzicato" from String Quartet no. 4 (iv)

The example from Webern demonstrated how contrasting harmonic profiles can operate as a means of formal delineation. In the pizzicato fourth movement of Bartók's Fourth String Quartet, we find similar contrasts being used for *stratification* of harmonic materials as well as formal delineation. The first section of the piece consists of fugal entries of a scalewise theme accompanied by ostinato-like patterns in the other instruments. Figure 3 shows the first entry and its accompaniment. The melody is written in the acoustic scale on A♭ (a collection favored by Bartók; see [6]), while the accompanimental collection is {DE♭GA♭}.

[1] I am indebted to Emmanuel Amiot for pointing this out and helping me improve upon a previous less elegant proof.

Fig. 3. The subject of Bartók's "Allegro Pizzicato"

Table 3 shows the DFT magnitudes for these two collections. Remarkably, the largest component of the accompaniment ($\hat{a}_2$) is a singularity for the acoustic scale, while the largest component for the acoustic scale ($\hat{a}_6$) is a singularity for the accompaniment. The acoustic scale also has a relatively high value on component 5 while {DE♭GA♭} has a relatively low value (contrary to what subset relations suggest – the (0156) tetrachord is a subset of the diatonic, but contains precisely the most marginal members of the diatonic on the circle of fifths). Note also that this accompanimental collection is the same set class as B from the Webern analysis above, and can be expressed as a product of dyads, ic1×ic5.

Table 3. Squared DFT magnitudes of pcsets in Bartók's "Allegro Pizzicato"

	Accompaniment						**Melody**					
	1	**2**	**3**	**4**	**5**	**6**	**1**	**2**	**3**	**4**	**5**	**6**
Mm. 6–12	1	**9**	4	1	1	0	**Acoustic scale:**					
Mm. 13–19	1	**13**	1	1	1	1	0.54	0	1	4	**7.46**	**9**
Mm. 20–27	2.27	3	1	1	**5.73**	**9**	**Whole-tone pentachord**:					
							1	1	1	1	1	**25**

As previously noted, high component 2 values typify the sonorous landscape of modernism, and its role here may reflect upon the Fourth Quartet's reputation for reflecting a turn towards a modernist aesthetic. The second accompanimental collection intensifies the focus on component 2, while the third shifts towards a closer match to the harmonic properties of the acoustic scale. This shift occurs precisely at the point where contrapuntal writing begins.

Taken by itself, the melodic subject realizes a harmonic motion from the acoustic scale (dominated by components 5 and 6) to its five-note whole-tone subset, where the presence of component 5 (diatonicity) is overtaken by 6 (whole-tone). (See Table 3.)

Bartók's stratification of hamonic materials is perhaps best viewed through the lens of Lewin's interval function [15, 16], which is a pcset product (see Section 3.2). The DFT magnitudes of this product are transposition-independent, just as they are for pcsets themselves, so phase is significant in determining the specific intervals between collections. Note that DFTs of interval functions in Table 4 shows are a product of the magnitudes and a difference of phases (as implied by (5)). In mm. 6–12, the singularities on components 2 and 6 annhilate these components in the product, leaving component 5 to predominate. This means that the most prevalent intervals tend to be fifth-related, which can be seen in the resulting interval function below. However, depending on the *phase* of component 5, these could be intervals of circle-of-fifths proximity (0, 5, 7, 10, 2, ...) or of circle-of-fifths remoteness (6, 1, 11, ...). The phase difference of component 5 ($\delta = 1$) is small but not minimal, shifting the interval function towards slightly more remote intervals. This small difference is most directly manifest in the mild polyscalar dissonance of the accompanimental G against the melodic G♭. In mm. 20–27 Bartók moves the accompanimental collection into near-perfect alignment with the melody, which is evident in the balance of the interval function around interval zero. The singularity in component 6 is also eliminated, leading to a strong imbalance in even versus odd intervals.

Table 4. Interval functions between melody and accompaniment

Component	**1**		**2**		**3**		**4**		**5**		**6**	
	mag^2	12φ	mag^2	12φ	mag^2	12φ	mag^2	12φ	mag^2	12φ	mag^2	12φ
Mm. 6–12, melody	0.54	8	0	–	1	6	4	2	**7.46**	**10**	9	0
accompaniment	1	7	9	8	4	3	1	4	**1**	**11**	0	–
Interval function	0.54	1	0	–	4	3	4	10	**7.46**	**11**	0	–
Mm. 20–27, melody	0.54	6	0	–	1	12	4	6	**7.46**	**0**	**9**	**0**
accompaniment	2.27	2.2	3	9	1	3	1	6	**5.73**	**11.7**	**9**	**0**
Interval function	1.22	3.83	0	–	1	9	4	0	**42.8**	**0.3**	**81**	**0**

Interval Functions

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
Mm. 6–12	**3**	2	2	**3**	2	2	2	**3**	2	2	**3**	2
Mm. 20–27	**5**	1	**4**	2	3	3	3	3	3	3	**4**	1

Bartók also, like Webern, uses harmonic contrasts for formal delineations in a three-part design. Example 4 shows two multiplications that make up the principal accompanimental and melodic material of the section. The first is the product of an ic2-generated trichord and ic1. The trichord represents the whole-tone saturated harmonic universe of the first part, but, as Table 5 shows, the multiplication ironically annihilates its sixth component altogether. A similar point can be made about the melodic construction, the product of an ic1-generated trichord and ic2. Both combinations result in intensely chromatic pcsets, reflected in the dominance of component 1.

References

1. Amiot, E. David Lewin and Maximally Even Sets. J. Math. Mus. 1, 157–172 (2007)
2. Amiot, E. Discrete Fourier Transform and Bach's Good Temperament. Mus. Theory Online 15 (2009)

Fig. 4. Transpositional combination in the middle section, mm. 47–49

Table 5. Transpositional combinations in the middle section

	1	**2**	**3**	**4**	**5**	**6**		**1**	**2**	**3**	**4**	**5**	**6**
{ABC♯}×	**4**	0	1	0	4	**9**	{EFF♯}×	**7.46**	4	1	0	0.54	**1**
ic1 =	**3.73**	3	2	1	0.27	**0**	ic2 =	**3**	1	0	1	3	**4**
{AB♭BCC♯D}	**14.9**	0	2	0	1.07	**0**	{EFF♯^{2}GA♭}	**22.4**	4	0	0	1.61	**4**

3. Amiot, E. The Torii of Phases. In: Yust, J., Wild, J., Burgoyne, J. A. (eds.) MCM 2013. LNAI, vol. 7937, pp. 1–18. Springer, Heidelberg (2013)
4. Amiot, E. Viewing Diverse Musical Features in Fourier Space: A Survey. Paper presented to the International Congress on Music and Mathematics, Puerto Vallarta, Nov. 28, 2014.
5. Amiot, E., Sethares, W.: An Algebra for Periodic Rhythms and Scales. J. Math. Mus. 5, 149–169 (2011)
6. Antokoletz, E.: Transformations of a Special Non-Diatonic Mode in Twentieth-Century Music: Bartók, Stravinsky, Scriabin and Albrecht. Mus. Analysis 12, 25–45 (1993)
7. Boulez, P.: Boulez on Music Today, trans. S. Bradshaw and R.R. Bennett. Harvard University Press, Cambridge, Mass. (1971).
8. Burkhart C.: The Symmetrical Source of Webern's Opus 5, No. 4. In:.Salzer, F. (ed.) Music Forum V, pp. 317–34. Columbia University Press, New York (1980)
9. Callender, C.: Continuous Harmonic Spaces. J. Mus. Theory 51, 277–332 (2007)
10. Cohn, R.: Transpositional Combination and Inversional Symmetry in Bartók. Mus. Theory Spectrum 10: 19–42 (1988)
11. Forte, A.: A Theory of Set-Complexes for Music. J. Mus. Theory 8, 136–183 (1964)
12. Forte, A.: The Structure of Atonal Music. Yale University Press, New Haven (1973)
13. Koblyakov, L.: Pierre Boulez: A World of Harmony. Harwood, New Haven (1990)
14. Lewin, D.: Re: Intervallic Relations between Two Collections of Notes. J. Mus. Theory 3, 298–301 (1959)
15. Lewin, D.: Special Cases of the Interval Function between Pitch-Class Sets X and Y. J. Mus. Theory 45, 1–29 (2001)
16. Lewin, D.: Generalized Musical Intervals and Transformations, 2d. Ed. Yale University Press, New Haven (2007)
17. Perle, G.: Serial Composition and Atonality: An Introduction to the Music of Schoenberg, Berg, and Webern, 6th Ed. UC Press, Berkeley, Calif. (1991)
18. Quinn, I.: General Equal-Tempered Harmony (in two parts). Perspectives of New Mus. 44, 114–159 and 45, 4–63 (2006)
19. Yust, J.: Schubert's Harmonic Language and Fourier Phase Space. J. Mus. Theory 59, 121–181.