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# Characterizations of slant helices in Euclidean 3-space

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**Abstract**

In this paper we investigate the relations between a general helix and a slant helix. Moreover, we obtain some differential equations which they are characterizations for a space curve to be a slant helix. Also, we obtain the slant helix equations and its Frenet aparatus.

**Key Words:** Slant helix, genaral helix, spherical helix, tangent indicatrix, principal normal indicatrix and binormal indicatrix.

## 1. Introduction

In differential geometry, a curve of constant slope or general helix in Euclidean 3-space $\mathbb{R}^3$ is defined by the property that the tangent makes a constant angle with a fixed straight line (the axis of the general helix). A classical result stated by *M. A. Lancret* in 1802 and first proved by *B. de Saint Venant* in 1845 (see [11, 13] for details) is: A necessary and sufficient condition that a curve be a general helix is that the ratio of curvature to torsion be constant. If both of $\varkappa$ and $\tau$ are non-zero constant it is, of course, a general helix. We call it a circular helix. Its known that straight line and circle are degenerate-helix examples ($\varkappa = 0$, if the curve is straight line and $\tau = 0$, if the curve is a circle).

The study of these curves in $\mathbb{R}^3$ as spherical curves is given by Monterde in [12]. The Lancret theorem was revisited and solved by Barros (in [2]) in 3-dimensional real space forms by using killing vector fields as along curves. Also in the same space-forms, a characterization of helices and Cornu spirals is given by Arroyo, Barros and Garay in [1].

On the studies of general helices in Lorentzian space forms, Lorentz-Minkowski spaces, semi-Riemannian manifolds, we refer to the papers [3, 4, 5, 6, 7, 9].

In [8], A slant helix in Euclidean space $\mathbb{R}^3$ was defined by the property that the principal normal makes a constant angle with a fixed direction. Moreover, Izumiya and Takeuchi showed that $\gamma$ is a slant helix in $\mathbb{R}^3$ if and only if the geodesic curvature of the principal normal of a space curve $\gamma$ is a constant function.

In [10], Kula and Yayli have studied spherical images of tangent indicatrix and binormal indicatrix of a slant helix and they showed that the spherical images are spherical helix.

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In this paper we consider the relationship between the curves slant helices and general helices in $\mathbb{R}^3$. We obtain the differential equations which are characterizations of a slant helix. Also, we give some slant helix examples in Euclidean 3-space

## 2. Preliminaries

We now recall some basic concepts on classical differential geometry of space curves and the definitions of general helix, slant helix in Euclidean 3-space. A curve $\gamma : I \subset R \to \mathbb{R}^3$, with unit speed, is a general helix if there is some constant vector $u$, so that $t.u = \cos\theta$ is constant along the curve, where $t(s) = \gamma^{'}(s)$ is a unit tangent vector of $\gamma$ at $s$. We define the curvature of $\gamma$ by $\varkappa(s) = \left\| \gamma^{''}(s) \right\|$. If $\varkappa(s) \neq 0$, then the unit principal normal vector $n(s)$ of the curve $\gamma$ at $s$ is given by $\gamma^{''}(s) = \varkappa(s)\, n(s)$. The unit vector $b(s) = t(s) \times n(s)$ is called the unit binormal vector of $\gamma$ at $s$. For the derivatives of the Frenet frame, the Frenet-Serret formulae hold:

$$\begin{aligned} t^{'}(s) &= \qquad \varkappa(s)\, n(s) \\ n^{'}(s) &= -\varkappa(s)\, t(s) + \tau(s)\, b(s) \\ b^{'}(s) &= \qquad -\tau(s)\, n(s), \end{aligned} \tag{2.1}$$

where $\tau(s)$ is the torsion of the curve $\gamma$ at $s$. It his known that curve $\gamma$ is a general helix if and only if $\left(\frac{\tau}{\varkappa}\right)(s)$ =constant. If both of $\varkappa(s) \neq 0$ and $\tau(s)$ are constant, we call as a circular helix.

**Definition 2.1.** *Let $\alpha$ be a unit speed regular curve in Euclidean 3-space with Frenet vectors $t$ , $n$ and $b$. The unit tangent vectors along the curve $\alpha$ generate a curve $(t)$ on the sphere of radius 1 about the origin. The curve $(t)$ is called the spherical indicatrix of $t$ or more commonly, $(t)$ is called tangent indicatrix of the curve $\alpha$. If $\alpha = \alpha(s)$ is a natural representation of $\alpha$, then $(t) = t(s)$ will be a representation of $(t)$. Similarly one considers the principal normal indicatrix $(n) = n(s)$ and binormal indicatrix $(b) = b(s)$ [13].*

**Definition 2.2.** *A curve $\gamma$ with $\varkappa(s) \neq 0$ is a slant helix if and only if the geodesic curvature of the spherical image of the principal normal indicatrix $(n)$ of $\gamma$*

$$\sigma_n(s) = \left( \frac{\varkappa^2}{(\varkappa^2 + \tau^2)^{3/2}} \left(\frac{\tau}{\varkappa}\right)^{'} \right)(s) \tag{2.2}$$

*is a constant function* [8].

*In this paper, by $D$ we denote the covariant differentiation of $\mathbb{R}^3$.*

**Remark 2.1** If the Frenet frame of the tangent indicatrix $(t)$ of a space curve $\gamma$ is $\{\mathcal{T}, \mathcal{N}, \mathcal{B}\}$, then we have the Frenet-Serret formulae:

$$\begin{aligned} D_{\mathcal{T}}\mathcal{T} &= \quad \varkappa_t \mathcal{N} \\ D_{\mathcal{T}}\mathcal{N} &= -\varkappa_t \mathcal{T} + \tau_t \mathcal{B} \\ D_{\mathcal{T}}\mathcal{B} &= \quad -\tau_t \mathcal{N}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \mathcal{T} &= n \\ \mathcal{N} &= \frac{1}{\sqrt{\varkappa^2+\tau^2}}\left(-\varkappa t + \tau b\right) \\ \mathcal{B} &= \frac{1}{\sqrt{\varkappa^2+\tau^2}}\left(\tau t + \varkappa b\right) \end{aligned} \tag{2.4}$$

and $\varkappa_t = \frac{\sqrt{\varkappa^2+\tau^2}}{\varkappa}$ is the curvature of $(t)$, $\tau_t = \frac{\varkappa\tau' - \varkappa'\tau}{\varkappa(\varkappa^2+\tau^2)}$ is the torsion of $(t)$.

**Remark 2.2.** If the Frenet frame of the principal normal indicatrix $(n)$ of a space curve $\gamma$ is $\{\mathsf{T}, \mathsf{N}, \mathsf{B}\}$, then we have the Frenet-Serret formulae:

$$\begin{aligned} D_{\mathsf{T}}\mathsf{T} &= \quad \varkappa_n \mathsf{N} \\ D_{\mathsf{T}}\mathsf{N} &= -\varkappa_n \mathsf{T} + \tau_n \mathsf{B} \\ D_{\mathsf{T}}\mathsf{B} &= \quad -\tau_n \mathsf{N}, \end{aligned} \tag{2.5}$$

where

$$\begin{aligned} \mathsf{T} &= \frac{1}{\sqrt{\varkappa^2+\tau^2}}(-\varkappa t + \tau b) \\ \mathsf{N} &= \frac{1}{\sqrt{(\varkappa^2+\tau^2)\left(\varkappa\tau' - \varkappa'\tau\right)^2 + (\varkappa^2+\tau^2)^4}}[(\varkappa\tau' - \varkappa'\tau)(\tau t + \varkappa b) - (\varkappa^2+\tau^2)^2 n] \\ \mathsf{B} &= \frac{1}{\sqrt{\left(\varkappa\tau' - \varkappa'\tau\right)^2 + (\varkappa^2+\tau^2)^3}}[(\varkappa^2+\tau^2)(\tau t + \varkappa b) + (\varkappa\tau' - \varkappa'\tau)n], \end{aligned} \tag{2.6}$$

the curvature of $(n)$ is

$$\varkappa_n = \frac{\sqrt{\left(\varkappa^2+\tau^2\right)^3 + \left(\varkappa\tau' - \varkappa'\tau\right)^2}}{\left(\varkappa^2+\tau^2\right)^{3/2}},$$

and the torsion of $(n)$ is

$$\tau_n = \frac{\left[\left(\varkappa\tau'' - \varkappa''\tau\right)\left(\varkappa^2+\tau^2\right) - 3\left(\varkappa\tau' - \varkappa'\tau\right)\left(\varkappa\varkappa' - \tau'\tau\right)\right]}{\left(\varkappa^2+\tau^2\right)^3 + \left(\varkappa\tau' - \varkappa'\tau\right)^2}.$$

**Remark 2.3.** If the Frenet frame of the binormal indicatrix $(b)$ of a space curve $\gamma$ is $\{\mathbb{T}, \mathbb{N}, \mathbb{B}\}$, then we have the Frenet-Serret formulae:

$$\begin{aligned} D_{\mathbb{T}}\mathbb{T} &= \quad \varkappa_b \mathbb{N} \\ D_{\mathbb{T}}\mathbb{N} &= -\varkappa_b \mathbb{T} + \tau_b \mathbb{B} \\ D_{\mathbb{T}}\mathbb{B} &= \quad -\tau_b \mathbb{N}, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} \mathbb{T} &= -n \\ \mathbb{N} &= \tfrac{1}{\sqrt{\varkappa^2+\tau^2}}\left(\varkappa t - \tau b\right) \\ \mathbb{B} &= \tfrac{1}{\sqrt{\varkappa^2+\tau^2}}\left(\tau t + \varkappa b\right) \end{aligned} \tag{2.8}$$

and $\varkappa_b = \frac{\sqrt{\varkappa^2+\tau^2}}{\tau}$ is the curvature of $(b)$, $\tau_b = \frac{-\left(\varkappa\tau' - \varkappa'\tau\right)}{\tau(\varkappa^2+\tau^2)}$ is the torsion of $(b)$.

## 3. Characterizations of slant helices

In this section, we give some characterizations for a unıt speed curve $\gamma$ in $\mathbb{R}^3$ to be a slant helix by using its *tangent indicatrix* $(t)$, *principal normal indicatrix* $(n)$ and *binormal indicatrix* $(b)$, respectively.

**Theorem 3.1.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$. $\gamma$ is a slant helix if and only if the principal normal vector field $\mathsf{N}$ of the principal normal indicatrix $(n)$ satisfies the equation*

$$D_{\mathsf{T}}^2\mathsf{N}+\varkappa_n^2\mathsf{N}= 0, \tag{3.1}$$

*where $\varkappa_n$ is curvature of the principal normal indicatrix $(n)$ of the curve $\gamma$.*

**Proof.** Suppose that $\gamma$ is a slant helix. From remark 2.2. the curvature of $(n)$ is

$$\varkappa_n = \sqrt{1+\sigma_n^2(s)} \tag{3.2}$$

and the torsion of $(n)$ is

$$\tau_n = \frac{\left(\varkappa^2+\tau^2\right)^{5/2}}{\left(\varkappa\tau' - \varkappa'\tau\right)^2 + \left(\varkappa^2+\tau^2\right)^3}\sigma_N'(s). \tag{3.3}$$

Since, $\sigma_n(s)$ is a constant function, we get

$$\varkappa_n = \text{non-zero constant, and } \ \tau_n = 0.$$

Hence the principal normal indicatrix of $\gamma$ is a circle. From frame equations (2.5), we obtain that

$$D_{\mathsf{T}}^2\mathsf{N} + \varkappa_n^2\mathsf{N} = 0.$$

Conversely, let us assume that (3.1) holds. We show that the curve $\gamma$ is a slant helix. From frame equations (2.5)

$$D_{\mathsf{T}}^2\mathsf{N} + \varkappa_n^2\mathsf{N} = -\varkappa_n'\mathsf{T} - \tau_n^2\mathsf{N} + \tau_n'\mathsf{B} = 0. \tag{3.4}$$

Then we see that

$$\varkappa_n \text{ is a constant and } \tau_n = 0,$$

which means that $\gamma$ is a slant helix. □

In the next six theorems, we obtain the differential equations of a slant helix according to the *tangent vector field* $\mathcal{T}$, *principal normal vector field* $\mathcal{N}$ and *binormal vector field* $\mathcal{B}$ *of the principal normal indicatrix* $(t)$ of the curve.

**Theorem 3.2.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$ The curve $\gamma$ is a slant helix if and only if the tangent vector field $\mathcal{T}$ of the tangent indicatrix $(t)$ of the curve $\gamma$ satisfies the following equation:*

$$D_{\mathcal{T}}^3\mathcal{T} - 3\frac{\varkappa_t'}{\varkappa_t}D_{\mathcal{T}}^2\mathcal{T} - \left\{\frac{\varkappa_t''}{\varkappa_t} - 3\left(\frac{\varkappa_t'}{\varkappa_t}\right)^2 - \lambda_1\varkappa_t^2\right\}D_{\mathcal{T}}\mathcal{T} = 0, \tag{3.5}$$

*where $\lambda_1 \in \mathbb{R}^+$ ( $\lambda_1 = 1 + \frac{1}{c_1^2}$ and $c_1 \in \mathbb{R}_0$ ) and $\varkappa_t$, is curvatures of the tangent indicatrix $(t)$ of the curve $\gamma$.*

**Proof.** Suppose that $\gamma$ is a slant helix. Thus the tangent indicatrix $(t)$ of $\gamma$ is a general helix. From (2.3), we have $D_{\mathcal{T}}\mathcal{T} = \varkappa_t\mathcal{N}$. By differentiating $D_{\mathcal{T}}\mathcal{T} = \varkappa_t\mathcal{N}$, we get

$$D_{\mathcal{T}}^3\mathcal{T} = -2\varkappa_t\varkappa_t'\mathcal{T} - \varkappa_t^2 D_{\mathcal{T}}\mathcal{T} + \varkappa_t''\mathcal{N} + \varkappa_t' D_{\mathcal{T}}\mathcal{N} + 2\varkappa_t'\tau_t\mathcal{B} + \varkappa_t\tau_t D_{\mathcal{T}}\mathcal{B}. \tag{3.6}$$

By using the frame equations in (2.3), we get (3.5).

Conversely let us assume that (3.5) holds. From (2.3), we have

$$\mathcal{B} = \frac{1}{\tau_t}D_{\mathcal{T}}\mathcal{N} + \frac{\varkappa_t}{\tau_t}\mathcal{T}. \tag{3.7}$$

Differentiating the last equality, we have

$$\begin{aligned} D_{\mathcal{T}}\mathcal{B} = {} & \tfrac{1}{\varkappa_t\tau_t}\left\{D_{\mathcal{T}}^3\mathcal{T} - 3\tfrac{\varkappa_t'}{\varkappa_t}D_{\mathcal{T}}^2\mathcal{T} - \left[\tfrac{\varkappa_t''}{\varkappa_t} - 3\left(\tfrac{\varkappa_t'}{\varkappa_t}\right)^2 - \varkappa_t^2 - \tau_t^2\right]D_{\mathcal{T}}\mathcal{T}\right\} \\ & + \tfrac{1}{\varkappa_t^2}\left(\tfrac{\varkappa_t}{\tau_t}\right)' D_{\mathcal{T}}^2\mathcal{T} - \left(\tfrac{\tau_t}{\varkappa_t} + \tfrac{\varkappa_t'}{\varkappa_t^3}\left(\tfrac{\varkappa_t}{\tau_t}\right)'\right)D_{\mathcal{T}}\mathcal{T} + \left(\tfrac{\varkappa_t}{\tau_t}\right)'\mathcal{T}. \end{aligned} \tag{3.8}$$

Using equations (2.3) and (3.5), we get

$$\left(\frac{\varkappa_t}{\tau_t}\right)' = 0 \text{ and } \frac{\varkappa_t}{\tau_t} = \sqrt{\frac{1}{\lambda_1 - 1}} = c_1(\text{non-zero constant}).$$

Thus, from (2.2), we obtain $\sigma_n = \frac{\tau_t}{\varkappa_t}$ =constant which means that $\gamma$ is a slant helix. □

By using the properties of general helix, we restate the theorem 3.2 according to the $\tau_t$ torsion of the tangent indicatrix $(t)$ of the curve $\gamma$ as follows.

**Theorem 3.3.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$. The curve $\gamma$ is a slant helix if and only if the tangent vector field $\mathcal{T}$ of the tangent indicatrix $(t)$ of the curve $\gamma$ satisfies the equation*

$$D^3_{\mathcal{T}}\mathcal{T} - 3\frac{\tau_t^{'}}{\tau_t}D^2_{\mathcal{T}}\mathcal{T} - \left\{ \frac{\tau_t^{''}}{\tau_t} - 3\left(\frac{\tau_t^{'}}{\tau_t}\right)^2 - \mu_1\tau_t^2 \right\} D_{\mathcal{T}}\mathcal{T} = 0, \tag{3.9}$$

*where $\mu_1 \in \mathbb{R}^+$ ( $\mu_1 = 1 + c_1^2$ and $c_1 \in \mathbb{R}_0$ ) and $\tau_t$, is torsion of the tangent indicatrix $(t)$ of the curve $\gamma$.*

**Theorem 3.4.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$. The curve $\gamma$ is a slant helix if and only if the principal normal vector field $\mathcal{N}$ of the tangent indicatrix $(t)$ of the curve $\gamma$ satisfies the equation*

$$D^2_{\mathcal{T}}\mathcal{N} - \frac{\varkappa_t^{'}}{\varkappa_t}D_{\mathcal{T}}\mathcal{N} + \lambda_1\varkappa_t^2\mathcal{N} = 0, \tag{3.10}$$

*where $\lambda_1 \in \mathbb{R}^+$ ( $\lambda_1 = 1 + \frac{1}{c_1^2}$ and $c_1 \in \mathbb{R}_0$ ) and $\varkappa_t$, is curvatures of the tangent indicatrix $(t)$ of the curve $\gamma$.*

**Proof.** Suppose that $\gamma$ is a slant helix. Thus the tangent indicatrix $(t)$ of $\gamma$ is a general helix. By differentiating $D_{\mathcal{T}}\mathcal{N} = -\varkappa_t\mathcal{T} + \tau_t\mathcal{B}$, we get

$$D^2_{\mathcal{T}}\mathcal{N} = -\varkappa_t^{'}\mathcal{T} + \tau_t^{'}\mathcal{B} - \left(\varkappa_t^2 + \tau_t^2\right)\mathcal{N}. \tag{3.11}$$

By using the frame equations in (2.3), equation (3.11) is reduced to (3.10).

Conversely, suppose that (3.10) holds. From (2.3), we have

$$\mathcal{T} = -\frac{1}{\varkappa_t}D_{\mathcal{T}}\mathcal{N} + \frac{\tau_t}{\varkappa_t}\mathcal{B}. \tag{3.12}$$

By differentiating equation (3.12), we get

$$D_{\mathcal{T}}\mathcal{T} = -\frac{1}{\varkappa_t}\left[D^2_{\mathcal{T}}\mathcal{N} - \frac{\varkappa_t^{'}}{\varkappa_t}D_{\mathcal{T}}\mathcal{N} + \left(\varkappa_t^2 + \tau_t^2\right)\mathcal{N}\right] + \varkappa_t\mathcal{N} + \left(\frac{\tau_t}{\varkappa_t}\right)^{'}\mathcal{B}. \tag{3.13}$$

Using equations (2.3) and (3.9), we get

$$\left(\frac{\tau_t}{\varkappa_t}\right)^{'} = 0 \text{ and } \frac{\varkappa_t}{\tau_t} = \sqrt{\frac{1}{\lambda - 1}} = c(\text{non-zero constant}).$$

Thus, from (2.2), we obtain $\sigma_n = \frac{\tau_t}{\varkappa_t}$ =constant, which means that $\gamma$ is a slant helix. This completes the proof. $\square$

By using the properties of general helix, we restate the theorem 3.4 according to the $\tau_t$ torsion of the tangent indicatrix $(t)$ of the curve $\gamma$ as follows.

**Theorem 3.5.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$. The curve $\gamma$ is a slant helix if and only if the principal normal vector field $\mathcal{N}$ of the tangent indicatrix $(t)$ of the curve $\gamma$ satisfies the equation*

$$D_{\mathcal{T}}^2\mathcal{N}-\frac{\tau_t^{'}}{\tau_t}D_{\mathcal{T}}\mathcal{N}+\mu_1\tau_t^2\mathcal{N}=0, \tag{3.14}$$

*where $\mu_1 \in \mathbb{R}^+$ ( $\mu_1 = 1+c_1^2$ and $c_1 \in \mathbb{R}_0$ ) and $\tau_t$, is torsion of the tangent indicatrix $(t)$ of the curve $\gamma$.*

We omit the proofs of the following theorems, since they are analogous to the proofs of the above theorems.

**Theorem 3.6.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$. The curve $\gamma$ is a slant helix if and only if the binormal vector field $\mathcal{B}$ of the principal normal indicatrix $(t)$ of the curve $\gamma$ satisfies the equation*

$$D_{\mathcal{T}}^3\mathcal{B}-3\frac{\varkappa_t^{'}}{\varkappa_t}D_{\mathcal{T}}^2\mathcal{B}-\left\{\frac{\varkappa_t^{''}}{\varkappa_t}-3\left(\frac{\varkappa_t^{'}}{\varkappa_t}\right)^2-\lambda_1\varkappa_t^2\right\}D_{\mathcal{T}}\mathcal{B}=0, \tag{3.15}$$

*where $\lambda_1 \in \mathbb{R}^+$ ( $\lambda_1 = 1+\frac{1}{c_1^2}$ and $c_1 \in \mathbb{R}_0$ ) and $\varkappa_t$, is curvatures of the tangent indicatrix $(t)$ of the curve $\gamma$.*

**Theorem 3.7.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$ The curve $\gamma$ is a slant helix if and only if the binormal vector field $\mathcal{B}$ of the principal normal indicatrix $(t)$ of the curve $\gamma$ satisfies the equation*

$$D_{\mathcal{T}}^3\mathcal{B}-3\frac{\tau_t^{'}}{\tau_t}D_{\mathcal{T}}^2\mathcal{B}-\left\{\frac{\tau_t^{''}}{\tau_t}-3\left(\frac{\tau_t^{'}}{\tau_t}\right)^2-\mu_1\tau_t^2\right\}D_{\mathcal{T}}\mathcal{B}=0, \tag{3.16}$$

*where $\mu_1 \in \mathbb{R}^+$ ( $\mu_1 = 1+c_1^2$ and $c_1 \in \mathbb{R}_0$ ) and $\tau_t$, is torsion of the tangent indicatrix $(t)$ of the curve $\gamma$.*

In the next six theorems, we obtain the differential equations of a slant helix according to the *tangent vector field* $\mathbb{T}$, *principal normal vector field* $\mathbb{N}$ and *binormal vector field* $\mathbb{B}$ *of the binormal indicatrix* $(b)$ of the curve.

**Theorem 3.8.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$. The curve $\gamma$ is a slant helix if and only if the tangent vector field $\mathbb{T}$ of the binormal indicatrix $(b)$ of the curve $\gamma$ satisfies the equation*

$$D_{\mathbb{T}}^3\mathbb{T}-3\frac{\varkappa_b^{'}}{\varkappa_b}D_{\mathbb{T}}^2\mathbb{T}-\left\{\frac{\varkappa_b^{''}}{\varkappa_b}-3\left(\frac{\varkappa_b^{'}}{\varkappa_b}\right)^2-\lambda_2\varkappa_b^2\right\}D_{\mathbb{T}}\mathbb{T}=0, \tag{3.17}$$

*where* $\lambda_2 \in \mathbb{R}^+$ *(* $\lambda_2 = 1 + \frac{1}{c_2^2}$ *and* $c_2 \in \mathbb{R}_0$ *) and* $\varkappa_b$*, is curvatures of the binormal indicatrix* $(b)$ *of the curve* $\gamma$.

**Proof.** Suppose that $\gamma$ is a slant helix. Hence the binormal indicatrix $(b)$ of $\gamma$ is a general helix. By differentiating $D_{\mathbb{T}}\mathbb{T} = \varkappa_b\mathbb{N}$, we get

$$D_{\mathbb{T}}^3\mathbb{T} = -2\varkappa_b\varkappa_b^{'}\mathbb{T} - \varkappa_b^2 D_{\mathbb{T}}\mathbb{T} + \varkappa_b^{''}\mathbb{N} + \varkappa_b^{'} D_{\mathbb{T}}\mathbb{N} + 2\varkappa_b^{'}\tau_b\mathbb{B} + \varkappa_b\tau_b D_{\mathbb{T}}\mathbb{B}. \tag{3.18}$$

By using the frame equations in (2.7), we get (3.17).

Conversely let us assume that (3.17) holds. From (2.7), we have

$$\mathbb{B} = \frac{1}{\tau_b} D_{\mathbb{T}}\mathbb{N} + \frac{\varkappa_b}{\tau_b}\mathbb{T}. \tag{3.19}$$

By differentiating equation (3.19),

$$\begin{aligned} D_{\mathbb{T}}\mathbb{B} = \tfrac{1}{\varkappa_b\tau_b}\left\{ D_{\mathbb{T}}^3\mathbb{T} - \tfrac{3\tau_b^{'}}{\tau_b} D_{\mathbb{T}}^2\mathbb{T} - \left[ \tfrac{\varkappa_b^{''}}{\varkappa_b} - \tfrac{3\left(\tau_b^{'}\right)^2}{\tau_b^2} - \varkappa_b^2 - \tau_b^2 \right] D_{\mathbb{T}}\mathbb{T} \right\} \\ + \tfrac{1}{\varkappa_b^2}\left(\tfrac{\varkappa_b}{\tau_b}\right)^{'} D_{\mathbb{T}}^2\mathbb{T} - \left( \tfrac{\tau_b}{\varkappa_b} + \tfrac{\varkappa_b^{'}}{\varkappa_b^3}\left(\tfrac{\varkappa_b}{\tau_b}\right)^{'} \right) D_{\mathbb{T}}\mathbb{T} + \left(\tfrac{\varkappa_b}{\tau_b}\right)^{'}\mathbb{T}. \end{aligned} \tag{3.20}$$

Using equations (2.7) and (3.17), we get

$$\left(\frac{\varkappa_b}{\tau_b}\right)^{'} = 0 \text{ and } \frac{\varkappa_b}{\tau_b} = \sqrt{\frac{1}{\lambda_2 - 1}} = c_2(\text{non-zero constant})$$

Since $\sigma_n = -\frac{\tau_b}{\varkappa_b}$, $\gamma$ is a slant helix. Thus the proof of theorem 3.8 is completed. □

**Theorem 3.9.** *Let* $\gamma$ *be a unit speed curve with Frenet vectors* $t$, $n$, $b$ *and with non-zero curvatures* $\varkappa$ *and* $\tau$ *in* $\mathbb{R}^3$ *The curve* $\gamma$ *is a slant helix if and only if the tangent vector field* $\mathbb{T}$ *of the binormal indicatrix* $(b)$ *of the curve* $\gamma$ *satisfies the equation*

$$D_{\mathbb{T}}^3\mathbb{T} - 3\frac{\tau_b^{'}}{\tau_b} D_{\mathbb{T}}^2\mathbb{T} - \left\{ \frac{\tau_b^{''}}{\tau_b} - 3\left(\frac{\tau_b^{'}}{\tau_b}\right)^2 - \mu_2\tau_b^2 \right\} D_{\mathbb{T}}\mathbb{T} = 0, \tag{3.21}$$

*where* $\mu_2 \in \mathbb{R}^+$ *(* $\mu_2 = 1 + c_2^2$ *and* $c_2 \in \mathbb{R}_0$ *) and* $\tau_b$*, is torsion of binormal indicatrix* $(b)$ *of the curve* $\gamma$.

**Theorem 3.10.** *Let* $\gamma$ *be a unit speed curve with Frenet vectors* $t$, $n$, $b$ *and with non-zero curvatures* $\varkappa$ *and* $\tau$ *in* $\mathbb{R}^3$ *The curve* $\gamma$ *is a slant helix if and only if the principal normal vector field* $\mathbb{N}$ *of the binormal indicatrix* $(b)$ *satisfies the equation*

$$D_{\mathbb{T}}^2\mathbb{N} - \frac{\tau_b^{'}}{\tau_b} D_{\mathbb{T}}\mathbb{N} + \mu_2\tau_b^2\mathbb{N} = 0, \tag{3.22}$$

*where* $\mu_2 \in \mathbb{R}^+$ *(* $\mu_2 = 1 + c_2^2$ *and* $c_2 \in \mathbb{R}_0$ *) and* $\tau_b$*, is torsion of binormal indicatrix* $(b)$ *of the curve* $\gamma$.

**Proof.** Suppose that $\gamma$ is a slant helix. Therefore the tangent indicatrix $(b)$ of $\gamma$ is a general helix. By differentiating $D_{\mathbb{T}}\mathbb{N} = -\varkappa_b\mathbb{T} + \tau_b\mathbb{B}$, we get

$$D_{\mathbb{T}}^2\mathbb{N} = -\varkappa_b^{'}\mathbb{T} + \tau_b^{'}\mathbb{B} - \left(\varkappa_b^2 + \tau_b^2\right)\mathbb{N}. \tag{3.23}$$

By using the frame equations in (1.7), equation (2.21) is reduced to (2.20).

Conversely suppose that (2.20) holds. From (1.7), we have

$$\mathbb{T} = -\frac{1}{\varkappa_b}D_{\mathbb{T}}\mathbb{N} + \frac{\tau_b}{\varkappa_b}\mathbb{B}. \tag{3.24}$$

Differentiating the last equality, we have

$$D_{\mathbb{T}}\mathbb{T} = -\frac{1}{\varkappa_b}\left[D_{\mathbb{T}}^2\mathbb{N} - \frac{\tau_b^{'}}{\tau_b}D_{\mathbb{T}}\mathbb{N} + \left(\varkappa_b^2 + \tau_b^2\right)\mathbb{N}\right] + \varkappa_b\mathbb{N} + \left(\frac{\tau_b}{\varkappa_b}\right)^{'}\mathbb{B}.$$

Using equations (1.7) and (2.20), we get

$$\left(\frac{\tau_b}{\varkappa_b}\right)^{'} = 0 \text{ and } \frac{\tau_b}{\varkappa_b} = \sqrt{\mu_2 - 1} = c_2(\text{non-zero constant})$$

Since $\sigma_n = -\frac{\tau_b}{\varkappa_b}$, $\gamma$ is a slant helix. This completes the proof of the theorem. □

**Theorem 3.11.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$ The curve $\gamma$ is a slant helix if and only if the principal normal vector field $\mathbb{N}$ of the binormal indicatrix $(b)$ of the curve $\gamma$ satisfies the equation*

$$D_{\mathbb{T}}^2\mathbb{N} - \frac{\varkappa_b^{'}}{\varkappa_b}D_{\mathbb{T}}\mathbb{N} + \lambda_2\varkappa_b^2\mathbb{N} = 0, \tag{3.25}$$

*where $\lambda_2 \in \mathbb{R}^+$ ( $\lambda_2 = 1 + \frac{1}{c_2^2}$ and $c_2 \in \mathbb{R}_0$) and $\varkappa_b$, is curvatures of the binormal indicatrix $(b)$ of the curve $\gamma$.*

With the similar proof, we have the following theorems.

**Theorem 3.12.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$. The curve $\gamma$* is a slant helix if and only if the *the binormal vector field $\mathbb{B}$ of the binormal indicatrix $(b)$ of the curve $\gamma$ satisfies the equation*

$$D_{\mathbb{T}}^3\mathbb{B} - 3\frac{\tau_b^{'}}{\tau_b}D_{\mathbb{T}}^2\mathbb{B} - \left\{\frac{\tau_b^{''}}{\tau_b} - 3\left(\frac{\tau_b^{'}}{\tau_b}\right)^2 - \mu_2\tau_b^2\right\}D_{\mathbb{T}}\mathbb{B} = 0, \tag{3.26}$$

*where $\mu_2 \in \mathbb{R}^+$ ( $\mu_2 = 1 + c_2^2$ and $c_2 \in \mathbb{R}_0$) and $\tau_b$, is torsion of binormal indicatrix $(b)$ of the curve $\gamma$.*

**Theorem 3.13.** *Let $\gamma$ be a unit speed curve with Frenet vectors $t$, $n$, $b$ and with non-zero curvatures $\varkappa$ and $\tau$ in $\mathbb{R}^3$. The curve $\gamma$* is a slant helix if and only if the *the binormal vector field $\mathbb{B}$ of the binormal indicatrix $(b)$ satisfies the equation*

$$D_{\mathbb{T}}^3\mathbb{B} - 3\frac{\varkappa_b^{'}}{\varkappa_b}D_{\mathbb{T}}^2\mathbb{B} - \left\{\frac{\varkappa_b^{''}}{\varkappa_b} - 3\left(\frac{\varkappa_b^{'}}{\varkappa_b}\right)^2 - \lambda_2\varkappa_b^2\right\}D_{\mathbb{T}}\mathbb{B} = 0, \tag{3.27}$$

*where $\lambda_2 \in \mathbb{R}^+$ ( $\lambda_2 = 1 + \frac{1}{c_2^2}$ and $c_2 \in \mathbb{R}_0$ ) and $\varkappa_b$, is curvatures of the binormal indicatrix $(b)$ of the curve $\gamma$.*

**Example.**

In [10], Kula and Yayli showed that the tangent indicatrix of a slant helix in Euclidean 3-space is a spherical general helix. The general equation of spherical helix obtained by Monterde in [11] as follows:

$$\begin{aligned}\beta_c(s) = \quad & (\cos s\cos(\omega s) + \tfrac{1}{\omega}\sin s\sin(\omega s), \\ & -\cos s\sin(\omega s) + \tfrac{1}{\omega}\sin s\cos(\omega s), \tfrac{1}{c\,\omega}\sin s),\end{aligned} \tag{3.28}$$

where $\omega = \frac{\sqrt{1+c^2}}{c}$ and $c \in \mathbb{R}_0$. Now we can easily obtained the general equation of a slant helix in Euclidean 3-space. Let $\alpha$ be a unit speed slant helix, then we have $\frac{d\alpha}{ds} = T = \beta_c(s)$. Thus by one integration we can easily obtained the family of slant helix according to the non-zero constant $c$ as follows. If we denote the family of slant helix by $\alpha_c$, then

$$\begin{aligned}\alpha_c(s) = \quad & (\tfrac{w+1}{2w(1-w)}\sin[(1-w)s] + \tfrac{w-1}{2w(1+w)}\sin[(1+w)s], \\ & \tfrac{w+1}{2w(1-w)}\cos[(1-w)s] + \tfrac{w-1}{2w(1+w)}\cos[(1+w)s], \tfrac{1}{wc}\cos s),\end{aligned}$$

where $\omega = \frac{\sqrt{1+c^2}}{c}$ and $c \in \mathbb{R}_0$. Also it is easily show that the slant helix fully lies in the hyperboloid of one sheet with equation

$$\frac{x^2}{4c^4} + \frac{y^2}{4c^4} - \frac{z^2}{4c^2} = 1.$$

Now we give an example of slant helices in Euclidean 3-space and draw pictures of tangent indicatricies, normal indicatricies of the family of slant helix for $c = \pm\frac{1}{4}, \pm1, \pm4, \pm6$.

(i) For $c = \pm\frac{1}{4}, \pm1, \pm4, \pm6$, normal indicatricies of the family of slant helix lie on the unit sphere which is rendered in Figure 1.

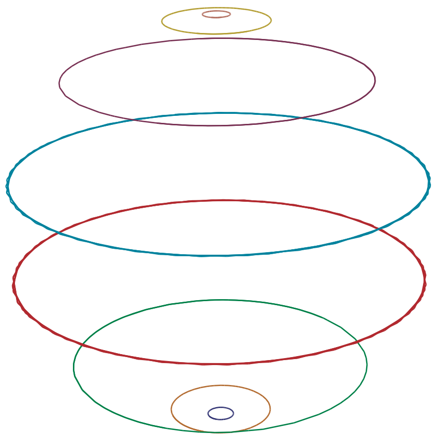


**Figure 1.** Normal indicatricies of the family of slant helix for $c = \pm\frac{1}{4}, \pm 1, \pm 4, \pm 6$.

(ii) For $c = \pm\frac{1}{4}$, tangent indicatricies of the family of slant helix lie on the unit sphere, which is rendered in Figure 2.

(iii) For $c = \pm 1$, tangent indicatricies of the family of slant helix lie on the unit sphere, which is rendered in Figure 3.

(iv) For $c = \pm 4$, tangent indicatricies of the family of slant helix lie on the unit sphere, which is rendered in Figure 4.

(v) For $c = \pm 6$, tangent indicatricies of the family of slant helix lie on the unit sphere, which is rendered in Figure 5.

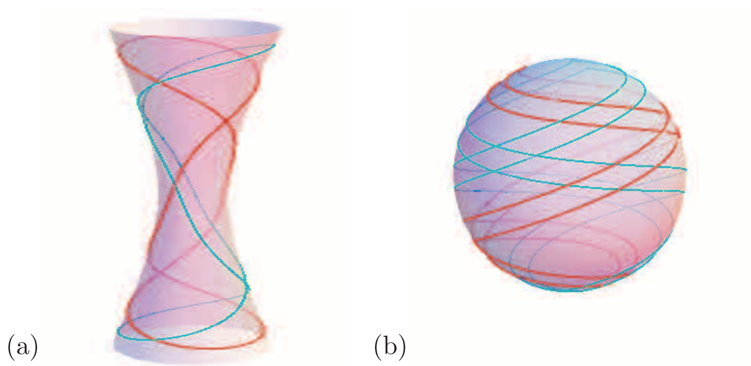
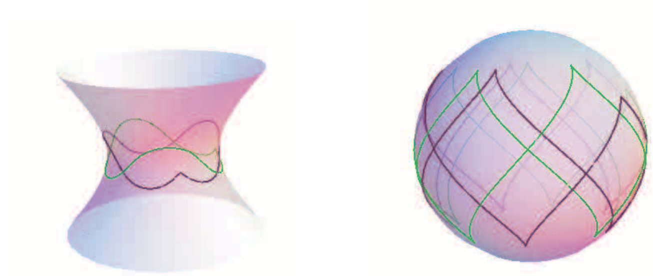
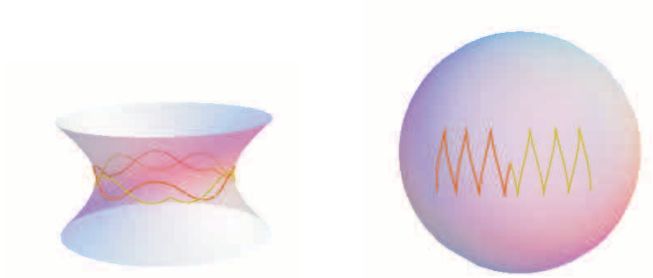



**Figure 2.** The slant helices for $c = \pm\frac{1}{4}$ (a) and tangent indicatricies of the slant helices for $c = \pm\frac{1}{4}$ (b)



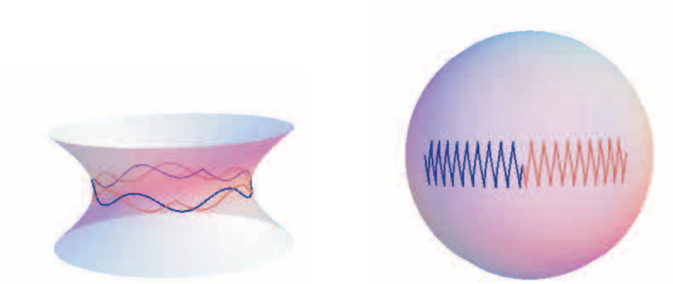
(a) (b)

**Figure 3.** The slant helices for $c = \pm 1$ (a) and tangent indicatricies of the slant helices for $c = \pm 1$ (b)



(a) (b)

**Figure 4.** The slant helices for $c = \pm 4$ (a) and tangent indicatricies of the slant helices for $c = \pm 4$ (b)



(a) (b)

**Figure 5.** The slant helices for $c = \pm 6$ (a) and tangent indicatricies of the slant helices for $c = \pm 6$ (b)

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