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Decision Making with Uncertainty Using Hesitant Fuzzy Sets

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Abstract Actual existing multi-criteria decision-making (MCDM) methods yield results that may be questionable and unreliable. These methods very often ignore the issue of uncertainty and rank reversal paradox, which are fundamental and important challenges of MCDM methods. In response to these challenges, the Characteristic Objects Method (COMET) was developed. Despite it being immune to the rank reversal paradox, classical COMET is not designed for uncertain, decisional problems. In this paper, we propose to extend COMET using hesitant fuzzy set (HFS) theory. Hesitant fuzzy set theory is a powerful tool to express the uncertainty that derives from an expert comparing characteristic objects and identifying membership functions for each criterion domain. We present the theoretical foundations and principles of COMET, and we provide an illustrative example to show how COMET handles uncertain decision problems both practically and effectively.

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1 Introduction

Together with the development of operational research, the multi-criteria decision-making (MCDM) methods have been observed as an alternative approach of assessment of alternatives in the field of decision problems. In our daily or professional lives, there are many conflicting criteria that need to be evaluated in making decisions, and it is an exactly task for MCDM methods [34]. Therefore, the use of these methods allows for organizing and analyzing complex decisions, based on mathematical principles and rules. Research on multi-criteria decision support developed two main groups of methods, i.e., American and European schools. Methods of the American school of decision support are based on a functional approach, or more accurately the utility or value function [3, 38]. These methods use two types of relationships between alternatives, i.e., indifference and preference, while they exclude incomparabilities of variants [3]. In this family, we can include the following methods: multi-attribute utility theory (MAUT), multi-attribute value theory (MAVT), analytic hierarchy process (AHP), analytic network process (ANP), simple multi-attribute rating technique (SMART), utility theory additive (UTA), measuring attractiveness by a categorical based evaluation technique (MACBETH), or technique for order preference by similarity to ideal solution (TOPSIS) [9, 13–15]. These approaches are criticized mainly by researchers from the European school. They emphasize the fact that these methods do not take into account the variability and uncertainty of expert judgements [7].

Methods of European school of decision support are based on relational model, where the most frequently are used relation of indifference, weak or strong preference, and incomparabilities. These methods use outranking relation in the preference aggregation process. This relationship is characterized by not transitive between pairs of decision variants. Among the methods of the European School most popular are ELECTRE family and PROMETHEE methods [11]. Additionally, we can indicate in this group following methods: Novel Approach to Imprecise Assessment and Decision Environment (NAIADE), ORESTE, REGIME, ARGUS, Treatment of the Alternatives according To the Importance of Criteria (TACTIC), MELCHIOR or PAMSSEM [15, 23].

Moreover, we can distinguish a number of methods for connecting multi-criteria approach of American and European schools decision support. We can indicate for example following methods: EVAMIX, QUALIFLEX, and group of PCCA methods (Criterion Pairwise Comparison Approach), i.e., MAPPAC, PRAGMA, PACMAN and IDRA [11, 19, 20]. The last group is the set of methods based strictly on the rules of decision making. These methods use the fuzzy sets theory (COMET) [37] and the rough set theory (DRSA) [12]. The methods in this group are built at the basis of decision rules [16]. It is worth to notice that in many of MCDM methods there is not taken into account the uncertainty, imprecision and ambiguity of data [38, 41]. However, the most common solution to this problem is to use granular mathematics, e.g., fuzzy sets theory [8, 24] or interval arithmetic [48].

The Characteristic Objects Method, i.e., the COMET, is a distance-based technique in dealing with MCDM problems [27, 35–37]. In methodological terms, it is a bit similar to the TOPSIS method [4, 30], because we are also using here reference points. However, we are using much more the characteristic points and so we can more accurately model the nonlinearity. The COMET method helps a decision maker organize the problems to be solved, and carry out analysis, comparisons and ranking of the alternatives, where the complexity of the algorithm is completely independent of the alternatives number. This method takes into account the existence of a correlation between components of MCDM function. Additionally, comparisons between the characteristic objects (COs) are easier than comparisons between alternatives. This is due to Weber–Fechner law, which determines that if a difference between two objects is too small, then the people cannot distinguish preferences between these objects [18, 22, 40]. The final ranking of the COMET is obtained on the basis of COs and their value of preferences. This ensures that the COMET is free of rank reversal phenomenon.

Since the introduction of fuzzy set theory by Zadeh [49], many research achievements have been made to enrich the fuzzy set theory. Interval-valued fuzzy set [50] and intuitionistic fuzzy set [2] are all well-known generalizations of fuzzy set and are extensively applied in many fields. In the practical applications, it is usually difficult to establish the degree of membership of fuzzy set because of the time pressure, lack of knowledge or data and some other reasons. Torra [39] introduced the concept of hesitant fuzzy set which permitted the membership having a set of possible values in order that hesitant fuzzy set can reflect the human's hesitancy more objectively than the other classical extensions of fuzzy set. To accommodate more complex environment, several extensions of HFS have been presented, such as interval-valued hesitant fuzzy set [5, 44], hesitant triangular fuzzy set [47, 51], hesitant multiplicative set [42], hesitant fuzzy linguistic term set [31], hesitant fuzzy uncertain linguistic set [53], dual hesitant fuzzy set [46, 54], generalized hesitant fuzzy set [28] and convex hesitant fuzzy set [29]. Meng et al. [21] discussed multiple attribute decision making under linguistic hesitant fuzzy environment, and Farhadinia presented the distance and similarity measures for hesitant fuzzy linguistic term sets and extended hesitant fuzzy set to the higher order hesitant fuzzy set [10]. The general state of the art and future directions for HFS can be found in [32]. When analyzing actual trends in MCDM research field, we can observe the growing popularity of HFS extensions of classical MCDM methods. For example, Zhang and Wei extended VIKOR and TOPSIS methods [52], whereas ELECTRE extensions with HFS are presented in [6, 25]. However, HFS has been also used to provide the new methodology, e.g., a segment-based approach [1]. It confirms the fact that HFS is a very useful tool to deal with uncertainty.

In this paper, the COMET is extended to solve decisional problems under uncertainty using hesitant fuzzy sets (HFS). The main motivation is that when expert is defining the membership of an element, the difficulty of establishing the membership degree is not because he has a margin of error (as in intuitionistic fuzzy sets), or some possibility distribution on the possible values (as in type 2 fuzzy sets), but because he has a set of possible values (as in HFS) [39]. This means that HFS can reflect decision hesitancy more completely than other extensions of fuzzy sets. Therefore, the paper presents theoretical foundations of the COMET extensions using HFS to better reflect the uncertainty. It is worth to notice that this connection eliminates the most important and dangerous paradoxes in decision-making areas.

Rest of the paper is organized as follows: In Sect. 2, some basic preliminary concepts are discussed. In Sect. 3, we introduced the notion of COMET under hesitant fuzzy environment. In Sect. 4, an example is given to show the practical feasibility study of the modified COMET. In Sect. 5, we conclude the paper.

2 Preliminaries

In this section, we recall some important concepts which are necessary to understand our proposed decision-making method. Torra [39] proposed a HFS, which is a more general fuzzy set and permits the membership to include a set of possible values.

Definition 1 [39] A hesitant fuzzy set A on X is a function h^A that when applied to X returns a finite subset of [0, 1], which can be represented as the following mathematical symbol:

$$A = \{(x, h^A(x))|x \in X\}, \tag{1}$$

where $h^A(x)$ is a set of some values in [0, 1], denoting the possible membership degrees of the element $x \in X$ to the set A. For convenience, Xia and Xu [45] named $h^A(x)$ a hesitant fuzzy element (HFE).

Definition 2 [39] For a hesitant fuzzy set represented by its membership function h, we define its complement as follows:

$$h^c(x) = \bigcup_{\gamma \in h(x)} \{1 - \gamma\}. \tag{2}$$

Definition 3 [45] For a HFE h, $Sc(h) = \frac{1}{l_h}\sum_{\gamma \in h}\gamma$, is called the score function of h, where l_h is the number of elements in h and $Sc(h) \in [0, 1]$. For two HFEs h_1 and h_2, if $Sc(h_1) > Sc(h_2)$, then $h_2 \prec h_1$, if $Sc(h_1) = Sc(h_2)$, then $h_1 \approx h_2$.

Xia and Xu [45] define some operations on the HFEs (h, h_1 and h_2) and the scalar number k:

$$kh = \bigcup_{\gamma \in h}\{1 - (1-\gamma)^k\} \tag{3}$$

$$h_1 \oplus h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\} \tag{4}$$

$$h_1 \otimes h_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2} \{\gamma_1\gamma_2\} \tag{5}$$

Definition 4 [55] Let L (and R) both be decreasing, shape functions from $\Re^+ = [, \infty)$ to $[0, 1]$ with $L(0) = \omega$; $L(x) < \omega$ for all $x < 1$; $L(1) = 0$or ($L(x) > 0$ for all x and $L(+\infty) = 0$) (and the same for R). A generalized fuzzy number is called L–R type if there are real numbers m, $\alpha > 0, \beta > 0$ and ω $(0 \leqslant \omega \leqslant 1)$with

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega L\left(\frac{m-x}{\alpha}\right), & x \leqslant m \\ \omega R\left(\frac{x-m}{\beta}\right), & x \geqslant m \end{cases} \tag{6}$$

where m is called the mean value of $\tilde{A}$ and α and β are called the left and right spreads, respectively. The L–R type generalized fuzzy number $\tilde{A}$ is symbolically denoted by $\tilde{A} = (m, \alpha, \beta; \omega)_{LR}$. If $\omega = 1$, then $\tilde{A}$ is called L–R type fuzzy number and simply denoted by $\tilde{A} = (m, \alpha, \beta)_{LR}$.

For an L–R type generalized fuzzy number $\tilde{A} = (m, \alpha, \beta; \omega)_{LR}$, if L and R are of the form

$$T(x) = \begin{cases} 1 - x, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Then $\tilde{A}$ is called a generalized triangular fuzzy number denoted by $\tilde{A} = (m, \alpha, \beta; \omega)_T$. Similarly for $\omega = 1$, $\tilde{A}$ is simply called a triangular fuzzy number denoted by $\tilde{A} = (m, \alpha, \beta)_T$.

A fuzzy number $\tilde{A}$ is called an L–R type generalized trapezoidal fuzzy number if there are real numbers $m_1, m_2, \alpha > 0$ and $\beta > 0$ with the following membership function

$$\mu_{\tilde{A}}(x) = \begin{cases} \omega L\left(\frac{m_1-x}{\alpha}\right), & x \leqslant m_1 \\ \omega, & m_1 \leqslant x \leqslant m_2 \\ \omega R\left(\frac{x-m_2}{\beta}\right), & x \geqslant m_2 \end{cases} \tag{8}$$

where m_1 and m_2 are called the mean values of $\tilde{A}$ and α, β are called the left and right spreads, respectively. Symbolically, $\tilde{A}$ is denoted by $(m_1, m_2, \alpha, \beta; \omega)_{LR}$. The L–R type generalized trapezoidal fuzzy number $\tilde{A} = (m_1, m_2, \alpha, \beta; \omega)_{LR}$ divides into three parts: left part, middle part and right part. The left, middle and right parts include the intervals $[m_1 - \alpha, m_1], [m_1, m_2]$ and $[m_2, m_2 + \beta]$, respectively.

If we take L and R to be of the form as mentioned in Eq. 7, then $\tilde{A}$ is called generalized trapezoidal fuzzy number denoted by $(m_1, m_2, \alpha, \beta; \omega)_T$. A generalized trapezoidal fuzzy number $\tilde{A}(m_1, m_2, \alpha, \beta; \omega)_T$ is simply called a trapezoidal fuzzy number denoted by $\tilde{A}(m_1, m_2, \alpha, \beta)_T$ when $\omega = 1$.

We know that L–R type fuzzy numbers are used to present real numbers in a fuzzy environment and trapezoidal fuzzy numbers are used to present fuzzy intervals that are widely applied in linguistic, knowledge representation, control systems, database, and so forth. Similarly, the L–R-type generalized fuzzy numbers are very general and allow one to represent the different types of information. For example, the *L*–*R* type generalized fuzzy number $\tilde{B} = (m, m, 0, 0; \omega)_{LR}$ with $m \in \Re = (-\infty, \infty)$ is used to denote a real number $\tilde{B}$ and the *L*–*R* type generalized fuzzy number $\tilde{C} = (m_1, m_2, 0, 0; \omega)_{LR}$ with $m_1, m_2 \in \Re$ and $m_1 < m_2$ is used to denote an interval $\tilde{C}$.

Definition 5 For a triangular fuzzy number $\tilde{A}$, we define

1. The support of $\tilde{A}$ is $S(\tilde{A}) = \{x : \mu_{\tilde{A}}(x) > 0\}$.
2. The core of $\tilde{A}$ is $C(\tilde{A}) = \{x : \mu_{\tilde{A}}(x) = 1\}$.

Definition 6 The fuzzy rule and the rule base:

1. The single fuzzy rule can be based on tautology modus ponens [26, 43]. The reasoning process uses logical connectives IF-THEN, OR and AND.
2. The rule base consists of logical rules determining causal relationships existing in the system between fuzzy sets of its inputs and outputs [33].

Definition 7 [17] A triangular norm (t-norm) is a binary operation $T : [0,1] \times [0,1] \rightarrow [0,1]$ satisfying $\forall x, y, z \in [0,1]$:

1. $T(x,y) = T(y,x)$ (commutativity),
2. $T(x,y) \leqslant T(x,z)$, if $y \leqslant z$(monotonicity),
3. $T(x, T(y,z)) = T(T(x,y),z)$ (associativity),
4. $T(x,1) = x$ (Neutrality of 1).

Throughout this paper, only the product is used as a t-norm operator, i.e., $P(\mu_{\alpha_1}(x), \mu_{\alpha_2}(y)) = \mu_{\alpha_1}(x).\mu_{\alpha_2}(y)$.

3 COMET for HFS

The classical COMET method is based on fuzzy sets theory. However, this approach does not completely solve the problem of the uncertainty of an expert's judgements. Sometimes, there is few possible values of membership degrees for the attribute of an alternative. Additionally, an expert's judgements can be uncertain, especially when the two characteristic objects are compared by an expert. Therefore, the framework of hesitant fuzzy sets is presented as the extension of the classical COMET approach, which can solve problems while account for the uncertainty of an expert's judgements.

Consider a MCDM problem in which the ratings of the alternative evaluations are expressed as HFSs. The solution procedure for the proposed MCDM approach is described below.

Let A_j $(j = 1, 2, \ldots, m)$ be the set of alternatives, and suppose a decision maker is asked to evaluate the given alternatives with respect to several criteria $C_i (i = 1, 2, \ldots, n)$.Suppose the evaluation characteristic of an alternative A_j $(j = 1, 2, \ldots, m)$ on a criteria $C_i (i = 1, 2, \ldots, n)$ is represented by the HFE h_{ij}.

The ranking algorithm of the COMET has the following five steps:

Step 1: Define the space of the problem as follows:

Let $\mathscr{F}$ be the collection of all *L*–*R*-type generalized fuzzy numbers, and $F_i^1, F_i^2, \ldots, F_i^q$ are different families of subsets of $\mathscr{F}$ (9):

$$\begin{aligned} F_i^1 &= \left\{F_{i1}^1, F_{i2}^1, \ldots, F_{ic_i}^1\right\} \\ F_i^2 &= \left\{F_{i1}^2, F_{i2}^2, \ldots, F_{ic_i}^2\right\} \\ &\vdots \\ F_i^q &= \left\{F_{i1}^q, F_{i2}^q, \ldots, F_{ic_i}^q\right\} \end{aligned} \tag{9}$$

where collections are established for each criterion C_i $(i = 1, 2, \ldots, n)$.

In this way, the following result is obtained (10):

$$\begin{aligned} C_1 &= \left\{\left\{F_{11}^1, F_{12}^1, \ldots, F_{1c_1}^1\right\}, \left\{F_{11}^2, F_{12}^2, \ldots, F_{1c_1}^2\right\}, \ldots, \left\{F_{11}^q, F_{12}^q, \ldots, F_{1c_1}^q\right\}\right\} \\ C_2 &= \left\{\left\{F_{21}^1, F_{22}^1, \ldots, F_{2c_2}^1\right\}, \left\{F_{21}^2, F_{22}^2, \ldots, F_{2c_2}^2\right\}, \ldots, \left\{F_{21}^q, F_{22}^q, \ldots, F_{2c_2}^q\right\}\right\} \\ &\vdots \\ C_n &= \left\{\left\{F_{n1}^1, F_{n2}^1, \ldots, F_{nc_n}^1\right\}, \left\{F_{n1}^2, F_{n2}^2, \ldots, F_{nc_n}^2\right\}, \ldots, \left\{F_{n1}^q, F_{n2}^q, \ldots, F_{nc_n}^q\right\}\right\} \end{aligned} \tag{10}$$

where $c_1, c_2, \ldots, c_n$ are numbers of fuzzy numbers in each family F_i^b $(1 \leqslant b \leqslant q, 1 \leqslant i \leqslant n)$ for all criteria.

Suppose among all F_i^b $(1 \leqslant b \leqslant q)$, one of them is a family of triangular fuzzy numbers (TFNs) F_i^t (say). The core of each criterion is defined as the core of each F_i^t $(1 \leqslant i \leqslant n)$, i.e.

$$\begin{aligned} C(C_1) &= \left\{C(F_{11}^t), C(F_{12}^t), \ldots, C\left(F_{1c_1}^t\right)\right\} \\ C(C_2) &= \left\{C(F_{21}^t), C(F_{22}^t), \ldots, C\left(F_{2c_2}^t\right)\right\} \\ &\vdots \\ C(C_n) &= \left\{C(F_{n1}^t), C(F_{n2}^t), \ldots, C\left(F_{nc_n}^t\right)\right\} \end{aligned} \tag{11}$$

Step 2: Generate the characteristic objects:

The COs are obtained by using the Cartesian product of all TFNs cores for each criteria as follows:

$$CO = C(C_1) \times C(C_2) \times \ldots \times C(C_n) \tag{12}$$

As the result of this, the ordered set of all COs is obtained:

$$\begin{aligned} CO_1 &= \left\{C(F_{11}^t), C(F_{21}^t), \ldots, C(F_{n1}^t)\right\} \\ CO_2 &= \left\{C(F_{11}^t), C(F_{21}^t), \ldots, C(F_{n2}^t)\right\} \\ &\vdots \\ CO_s &= \left\{C(F_{1c_1}^t), C(F_{2c_2}^t), \ldots, C(F_{nc_n}^t)\right\} \end{aligned} \tag{13}$$

where $s = \prod_{i=1}^{n} c_i$ is a number of COs.

Step 3: Rank and evaluate the characteristic objects:

Determine the Matrix of Expert Judgment (MEJ). This is a result of comparison of COs by the knowledge of expert. The MEJ structure is as follows:

$$\mathrm{MEJ} = \begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \cdots & \tilde{h}_{1s} \\ \tilde{h}_{21} & \tilde{h}_{22} & \cdots & \tilde{h}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{h}_{s1} & \tilde{h}_{s2} & \cdots & \tilde{h}_{ss} \end{bmatrix} \tag{14}$$

where $\tilde{h}_{ij}$ is HFE obtained in result of comparing CO_i and CO_j by the expert. The more preferred CO obtains a stronger hesitant degree denoted by HFE $\tilde{h}_s$, and the second object obtains a weaker hesitant degree denoted by HFE $\tilde{h}_w$. If the preferences are balanced, the both objects get a hesitant degree denoted by HFE $\tilde{h}_f$. The selection of HFEs $\tilde{h}_s, \tilde{h}_w$ and $\tilde{h}_f$ depends solely on the knowledge and opinion of the expert and can be presented as follows:

$$\tilde{h}_{ij} = f(CO_i, CO_j) = \begin{cases} \tilde{h}_w, & f_{\exp}(CO_i) < f_{\exp}(CO_j) \\ \tilde{h}_f, & f_{\exp}(CO_i) = f_{\exp}(CO_j) \\ \tilde{h}_s, & f_{\exp}(CO_i) > f_{\exp}(CO_j) \end{cases} \tag{15}$$

where $f_{\exp}$ is an expert judgement function.

Suppose $\tilde{H}_i = \oplus_{j=1}^{s} \tilde{h}_{ij}$, where each $\tilde{H}_i$ is HFE.

Afterward, we get a vertical vector SJ of the Summed Judgments where $SJ_i = Sc(\tilde{H}_i) = \frac{1}{l_{\tilde{H}_i}} \sum_{\gamma \in \tilde{H}_i} \gamma$ (see Definition 3).

Finally, we use the same MATLAB code as used by Sałabun in [37] to assign for each CO the approximate value of preference. As a result, we get a vertical vector P, where i^{th} row of P contains the approximate value of preference for CO_i

Step 4: The rule base:

Each characteristic object and value of preference is converted to a fuzzy rule as follows:

$$\mathrm{IF}\, \mathrm{CO}_i\, \mathrm{THEN}\, P_i \tag{16}$$

$$\mathrm{IF}\, C(F_{1i}^t)\, \mathrm{AND}\, C(F_{2i}^t)\, \mathrm{AND} \ldots \mathrm{THEN}\, P_i \tag{17}$$

In this way, the complete fuzzy rule base is obtained, which can be presented as follows:

$$\begin{array}{l} \mathrm{IF}\, \mathrm{CO}_1\, \mathrm{THEN}\, P_1 \\ \mathrm{IF}\, \mathrm{CO}_2\, \mathrm{THEN}\, P_2 \\ \quad\vdots \\ \mathrm{IF}\, \mathrm{CO}_s\, \mathrm{THEN}\, P_s \end{array} \tag{18}$$

Step 5: Inference in a fuzzy model and final ranking:

Each alternative activates the specified number of fuzzy rules, where for each one is determined the fulfillment degree of the conjunctive complex premise. Fulfillment degree of each activated rule corresponding to each element of F_i^b $(1 \leqslant b \leqslant q, 1 \leqslant i \leqslant n)$ of same type sum to one. The each one alternative is a set of crisp number, which corresponds to criteria $C_1, C_2, \ldots, C_n$. It can be presented as follows (19), where the following condition (20) must be satisfied.

$$A_j = \{a_{1j}, a_{2j}, \ldots, a_{nj}\} \tag{19}$$

$$\begin{array}{l} a_{1j} \in \left[C(F_{11}^t), C\left(F_{1c_1}^t\right)\right] \\ a_{2j} \in \left[C(F_{21}^t), C\left(F_{2c_2}^t\right)\right] \\ \quad\vdots \\ a_{nj} \in \left[C(F_{n1}^t), C\left(F_{nc_n}^t\right)\right] \end{array} \tag{20}$$

To infer the final ranking of the alternatives corresponding to each criterion, we proceed as follows:

$$\begin{array}{l} a_{1j} \in \left[C\left(F_{1k_1}^t\right), C\left(F_{1(k_1+1)}^t\right)\right] \\ a_{2j} \in \left[C\left(F_{2k_2}^t\right), C\left(F_{2(k_2+1)}^t\right)\right] \\ \quad\vdots \\ a_{nj} \in \left[C\left(F_{nk_n}^t\right), C\left(F_{n(k_n+1)}^t\right)\right] \end{array} \tag{21}$$

where for each $j = 1, 2, \ldots, m$, $k_i = 1, 2, \ldots, (c_i - 1)$, $(1 \leqslant i \leqslant n)$.The activated rules (COs), i.e., the group of those COs where the membership function of each alternative A_j $(1 \leqslant j \leqslant m)$ is nonzero is

$$\begin{array}{l} \left(C\left(F_{1k_1}^t\right), C\left(F_{2k_2}^t\right), \ldots, C\left(F_{nk_n}^t\right)\right) \\ \left(C\left(F_{1k_1}^t\right), C\left(F_{2k_2}^t\right), \ldots, C\left(F_{n(k_n+1)}^t\right)\right) \\ \quad\vdots \\ \left(C\left(F_{1(k_1+1)}^t\right), C\left(F_{2(k_2+1)}^t\right), \ldots, C\left(F_{n(k_n+1)}^t\right)\right) \end{array} \tag{22}$$

The number of COs is obviously 2^n and $1 \leqslant 2^n \leqslant s$.

Let $p_1, p_2, \ldots, p_{2^n}$ be the approximate values of preference of the activated rules (COs) which were already calculated in Step 3.

We denote the HFE at the point $x \in A_j$ $(1 \leqslant j \leqslant m)$ as

$$h_{ij}(x) = \left\{F_{ij}^1(x), F_{ij}^2(x), \ldots, F_{ij}^q(x)\right\} \tag{23}$$

for each criterion C_i $(i = 1, 2, .., n)$.

Let $\mathbf{A}_j$ be HFE which is computed as sum of the product of all activated rules, as their fulfillment degrees and their values of the preference, i.e.

$$\mathbf{A}_j = \left(h_{1k_1}(a_{1j}) \otimes h_{2k_2}(a_{2j}) \otimes \ldots h_{nk_n}(a_{nj})\right)p_1 \oplus \left(h_{1k_1}(a_{1j}) \otimes h_{2k_2}(a_{2j}) \otimes \ldots h_{n(k_n+1)}(a_{nj})\right)p_2 \oplus \ldots \left(h_{1(k_1+1)}(a_{1j}) \otimes h_{2(k_2+1)}(a_{2j}) \otimes \ldots h_{n(k_n+1)}(a_{nj})\right)p_{2^n} \quad (24)$$

The preference of each alternative A_j $(1 \leqslant j \leqslant m)$ can be found by finding the score of the corresponding HFE $\mathbf{A}_j (1 \leqslant j \leqslant m)$ as follows:

$$Sc(\mathbf{A}_j) = \frac{1}{l_{\mathbf{A}_j}} \sum_{y \in \mathbf{A}_j} y \quad (25)$$

Rank the alternatives in accordance with the preference values of each alternative. Greater the preference value, better the alternative A_j $(1 \leqslant j \leqslant m)$.

4 Illustrative Example

In this section, we study the same problem as in [37]. The decision problem is defined as a ranking of the electrical resistance of 12 alternatives with respect to two criteria, the electric current C_1 and the potential difference C_2. On the basis of Ohm's law $R = \frac{\text{Potentialdifference}}{\text{Current}} = \frac{U}{I}$, the resistance of an alternative can be easily obtained. This law is a perfect reference for the true ranking of selected alternatives. Table 1 presents the group of alternatives, values of the potential difference, values of the electric current, values of the resistance and the original ranking (a smaller resistance is better), which is reference to the rest of the ranking.

Suppose F_1^1, F_1^2 and F_1^3 are three different families of subsets of $\mathscr{F}$ for the criteria C_1 where

Table 1 Original ranking of alternatives (by Ohm's law)

Alternatives	Current A	Voltage V	Resistance Ω	Original rank
A_1	0.125	5	40	8
A_2	0.125	10	80	9
A_3	0.125	20	160	10
A_4	0.125	30	240	11
A_5	1	5	5	3
A_6	1	10	10	5
A_7	1	20	20	6
A_8	1	30	30	7
A_9	4	5	1.25	1
A_{10}	4	10	2.5	2
A_{11}	4	20	5	3
A_{12}	4	30	7.5	4

$$\begin{aligned} F_1^1 &= \{F_{11}^1, F_{12}^1, F_{13}^1\} = \{(0.1, 0.1, 1.5),\ (0.1, 1.5, 4.1),\ (1.5, 4.1, 4.1)\} \\ F_1^2 &= \{F_{11}^2, F_{12}^2, F_{13}^2\} = \{(0.1, 0.1, 0.1, 1.3),\ (0.1, 1.3, 2.5, 4.1),\ (2.5, 4.1, 4.1, 4.1)\} \\ F_1^3 &= \{F_{11}^3, F_{12}^3\} = \{0.1934A^2 - 0.8224A + 1.0247,\ -0.1934A^2 + 0.8224A - 0.0247\} \end{aligned} \quad (26)$$

Similarly, suppose the families F_2^1, F_2^2 and F_2^3 of subsets of $\mathscr{F}$ for the criteria C_2 are:

$$\begin{aligned} F_2^1 &= \{F_{21}^1, F_{22}^1, F_{23}^1\} = \{(3, 3, 15),\ (3, 15, 33),\ (15, 33, 33)\} \\ F_2^2 &= \{F_{21}^2, F_{22}^2, F_{23}^2\} = \{(3, 3, 3, 13),\ (3, 13, 18, 33),\ (18, 33, 33, 33)\} \\ F_2^3 &= \{F_{21}^3, F_{22}^3\} = \{-0.0038V^2 + 0.1359V - 0.3732,\ 0.0038V^2 - 0.1359V + 1.3732 \end{aligned} \quad (27)$$

The graphs of L–R-type generalized fuzzy numbers of the families mentioned above for both the criteria C_1 and C_2 are shown in Figs. 1 and 2, respectively. We can see that each element from criterion domain has a set of possible membership degree values. The expert identified three membership functions for each criterion.

The set of cores of F_1^1 and F_2^1 are, respectively $C(F_1^1) = \{0.1, 1.5, 4.1\}$ and $C(F_2^1) = \{3, 15, 33\}$. The solution of the COMET is obtained for different number of COs. The simplest solution involves the use of nine COs which are presented as follows (27):

$$\begin{aligned} &\mathrm{CO}_1 = \{0.1, 3\}, \mathrm{CO}_2 = \{0.1, 15\}, \mathrm{CO}_3 = \{0.1, 33\}, \\ &\mathrm{CO}_4 = \{1.5, 3\}, \mathrm{CO}_5 = \{1.5, 15\}, \mathrm{CO}_6 = \{1.5, 33\}, \\ &\mathrm{CO}_7 = \{4.1, 3\}, \mathrm{CO}_8 = \{4.1, 15\}, \mathrm{CO}_9 = \{4.1, 33\}. \end{aligned} \quad (28)$$

To rank and evaluate the COs, suppose the expert gives his/her assessments by providing the following HFEs:

$$\tilde{h}_s = \{0.8, 1\},\ \tilde{h}_w = \{0, 0.2\},\ \tilde{h}_f = \{0.5\} \quad (29)$$

The matrix of expert judgement (MEJ) is given in Table 2.

On the basis of MEJ, the vector SJ is obtained as follows:

$$\mathrm{SJ} = [0.991219, 0.952583, 0.760839, 0.999996, 0.999552, 0.997900, 0.999999, 0.999983, 0.999900]^{\mathrm{T}} \quad (30)$$

Normalize the vector SJ, we obtain a vertical vector P which transforms to approximate values of the preference for the generated COs as follows:

$$P = [0.25, 0.125, 0, 0.875, 0.5, 0.375, 1, 0.75, 0.625]^{\mathrm{T}} \quad (31)$$

Each CO and the value of preference p_i is converted to a fuzzy rule, as follows:

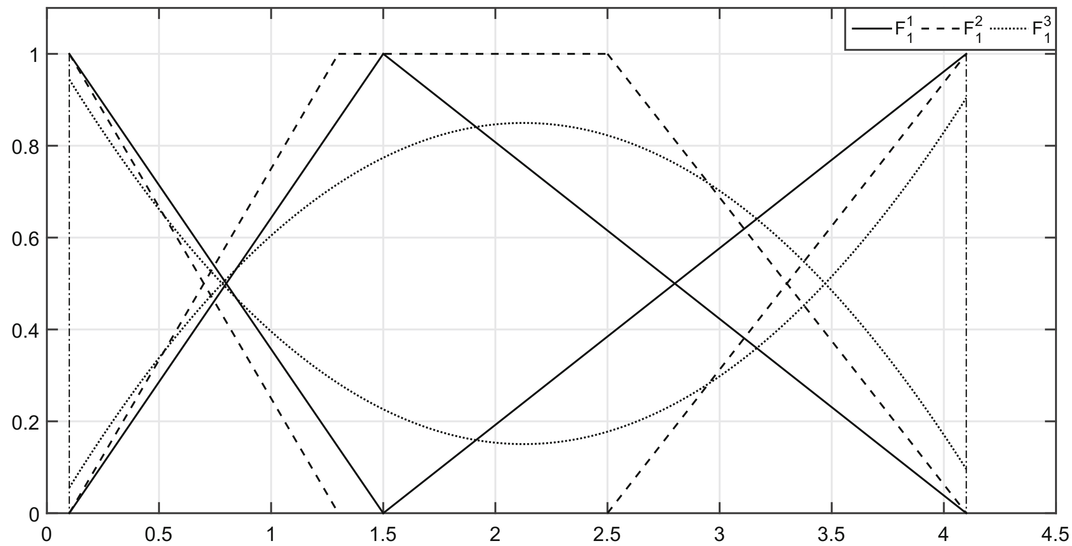

Fig. 1 Graphs of *L–R*-type generalized fuzzy numbers for the criterion C_1

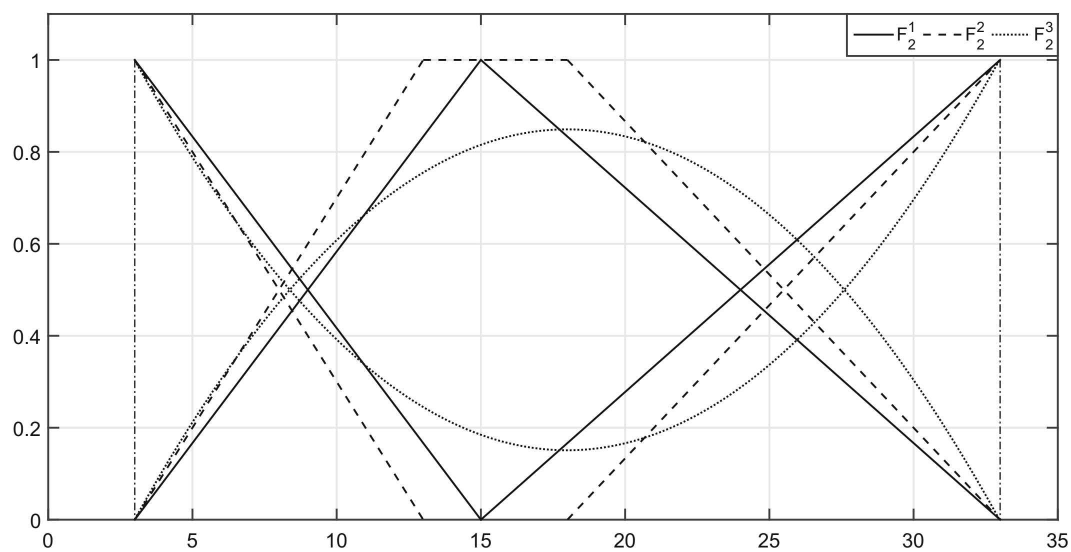

Fig. 2 Graphs of *L–R*-type generalized fuzzy numbers for the criterion C_2

Table 2 Matrix of expert judgment (MEJ)

	CO_1	CO_2	CO_3	CO_4	CO_5	CO_6	CO_7	CO_8	CO_9	SJ
CO_1	$\tilde{h}_f$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	0.991219
CO_2	$\tilde{h}_w$	$\tilde{h}_f$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	0.952583
CO_3	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_f$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	0.760839
CO_4	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_f$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_s$	$\tilde{h}_s$	0.999996
CO_5	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_f$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	0.999552
CO_6	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_f$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_w$	0.997900
CO_7	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_f$	$\tilde{h}_s$	$\tilde{h}_s$	0.999999
CO_8	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_f$	$\tilde{h}_s$	0.999983
CO_9	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_s$	$\tilde{h}_s$	$\tilde{h}_w$	$\tilde{h}_w$	$\tilde{h}_f$	0.999900

IF	$A \sim 0.1$	AND	$V \sim 3$	THEN	$P \sim 0.25$
IF	$A \sim 0.1$	AND	$V \sim 15$	THEN	$P \sim 0.125$
IF	$A \sim 0.1$	AND	$V \sim 33$	THEN	$P \sim 0$
IF	$A \sim 1.5$	AND	$V \sim 3$	THEN	$P \sim 0.875$
IF	$A \sim 1.5$	AND	$V \sim 15$	THEN	$P \sim 0.5$
IF	$A \sim 1.5$	AND	$V \sim 33$	THEN	$P \sim 0.375$
IF	$A \sim 4.1$	AND	$V \sim 3$	THEN	$P \sim 1$
IF	$A \sim 4.1$	AND	$V \sim 15$	THEN	$P \sim 0.75$
IF	$A \sim 4.1$	AND	$V \sim 33$	THEN	$P \sim 0.625$

(32)

In respect of Model (32) for the alternative $A_1 = \{0.125, 5\}$, we have nine rules (COs), but the activated rules are CO_1, CO_2, CO_4, CO_5. The approximate values of preference of corresponding COs are $p_1 \sim 0.25$, $p_2 \sim 0.125$, $p_4 \sim 0.875$, $p_5 \sim 0.500$. Since $0.125 \in [C(F^1_{11}), C(F^1_{12})]$, $5 \in [C(F^1_{21}), C(F^1_{22})]$. The corresponding HFE $\mathbf{A}_1$ and the preference value of the alternative A_1 are given, respectively, as follows:

$$\mathbf{A}_1 = p_1(h_{11}(0.125) \otimes h_{21}(5)) \oplus p_2(h_{11}(0.125) \otimes h_{22}(5)) \oplus p_4(h_{12}(0.125) \otimes h_{21}(5)) \oplus p_5(h_{12}(0.125) \otimes h_{22}(5)) \quad (33)$$

$$Sc(\mathbf{A}_1) = \frac{1}{l_{\mathbf{A}_1}} \sum_{y \in \mathbf{A}_1} y = 0.3513 \quad (34)$$

Table 3. presents the detailed preference values and rankings for considered alternatives by using the TOPSIS method, the classical COMET (TFNs) and the proposed extension (HFSs). Calculation details for TOPSIS and classical COMET are presented in [37]. We can see that extended and classical COMET have very similar rankings. Differences are observed in the order of alternatives pairs $A_{10} - A_{11}$ and $A_1 - A_8$. The reason is that the range of uncertainty for membership values of these two alternatives was quite high, e.g., difference between the highest and lowest membership values from h_{22} for A_{11} is equal to 0.1443. This fact may explain the observed differences in rankings. However, it is natural that increasing level of uncertainty makes it difficult to find the optimal ranking. In the presented example, the ranking obtained by TOPSIS method is definitely worse than the other. Additionally, we calculate the most popular measures of similarity degree between each obtained ranking and reference ranking (results are presented in Table 4). The all measures show the same relationship between rankings, i.e., Spearman's ρ, Kendall's τ and Gamma γ values are the highest for the classical COMET and the worse for TOPSIS. This comparison confirms that rankings obtained by using classical and extended COMET are better than ranking obtained by TOPSIS.

Table 4 Comparison of rank correlation measurement (in respect of original ranking)

The used method	Measure of rank correlation		
	Spearman's ρ	Kendall's τ	Gamma γ
Classical COMET	0.9877	0.9692	0.9619
Proposed extension	0.9702	0.9077	0.9008
TOPSIS	0.9017	0.8125	0.8000

Table 3 Comparison of results between TOPSIS, COMET (using TFNs and HFNs) and the original ranking

Alter	Original rank by Ohm's law	Pref. values using TOPSIS	Pref. values using TFNs	Pref. values using HFSs	Ranking using TOPSIS	Ranking using TFNs	Ranking using HFSs
A_1	8	0.5000	0.3027	0.3513	6	9	8
A_2	9	0.4396	0.1754	0.2231	7	10	10
A_3	10	0.2554	0.1039	0.1897	9	11	11
A_4	11	0.0000	0.0377	0.0452	11	12	12
A_5	3	0.5697	0.6647	0.5645	4	5	5
A_6	5	0.5097	0.4937	0.4419	5	6	6
A_7	6	0.3192	0.3971	0.381	8	7	7
A_8	7	0.1515	0.3162	0.3061	10	8	9
A_9	1	1.0000	0.9493	0.7866	1	1	1
A_{10}	2	0.8649	0.8451	0.6703	2	2	3
A_{11}	3	0.6422	0.7004	0.6889	3	3	2
A_{12}	4	0.5000	0.6029	0.6248	6	4	4

5 Conclusion

The main contribution of the paper is a proposal of the new extension of the COMET method of decision making under uncertainty. For this purpose, the hesitant fuzzy set theory is used, which is a generalization of fuzzy set theory. The hesitant fuzzy set theory is a useful tool to deal with uncertainty in decision-making problems, which is proved by many scientific papers. This approach represents the situation in which different membership functions are considered possible in respect of decision situation.

The paper presents a theoretical foundation of proposed approach, which ensures that a new extension is free of rank reversal phenomenon and allows for making decisions under imperfect information from experts. This approach facilitates a decision making under uncertainty because it permits establishing a membership degree as a set of possible values. The proposed approach is also included in accordance with actual research trends in the terms of methodological backgrounds (actuality of HFS in decision making) as well MCDM methods development directions.

The result of the presented numeric example is compared with the TOPSIS method and the classical COMET approach. Despite the fact that uncertainty appeared in the expert's answers, the final ranking is very convergent to the original. This means that the hesitant fuzzy set can reflect decision hesitancy more completely than the classical fuzzy sets.

During the research, some improvement areas have been identified. The future work directions should concentrate on:

– Practical exploitation of the application areas of proposed extension and wider comparison of the obtained results with classical COMET method.
– Searching for more accurate dealing with uncertainty data (i.e., data that contain noise that makes it deviate from the correct, intended or original values).
– Preparing a complete, COMET based, decision support system with knowledge base, including practical cases.

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