Determination of the blow-up rate for a critical semilinear wave equation

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Abstract : In this paper, we determine the blow-up rate for the semilinear wave equation with critical power nonlinearity related to the conformal invariance.

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1 introduction

We are concerned in this paper with blow-up solutions for the following semilinear wave equation:

$$\begin{cases} u_{tt} = \Delta u + |u|^{p-1}u, \\ u(0) = u_0 \text{ and } u_t(0) = u_1, \end{cases} \tag{1}$$

where $u(t) : x \in \mathbb{R}^N \to u(x,t) \in \mathbb{R}$, $u_0 \in \mathrm{H}^1_{\mathrm{loc,u}}$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc,u}}$. The space $\mathrm{L}^2_{\mathrm{loc,u}}$ is the set of all v such that $\|v\|^2_{\mathrm{L}^2_{\mathrm{loc,u}}} \equiv \sup_{a\in\mathbb{R}^N} \int_{|x-a|<1} |v(x)|^2 dx < +\infty$ and the space $\mathrm{H}^1_{\mathrm{loc,u}} = \{v \mid v, \nabla v \in \mathrm{L}^2_{\mathrm{loc,u}}\}$.

The Cauchy problem for equation in the space $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$ follows from the finite speed of propagation and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2(\mathbb{R}^N)$ for $1 < p < \frac{N+2}{N-2}$. See for instance Lindblad and Sogge [11], Shatah and Struwe [18] and their references (for the local in time wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$). The existence of blow-up solutions for equation (1) is a consequence of the finite speed of propagation and ODE techniques (see for example John [8], Caffarelli and Friedman [4], Alinhac [1], Kichenassamy and Littman [9], [10]). Given a solution u of (1) that blows up at time $T > 0$, we aim at understanding the behavior of its blow-up norm in $\mathrm{H}^1_{\mathrm{loc,u}}$.

Unlike previous work where the considered question was to construct blow-up solutions with explicit blow-up behavior (for example, the authors construct in [9] and [10] a solution that blows-up on a prescribed analytic space-like hypersurface), the question we address in this paper is about classification of blow-up behavior (see for example Merle and Raphaël [13], [12], [14] for the nonlinear Schrödinger equation (NLS), see Giga and Kohn [6], and Merle and Zaag [16] for the semilinear heat equation). By classification, we mean that we consider an arbitrary blow-up solution and we want to know about its properties, in particular, the blow-up rate in H^1. Previous results about this question are for special initial data (the result of Caffarelli and Friedman [4] holds for $N \leq 3$ and initial data that ensure that $u \geq 0$ an $\partial_t u > |\nabla u|$; the result of Antonini and Merle [3] is for positive solutions with restrictions on the exponent). Under these special conditions, authors in earlier work prove that the blow-up rate is given by the associated ODE ($u'' = u^p$).

In [17], we have determined the blow-up rate in the *subcritical* case

$$1 < p < p_c \equiv 1 + \frac{4}{N-1}.$$

Let us note that very few things were known for the semilinear equation (1). More results are available for quasilinear wave equations, (see Alinhac [1]). More precisely, we showed in [17] that the blow-up rate of u is given by v, the solution of the associated ODE :

$$v_{tt} = v^p, \ \ v(T) = +\infty,$$

that is $v(t) \sim \kappa (T-t)^{-\frac{2}{p-1}}$ where $\kappa = \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}$. If we introduce for each $a \in \mathbb{R}^N$ the following self-similar change of variables :

$$w_a(y,s) = (T-t)^{\frac{2}{p-1}} u(x,t), \ \ y = \frac{x-a}{T-t}, \ \ s = -\log(T-t), \tag{2}$$

then our result for $1 < p < p_c$ [17] reads as:

For all $a \in \mathbb{R}^N$ *and* $s \geq -\log T + 1$,

$$\epsilon_0(N,p) \leq \sup_{a \in \mathbb{R}^N} \|w_a(s)\|_{\mathrm{H}^1(B(0,1))} + \|\partial_s w_a(s)\|_{\mathrm{L}^2(B(0,1))} \leq K$$

where K *depends only on* N, p *and on bounds on* T *and the norm of initial data.*

Let us mention that our result in this paper and in [17] is a fundamental step towards the proof given in [15] of the same rates near the space-time blow-up curve $a \to T(a)$, for general solutions. The result of [15] generalizes the result of Caffarelli and Friedman [4] which was valid under restrictive conditions on initial data.

In this paper, we consider the critical case of our previous work [17]

$$p = p_c \equiv 1 + \frac{4}{N-1} \text{ where } N \geq 2. \tag{3}$$

Thus, equation (1) becomes

$$u_{tt} = \Delta u + |u|^{\frac{4}{N-1}} u \tag{4}$$

and we are able to show the same estimates as in the subcritical case. We claim the following:

Theorem 1 *If* u *is a solution of (4) that blows up at time* T*, then there exist* $\epsilon_N > 0$ *and* $K > 0$ *which depends only on* N *and on bounds on* T *and the initial data in* $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$

such that:
i) **(Uniform bounds on** w**)** *For all $a \in \mathbb{R}^N$ and $s \geq -\log T + 1$,*

$$\epsilon_N \leq \sup_{a\in\mathbb{R}^N} \|w_a(s)\|_{\mathrm{H}^1(B)} + \|\partial_s w_a(s)\|_{\mathrm{L}^2(B)} \leq K$$

where w_a is defined in (2) and B is the unit ball of $\mathbb{R}^N$.
ii) **(Uniform bounds on** u**)** *For all $t \in [T(1-e^{-1}), T)$,*

$$\epsilon_N \leq (T-t)^{\frac{2}{p_c-1}} \|u\|_{\mathrm{L}^2_{\mathrm{loc,u}}} + (T-t)^{\frac{2}{p_c-1}+1} \left(\|u_t\|_{\mathrm{L}^2_{\mathrm{loc,u}}} + \|\nabla u\|_{\mathrm{L}^2_{\mathrm{loc,u}}} \right) \leq K$$

Remark: Let us remark that as in [17] for the subcritical case, the lower bound in the theorem follows by standard techniques from scaling arguments and the wellposedness in $\mathrm{H}^1 \times \mathrm{L}^2$.
Remark: In blow-up problems, there are two key questions : the blow-up rate and the blow-up profile. The blow-up rate is the first step towards the obtaining of the blow-up profile. See Antonini [2], where the first results about the blow-up profile are derived using estimates from Theorem 1.

In the subcritical case [17], the proof relies on three ideas:
- the existence of a Lyapunov functional in the $w(y,s)$ formulation and some energy estimates related to this structure,
- the improvement of regularity estimates by interpolation,
- some Gagliardo-Nirenberg type argument.

It happens that the dissipation of the Lyapunov functional becomes degenerate when $p = p_c$. This is one of some major difficulties in adapting the proof to the critical case. Notice that in the supercritical case, the Lyapunov functional is not even well-defined, which makes $p = p_c$ a critical case. The second major difficulty comes from the fact that critical exponents appear in the Gagliardo-Nirenberg argument. It will be overcome by a non concentration property of the solution. We will explain in each step of the proof the difficulties related to the criticality and how to deal with them. Let us remark that our paper is not just a technical adaptation of the subcritical case of [17]. Indeed, even though we follow the same pattern as in [17], we introduce in our paper new ideas such as an averaging estimate in Proposition 2.4, a non concentration estimate in Proposition 3.1, and a sharp use of a local Gagliardo-Nirenberg estimate below a critical threshold. For the reader's convenience, we emphasize in this paper the new ideas and omit technical steps which are similar to [17] (however, we keep the very short similarities for the sake of preserving a clear structure in the proof). Note also another degeneracy in the problem due to the conformal invariance of the equation (4). More precisely, if $p = p_c$ and $U(\xi,\tau)$ is defined by

$$U(\xi,\tau) = (|x|^2 - t^2)^{\frac{N-1}{2}} u(x,t), \quad \xi = \frac{x}{|x|^2 - t^2}, \quad \tau = \frac{t}{|x|^2 - t^2}, \tag{5}$$

then U satisfies the same equation (4) as u. Nevertheless, this conformal invariance does not change the fact that the solution blows up in a self similar way and will not be relevant for the proof we present. We would like to emphasize the fact that for $p \leq p_c$, the Lyapunov functional we introduce is different from the conformal energy and not related to it.

The wave equation has no new blow-up rate at the critical case, unlike NLS where new blow-up rates appear at the critical case (with respect to the conformal invariance), see Merle and Raphaël [13], [12], [14], or the semilinear heat equation where we have new blow-up rates at the critical case (with respect to Sobolev injection), see Filippas, Herrero and Velázquez [5].

2 Local energy estimates

Throughout this section, w stands for any w_a defined in (2) where $a \in \mathbb{R}^N$. All estimates we obtain are in fact independent of $a \in \mathbb{R}^N$. We denote by C_0 a constant which depends only on bounds on T and the initial data of (4) in $\mathrm{H}^1_{\mathrm{loc,u}} \times \mathrm{L}^2_{\mathrm{loc,u}}$. We denote the unit ball of $\mathbb{R}^N$ by B and its boundary by ∂B.

2.1 A Lyapunov functional in the $w(y,s)$ formulation

As in the subcritical case, the function w defined in (2) satisfies the following equation for all $y \in \mathbb{R}^N$ and $s \in (-\log T, +\infty)$:

$$\partial^2_{ss} w - \mathrm{div}\,(\nabla w - (y.\nabla w) y) + \frac{2(p_c+1)}{(p_c-1)^2} w - |w|^{p_c-1} w = -\frac{p_c+3}{p_c-1}\partial_s w - 2y.\nabla \partial_s w. \tag{6}$$

Note also that s goes to infinity as t goes to T.

As we have recalled in the introduction, the starting point of the analysis in [3] and [17] was a dispersive effect in the light cone with vertex (a,T) in the $u(x,t)$ formulation, which reads in the $w(y,s)$ variable as the existence of a Lyapunov functional called $E(w)$ defined for y in the unit ball.
Note that E is not even defined for $p > p_c$, which makes $p = p_c$ a critical case. For $p = p_c$, E is well defined and decreases in time. However, its dissipation degenerates (the dissipation is supported on ∂B and not on B). More precisely,

Lemma 2.1 *If we define*

$$E(w) = \int_B \left(\frac{1}{2}(\partial_s w)^2 + \frac{1}{2}|\nabla w|^2 - \frac{1}{2}(y.\nabla w)^2 + \frac{(p_c+1)}{(p_c-1)^2} w^2 - \frac{1}{p_c+1}|w|^{p_c+1} \right) dy, \tag{7}$$

then for all $s_1 < s_2$,

$$E(w(s_2)) - E(w(s_1)) = -\int_{s_1}^{s_2} \int_{\partial B} \partial_s w(\sigma, s)^2 d\sigma ds.$$

Remark: The functional $E(w)$ is different from the energy of the conformal transformation of u defined in (5).

Proof of Lemma 2.1: The proof is the same as in the subcritical case [3], except for one integration by parts where the boundary term can no longer be neglected. If we multiply equation (6) by $\partial_s w$ and integrate over B, then we get:

$$\begin{aligned} &\int_B dy \partial_s w \left(\partial^2_s w - \mathrm{div}\,(\nabla w - (y.\nabla w) y) + \frac{2(p_c+1)}{(p_c-1)^2} w - |w|^{p_c-1} w \right) \\ = &-\int_B dy \partial_s w \left(\frac{p_c+3}{p_c-1} \partial_s w + 2y.\nabla \partial_s w \right). \end{aligned} \tag{8}$$

As in the subcritical case [3], the left-hand side is simply $\frac{d}{ds}E(w(s))$. As for the right-hand side, we integrate by parts as follows (note that for $\sigma \in \partial B$, the outer normal is σ and $|\sigma| = 1$):

$$
\begin{aligned}
& -\int_B dy \partial_s w \left(\frac{p_c+3}{p_c-1}\partial_s w + 2y.\nabla \partial_s w \right) = -\frac{p_c+3}{p_c-1}\int_B (\partial_s w)^2 - \int_B y.\nabla (\partial_s w)^2 \\
= & -\frac{p_c+3}{p_c-1}\int_B (\partial_s w)^2 + \int_B \operatorname{div} y\, (\partial_s w)^2 - \int_{\partial B} d\sigma \sigma.\sigma\, (\partial_s w(\sigma,s))^2 \\
= & -\int_{\partial B} d\sigma\, (\partial_s w(\sigma,s))^2
\end{aligned}
$$

because $\operatorname{div} y \equiv N = \frac{p_c+3}{p_c-1}$ (see (3)). Therefore, (8) implies

$$\frac{d}{ds}E(w(s)) = -\int_{\partial B} d\sigma\, (\partial_s w(\sigma,s))^2,$$

which yields the conclusion of Lemma 2.1 by integration in time. ■

This degeneracy will change quite a bit the further analysis of the problem, nevertheless, let us recall some common features with the subcritical case.

Corollary 2.2 (Blow-up criterion for equation (6)) *If a solution W of equation (6) satisfies $E(W(s_0)) < 0$ for some $s_0 \in \mathbb{R}$, then W blows up in finite time $S^* > s_0$.*

Proof: See [3]. ■

Since w is by definition defined for all $s \geq -\log T$, we get the following bounds:

Corollary 2.3 (Energy bounds) *For all $a \in \mathbb{R}^N$, $s \geq -\log T$, $s_2 \geq s_1 \geq -\log T$, the following identities hold:*

$$0 \leq E(w_a(s)) \leq E(w_a(-\log T)) \leq C_0, \tag{9}$$

$$\int_{s_1}^{s_2}\int_{\partial B} \partial_s w_a(y,s)^2 dyds \leq C_0. \tag{10}$$

2.2 Control of $\partial_s w$ on B

In this step, we overcome the first major difficulty coming from criticality. As we have seen, we have in (10) an estimate of $\partial_s w$ on ∂B and not on B as it is the case in the subcritical case. Nevertheless, the fact that (10) is uniform with respect to $a \in \mathbb{R}^N$ allows us through an averaging formula to obtain an estimate of $\partial_s w$ on B.

Proposition 2.4 (Control of $\partial_s w$ on B) *For all $a \in \mathbb{R}^N$ and $s_2 \geq s_1 \geq -\log T$ such that $s_2 - s_1 \leq 10$,*

$$\int_{s_1}^{s_2}\int_B \partial_s w_a(y,s)^2 dyds \leq C_0. \tag{11}$$

Proof: From translation invariance, we can take $a = 0$. Note that the definition of w_a implies that for all $b \in \mathbb{R}^N$, $y \in \mathbb{R}^N$ and $s \geq -\log T$,

$$w_b(y,s) = w_0(y+be^s,s) \text{ and } \partial_s w_b(y,s) = \partial_s w_0(y+be^s,s) + be^s.\nabla w_0(y+be^s,s).$$

Therefore, for all $z \in \mathbb{R}^N$, $s \geq -\log T$ and $b \in \mathbb{R}^N$,

$$\partial_s w_0(z,s) + be^s.\nabla w_0(z,s) = \partial_s w_b(z-be^s,s). \tag{12}$$

If we introduce for all σ and z in $\mathbb{R}^N$,

$$P(\sigma,z) = 1 - \frac{\sigma.z}{|z|^2+1/N}, \tag{13}$$

then we see from straightforward computations (see Appendix A for details) that

$$\forall z \in \mathbb{R}^N,\ \int_{\partial B(z,1)} d\sigma P(\sigma,z)\sigma \ = \ 0, \tag{14}$$

$$\forall z \in \mathbb{R}^N,\ \frac{\int_{\partial B(z,1)} d\sigma P(\sigma,z)^2}{\left(\int_{\partial B(z,1)} d\sigma P(\sigma,z)\right)^2} \ = \ \left(\int_{\partial B(z,1)} d\sigma P(\sigma,z)\right)^{-1} = \frac{N|z|^2+1}{|\partial B|}. \tag{15}$$

Note that (14) means that for all $z \in \mathbb{R}^N$, $1-P(.,z)$ is the projection of the constant function 1 on the subspace of $\mathrm{L}^2(\partial B(z,1))$ generated by σ_1, ..., σ_N.

Let us express $\partial_s w_0(z,s)$ for given $z \in \mathbb{R}^N$ and $s \geq -\log T$ as an average on the sphere $\partial B(z,1)$. Using (14), we have

$$\begin{aligned}&\int_{\partial B(z,1)} d\sigma P(\sigma,z)\left(\partial_s w_0(z,s) + \sigma.\nabla w_0(z,s)\right)\\
= \ &\partial_s w_0(z,s)\int_{\partial B(z,1)} d\sigma P(\sigma,z) + \nabla w_0(z,s).\int_{\partial B(z,1)} d\sigma P(\sigma,z)\sigma = \partial_s w_0(z,s)\int_{\partial B(z,1)} d\sigma P(\sigma,z).\end{aligned}$$

Therefore, using the Cauchy-Schwarz inequality, we write for all $z \in B(0,3)$ and $s \geq -\log T$,

$$\begin{aligned}(\partial_s w_0(z,s))^2 \ &= \ \frac{1}{\left(\int_{\partial B(z,1)} d\sigma P(\sigma,z)\right)^2}\left(\int_{\partial B(z,1)} d\sigma P(\sigma,z)\left(\partial_s w_0(z,s) + \sigma.\nabla w_0(z,s)\right)\right)^2\\
&\leq \ \frac{\int_{\partial B(z,1)} d\sigma P(\sigma,z)^2}{\left(\int_{\partial B(z,1)} d\sigma P(\sigma,z)\right)^2}\int_{\partial B(z,1)} d\sigma\left(\partial_s w_0(z,s) + \sigma.\nabla w_0(z,s)\right)^2\end{aligned}$$

Using (15), the change of variables $\sigma = be^s$ and the identity (12), we write

$$\begin{aligned}(\partial_s w_0(z,s))^2 \ &\leq \ \frac{N|z|^2+1}{|\partial B|}e^{(N-1)s}\int_{\partial B(ze^{-s},e^{-s})} db\left(\partial_s w_0(z,s) + be^s.\nabla w_0(z,s)\right)^2,\\
&\leq \ Ce^{(N-1)s}\int_{\partial B(ze^{-s},e^{-s})} db\left(\partial_s w_b(z-be^s,s)\right)^2.\end{aligned} \tag{16}$$

If we integrate this identity for $(z,s)\in B\times[s_1,s_2]$ and use Fubini's property, then we get

$$\begin{aligned}
\int_{s_1}^{s_2} ds\int_B dz\,(\partial_s w_0(z,s))^2 &\le C\int_{s_1}^{s_2} ds\int_B dz e^{(N-1)s}\int_{\partial B(ze^{-s},e^{-s})} db\,(\partial_s w_b(z-be^s,s))^2\\
&= C\int_{s_1}^{s_2} ds\int_{B(0,2e^{-s})} db e^{Ns}\int_{\partial B(be^s,1)\cap B(0,1)} dz\,(\partial_s w_b(z-be^s,s))^2\\
&= C\int_{B(0,2e^{-s_1})} db\int_{s_1}^{s_2} ds 1_{\{|b|<2e^{-s}\}}e^{Ns}\int_{\partial B(be^s,1)\cap B(0,1)} dz\,(\partial_s w_b(z-be^s,s))^2\\
&\le Ce^{Ns_2}\int_{B(0,2e^{-s_1})} db\int_{s_1}^{s_2} ds\int_{\partial B(0,1)} dy\,(\partial_s w_b(y,s))^2
\end{aligned}$$

where we made the change of variables $y=z-be^s$. Using the estimate on the sphere stated in (10) with b, we end up with

$$\int_{s_1}^{s_2} ds\int_B dz\,(\partial_s w_0(z,s))^2 \le Ce^{Ns_2}\int_{B(0,2e^{-s_1})} d\sigma C_0 = C_0 e^{N(s_2-s_1)}\le C_0$$

if $s_2-s_1\le 10$. This concludes the proof of Proposition 2.4. ■

2.3 Space-time estimates for w

Unlike it may appear, this step is different from the corresponding one in [17]. Indeed, we knew in [17] from the energy dissipation that for $p<p_c$

$$\int_{s_1}^{s_2}\int_B (\partial_s w)^2\left(1-|y|^2\right)^{\alpha-1} dyds \le C_0 \text{ where } \alpha=\frac{2}{p-1}-\frac{N-1}{2}>0,$$

and we are unable to prove the same thing for $p=p_c$, a case which makes $\alpha=0$. We only have estimate (11), which is weaker. We claim the following:

Proposition 2.5 (Bound on space-time norms of the solution) *For all $a\in\mathbb{R}^N$ and $s\ge -\log T+1$, the following identities hold*

$$\int_s^{s+1}\int_B \left(|w_a|^{p_c+1}+\partial_s w_a(y,s)^2+|\nabla w_a|^2\right) dyds \le C_0,$$

$$\int_B w_a(y,s)^2 dy \le C_0.$$

Since the L^2 estimate of $\partial_s w$ has been shown in Proposition 2.4, we claim that this proposition can be reduced to the following:

Proposition 2.6 (Control of w in $L^{p_c+1}_{\rm loc}$ and ∇w in $L^2_{\rm loc}$) *For all $a\in\mathbb{R}^N$ and $s\ge -\log T+1$,*

$$\int_s^{s+1}\int_B \left(|w|^{p_c+1}dyds+|\nabla w|^2\right) dyds \le C_0.$$

Proposition 2.6 implies Proposition 2.5: It just remains to prove the estimate on $\int w^2$. We claim that for all $\epsilon_2 \in (0,1)$, $s_2 \geq s_1 \geq -\log T$ such that $1 \leq s_2 - s_1 \leq 3$,

$$\sup_{s_1 \leq s \leq s_2} \int_B w(y,s)^2 dy \leq \frac{C_0}{\epsilon_2} + C\epsilon_2 \int_{s_1}^{s_2} \int_B |w|^{p_c+1} dyds. \tag{17}$$

Indeed, as in [17], let $g(s) = \left(\int_B w(y,s)^2 dy\right)^{\frac{1}{2}}$. From the Sobolev injection in one dimension, it is enough to prove that $\|g\|_{\mathrm{L}^2(s_1,s_2)}$ and $\|\frac{dg}{ds}\|_{\mathrm{L}^2(s_1,s_2)}$ satisfy the desired bound in (17). Using Jensen's inequality, the Cauchy-Schwarz inequality and the estimate (11) on $\partial_s w$, we write

$$\begin{aligned}
\|g\|_{\mathrm{L}^2(s_1,s_2)} &= \left(\int_{s_1}^{s_2}\int_B w^2 dyds\right)^{\frac{1}{2}} \leq C\left(\int_{s_1}^{s_2}\int_B |w|^{p_c+1} dyds\right)^{\frac{1}{p_c+1}} \\
&\leq \frac{C}{\epsilon_2} + C\epsilon_2 \int_{s_1}^{s_2}\int_B |w|^{p_c+1} dyds. \\
\|\frac{dg}{ds}\|^2_{\mathrm{L}^2(s_1,s_2)} &= \int_{s_1}^{s_2} ds \frac{\left(\int_B w\partial_s w dy\right)^2}{4\int_B w^2 dy} \leq \frac{1}{4}\int_{s_1}^{s_2} ds \int_B (\partial_s w)^2 dy \leq C_0.
\end{aligned}$$

■

Proof of Proposition 2.6: Consider $s_2 \geq s_1 \geq -\log T$ such that $1 \leq s_2 - s_1 \leq 3$. Let us first derive two relations.

The first is obtained by integrating in time between s_1 and s_2, the expression (7) of $E(w)$:

$$\begin{aligned}
2\int_{s_1}^{s_2} E(w(s))ds &= \int_{s_1}^{s_2}\int_B \left((\partial_s w)^2 + \frac{2(p_c+1)}{(p_c-1)^2}w^2 - \frac{2}{p_c+1}|w|^{p_c+1}\right) dsdy \\
&+ \int_{s_1}^{s_2}\int_B \left(|\nabla w|^2 - (y.\nabla w)^2\right) dsdy.
\end{aligned} \tag{18}$$

We derive the second relation by multiplying the equation (6) by w, integrating both in time and space over $B \times (s_1, s_2)$, integrating by parts, and then by using (18) to eliminate the term $\iint |\nabla w|^2$ (see section 2.2 in [17] for a similar computation):

$$\begin{aligned}
\frac{(p_c-1)}{2(p_c+1)}\int_{s_1}^{s_2}\int_B |w|^{p_c+1} dyds &= \int_{s_1}^{s_2} E(w(s))ds + \int_{s_1}^{s_2}\int_B \left(-(\partial_s w)^2 - \partial_s w y.\nabla w\right) dyds \\
+ \ 2\int_{s_1}^{s_2}\int_{\partial B} \partial_s w w d\sigma ds &+ \frac{1}{2}\left[\int_B \left(w\partial_s w + \left(\frac{p_c+3}{2(p_c-1)} - N\right) w^2\right) dy\right]_{s_1}^{s_2}.
\end{aligned} \tag{19}$$

We claim that all the terms on the right hand side of the relation (19) can be controlled in terms of $\iint |w|^{p_c+1}$ and $\iint |\nabla w|^2$, which will allow us to conclude. Note that for $p < p_c$, everything could be controlled in terms of the first integral. Note also that we proceed as in the subcritical case, except for terms involving $\partial_s w$ or integrals on ∂B which have to be controlled differently. In particular, the following trace formula will be used for the control of the latter:

$$\text{For all } v \in \mathrm{H}^1(B), \ \ \|v\|_{\mathrm{L}^2(\partial B)} \leq C\|v\|_{\mathrm{H}^1(B)}. \tag{20}$$

We then conclude the proof in a different way from the subcritical case in order to control $\iint |w|^{p_c+1}$ and $\iint |\nabla w|^2$ at the same time. In the following, ϵ_1 and ϵ_2 are arbitrary numbers in $(0,1)$.

Step 1 : Control of the terms on the right hand side of the relation (19)
In this step, we prove the following identity: for all $\epsilon_1 > 0$,

$$\int_{s_1}^{s_2}\int_B |w|^{p_c+1}dyds \le \frac{C_0}{\epsilon_1} + \int_B \left(\partial_s w(y,s_1)^2 + \partial_s w(y,s_2)^2\right) dy + C_0\epsilon_1 \int_{s_1}^{s_2}\int_B |\nabla w|^2 dyds. \tag{21}$$

In the following we use the Cauchy-Schwarz inequality and the estimates (10) and (11) on $\partial_s w$.

a) Control of $\int_{s_1}^{s_2}\int_B \partial_s wy.\nabla wdyds$:

$$\begin{aligned}\left|\int_{s_1}^{s_2}\int_B \partial_s wy.\nabla wdyds\right| &\le \left(\int_{s_1}^{s_2}\int_B (\partial_s w)^2 dyds\right)^{1/2}\left(\int_{s_1}^{s_2}\int_B |\nabla w|^2 dyds\right)^{1/2}\\ &\le \frac{C_0}{\epsilon_1} + C_0\epsilon_1\int_{s_1}^{s_2}\int_B |\nabla w|^2 dyds. \end{aligned}\tag{22}$$

b) Control of $\int_{s_1}^{s_2}\int_{\partial B} \partial_s wwd\sigma ds$:

$$\begin{aligned}\left|\int_{s_1}^{s_2}\int_{\partial B} \partial_s wwd\sigma ds\right| &\le \left(\int_{s_1}^{s_2}\int_{\partial B} (\partial_s w)^2 d\sigma ds\right)^{1/2}\left(\int_{s_1}^{s_2}\int_{\partial B} w^2 d\sigma ds\right)^{1/2}\\ &\le \frac{C_0}{\epsilon_1} + C_0\epsilon_1\int_{s_1}^{s_2}\int_{\partial B} w^2 d\sigma ds.\end{aligned}$$

Using the trace identity (20) and the bound on the L^2 norm (17), we end up with

$$\left|\int_{s_1}^{s_2}\int_{\partial B} \partial_s wwd\sigma ds\right| \le \frac{C_0}{\epsilon_1} + C_0\epsilon_1\int_{s_1}^{s_2}\int_B |\nabla w|^2 dyds + \frac{C_0}{\epsilon_2} + C_0\epsilon_2\int_{s_1}^{s_2}\int_B |w|^{p_c+1}dyds. \tag{23}$$

c) Control of $\int_B w\partial_s wdy$:

$$\left|\int_B w\partial_s wdy\right| \le \int_B (\partial_s w)^2 dy + \int_B w^2 dy \le \int_B (\partial_s w)^2 dy + \frac{C}{\epsilon_2} + C\epsilon_2\int_{s_1}^{s_2}\int_B |w|^{p_c+1}dyds. \tag{24}$$

Now, we are able to conclude the proof of the identity (21) from the relation (19). For this, we bound all the terms on the right hand side of (19) (use (9), (22), (23), (24) and (17) for the other terms) to get:

$$\begin{aligned}\int_{s_1}^{s_2}\int_B |w|^{p_c+1}dyds &\le \frac{C_0}{\epsilon_2} + C_0\epsilon_2\int_{s_1}^{s_2}\int_B |w|^{p_c+1}dyds + \frac{C_0}{\epsilon_1} + C_0\epsilon_1\int_{s_1}^{s_2}\int_B |\nabla w|^2 dyds\\ &+ \int_B \left(\partial_s w(y,s_1)^2 + \partial_s w(y,s_2)^2\right) dy.\end{aligned}$$

Taking $\epsilon_2 = 1/2C_0$ yields identity (21). ■

Step 2 : Conclusion of the proof

From (21), we write for all $s_2 \geq s_1 \geq -\log T$ such that $1 \leq |s_2 - s_1| \leq 3$, for all $\tau \in [0,1]$,

$$\int_{s_1+\tau}^{s_2+\tau}\int_B |w|^{p_c+1}dyds \leq \frac{C_0}{\epsilon_1}+\int_B \left(\partial_s w(y,s_1+\tau)^2+\partial_s w(y,s_2+\tau)^2\right)dy+C_0\epsilon_1\int_{s_1+\tau}^{s_2+\tau}\int_B |\nabla w|^2 dyds.$$

Integrating this identity for $\tau \in (0,1)$, the two terms with $\int_B \partial_s w(y,s_1+\tau)^2 dy$ become bounded thanks to the estimate on $\partial_s w$ in (11) and we get for all $a \in \mathbb{R}^N$, $s_2 \geq s_1$ with $1 \leq s_2 - s_1 \leq 3$ and $\epsilon_1 \in (0,1)$,

$$\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|w_a(y,s)|^{p_c+1} \leq \frac{C}{\epsilon_1}+C\epsilon_1\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_a(y,s)|^2 \tag{25}$$

where $\chi_{s_1,s_2}(s) = \displaystyle\int_0^1 d\tau 1_{\{s_1+\tau\leq s\leq s_2+\tau\}}$. Note that

$$\chi_{s_1,s_2}(s) \geq 0 \text{ and } \chi_{s_1,s_2}(s) = 1 \text{ whenever } s_1+1 \leq s \leq s_2. \tag{26}$$

If $a_0 = a_0(s_1,s_2)$ is chosen such that

$$\begin{aligned} M(s_1,s_2) &\equiv \int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_{a_0}(y,s)|^2(1-|y|^2) \\ &\geq \frac{1}{2}\sup_{a\in\mathbb{R}^N}\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_a(y,s)|^2(1-|y|^2), \end{aligned} \tag{27}$$

then it holds through a covering argument and the relation between w_a and w_{a_0} that

$$\forall a \in \mathbb{R}^N,\quad \int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_a(y,s)|^2 \leq CM(s_1,s_2). \tag{28}$$

Indeed, fix an arbitrary $a \in \mathbb{R}^N$. Since $1-|y|^2 \geq \frac{3}{4}$ whenever $|y| \leq \frac{1}{2}$, we have from (27) that for all $b \in \mathbb{R}^N$,

$$\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_{B(0,\frac{1}{2})} dy|\nabla w_b(y,s)|^2 \leq CM(s_1,s_2).$$

Then, since the definition of w_b (2) implies that for all b, $y \in \mathbb{R}^N$ and $s \geq -\log T$,

$$w_b(y,s) = w_a(y+(b-a)e^s,s) \text{ and } \nabla w_b(y,s) = \nabla w_a(y+(b-a)e^s,s), \tag{29}$$

this yields for all $b \in \mathbb{R}^N$, $\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_{B((b-a)e^s,\frac{1}{2})} dz|\nabla w_a(z,s)|^2 \leq CM(s_1,s_2)$. Since $s_2 - s_1 \leq 3$, we can cover $B(0,1)\times[s_1,s_2]$ by a finite number (depending only on N) of domains of the type $\{(y,s) \mid s \in [s_1,s_2] \text{ and } y \in B((b-a)e^s,\frac{1}{2})\}$. Thus, estimate (28) follows.

If we use estimates (25) and (28) with $a = a_0$, then we get for all $\epsilon_1 > 0$,

$$\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|w_{a_0}(y,s)|^{p_c+1} \le \frac{C_0}{\epsilon_1} + C_0\epsilon_1 \int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_{a_0}(y,s)|^2(1-|y|^2).$$

Now, if we multiply the functional E with $a = a_0$ (7) by $\chi_{s_1,s_2}(s)$ and integrate in time, then we get

$$\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_{a_0}(y,s)|^2(1-|y|^2) \le C_0 + \frac{2}{p_c+1}\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|w_{a_0}(y,s)|^{p_c+1}.$$

Therefore,

$$\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_{a_0}(y,s)|^2(1-|y|^2) \le \frac{C_0}{\epsilon_1} + C_0\epsilon_1 \int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_{a_0}(y,s)|^2(1-|y|^2). \tag{30}$$

Using (30) with $\epsilon_1 = 1/2C_0$, we get $M(s_1, s_2) \le C_0$ and by (28)

$$\sup_{a\in\mathbb{R}^N}\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|\nabla w_a(y,s)|^2 \le C_0.$$

Therefore, (25) implies that $\sup_{a\in\mathbb{R}^N}\int_{s_1}^{s_2+1} ds\chi_{s_1,s_2}(s)\int_B dy|w_a(y,s)|^{p_c+1} \le C_0$. Using (26), it follows that $\sup_{a\in\mathbb{R}^N}\int_{s_1+1}^{s_2} ds\int_B dy\left(|w_a(y,s)|^{p_c+1} + |\nabla w_a(y,s)|^2\right) \le C_0$. Taking $s_1 = s-1$ and $s_2 = s+1$, this concludes the proof of Proposition 2.6. ■

3 Control of the $\mathrm{H}^1_{\mathrm{loc,u}}$ norm of the solution

In this section, we conclude the proof of Theorem 1. As in the subcritical case, we proceed in two steps:

In the first step, we use the uniform local bounds we obtained in the previous section to gain more control on the solution by interpolation (control of the $\mathrm{L}^q_{\mathrm{loc}}$ norm of the solution, where $q = \frac{p_c+3}{2}$).
Then, in the second step, for a given s and a given ball, we use a Gagliardo-Nirenberg inequality to interpolate the $\mathrm{L}^{p_c+1}_{\mathrm{loc}}$ norm between the $\mathrm{L}^q_{\mathrm{loc}}$ and the $\mathrm{H}^1_{\mathrm{loc}}$ norms.

Unfortunately, as stated, the subcritical argument breaks down. Indeed, the Gagliardo-Nirenberg inequality yields a critical exponent, which does not allow us to conclude, unless $\|w\|_{\mathrm{L}^q_{\mathrm{loc}}}$ is small in terms of the dimension.
The strategy to avoid this difficulty is to remove the possibility of concentration of the L^q norm first, and then to use the Gagliardo-Nirenberg inequality locally in a small ball where the local L^q norm is smaller than the critical constant.

We first claim that a covering argument allows us to extend the integration domain in Proposition 2.5 to $B(0,3)$ and get

$$\forall a \in \mathbb{R}^N,\ \ \forall s \ge -\log T + 1,\ \ \int_{B(0,3)} |w_a|^2 + \int_s^{s+1}\int_{B(0,3)} |w_a|^{p_c+1} dyds \le C_0. \tag{31}$$

Indeed, fix an arbitrary $a \in \mathbb{R}^N$ and recall from Proposition 2.5 that

$$\forall b \in \mathbb{R}^N,\ \ \forall s \geq -\log T,\ \ \int_B |w_b(y,s)|^2 dy + \int_s^{s+1} \int_B |w_b(y,s)|^{p_c+1} dyds \leq C_0.$$

Using (29), this yields

$$\forall b \in \mathbb{R}^N,\ \ \forall s \geq -\log T,\ \ \int_{B((b-a)e^s,1)} |w_a(z,s)|^2 dz + \int_s^{s+1} \int_{B((b-a)e^s,1)} |w_a(z,s)|^{p_c+1} dzds \leq C_0. \tag{32}$$

Since we can cover $B(0,3)$ with $k(N)$ balls of radius 1 and $B(0,3)\times[s,s+1]$ by $k'(N)$ sets of the form $\{(y,s)\ \mid\ s \in [s_1,s_2] \text{ and } y \in B((b-a)\,e^s,1)\}$, (31) follows from (32).

Step 1: Control of $w_a(s)$ in $\mathrm{L}^q_{\mathrm{loc}}$ (non concentration behavior of w in L^q)

Proposition 3.1 *Let $q = \frac{p_c+3}{2} = 2 + \frac{2}{N-1}$. For all $s \geq -\log T + 1$, $a \in \mathbb{R}^N$, $d \in B(0,2)$ and $r_0 \in (0,1)$,*

$$\int_{B(d,r_0)} |w_a(y,s)|^q dy \leq C_0 {r_0}^{\frac{1}{4}}. \tag{33}$$

Remark: The power 1/4 is not optimal. With the same proof, one can show that the optimal power is $N/(3N+1)$.
Proof: From translation invariance, we can take $a = 0$.

i) The averaging technique of Proposition 2.4 gives in fact the non concentration of the L^2 norm of $\partial_s w_a$:

$$\text{For all } d \in B(0,2),\ \ s \geq -\log T \text{ and } r_0 \in (0,1),\ \ \int_s^{s+r_0^{\frac{1}{2}}} ds \int_{B(d,r_0)} dz\, (\partial_s w_0(z,s))^2 \leq C_0 r_0^{\frac{1}{2}}. \tag{34}$$

Indeed, if $(z,s') \in B(b,r_0)\times[s,s+r_0]$, then we have from (16)

$$\left(\partial_s w_0(z,s')\right)^2 \ \leq\ Ce^{(N-1)s'} \int_{\partial B(ze^{-s'},e^{-s'})} db \left(\partial_s w_b(z-be^{s'},s')\right)^2$$

since $|z| \leq |b| + r_0 \leq 3$. If we integrate this and use Fubini's property, then we get

$$\begin{aligned}
&\int_s^{s+r_0^{\frac{1}{2}}} ds' \int_{B(d,r_0)} dz\, \left(\partial_s w_0(z,s')\right)^2 \\
\leq\ & C\int_s^{s+r_0^{\frac{1}{2}}} ds' \int_{B(d,r_0)} dz e^{(N-1)s'} \int_{\partial B(ze^{-s'},e^{-s'})} db \left(\partial_s w_b(z-be^{s'},s')\right)^2 \\
\leq\ & C\int_s^{s+r_0^{\frac{1}{2}}} ds' \int_{D_{s'}} db e^{Ns'} \int_{\partial B(ze^{-s'},e^{-s'})} db \left(\partial_s w_b(z-be^{s'},s')\right)^2 \\
\leq\ & Ce^{N(s+r_0^{\frac{1}{2}})} \int_{\{\bigcup_{s\leq s'\leq s+r_0^{\frac{1}{2}}} D_{s'}\}} db \int_s^{s+r_0^{\frac{1}{2}}} ds' \int_{\partial B(ze^{-s'},e^{-s'})} db \left(\partial_s w_b(z-be^{s'},s')\right)^2
\end{aligned}$$

where $D_{s'} = B\left(de^{-s'}, (1+r_0)e^{-s'}\right) \setminus B\left(de^{-s'}, (1-r_0)e^{-s'}\right)$. Since $|D_{s'}| \le Cr_0 e^{-Ns'}$, it holds that $\left|\cup_{s \le s' \le s+r_0^{\frac{1}{2}}} D_{s'}\right| \le C r_0^{\frac{1}{2}} e^{-Ns}$. Using the estimate on the sphere stated in (10), we end up with

$$\begin{aligned}
\int_s^{s+r_0^{\frac{1}{2}}} ds' \int_B dz \left(\partial_s w_0(z,s')\right)^2 &\le Ce^{N(s+r_0^{\frac{1}{2}})} \int_{\{\cup_{s\le s'\le s+r_0^{\frac{1}{2}}} D_{s'}\}} d\sigma C_0 \\
&= Ce^{N(s+r_0^{\frac{1}{2}})} |\cup_{s\le s'\le s+r_0^{\frac{1}{2}}} D_{s'}| C_0 \le C_0 r_0^{\frac{1}{2}}.
\end{aligned}$$

Thus estimate (34) follows.

ii) Using the mean value theorem and (31), we derive the existence of $\tau(s,r_0) \in [s, s+r_0^{\frac{1}{2}}]$ such that

$$\int_{B(0,3)} |w(y,\tau)|^{p_c+1} dy = r_0^{-\frac{1}{2}} \int_s^{s+r_0^{\frac{1}{2}}} \int_{B(0,3)} |w|^{p_c+1} dy ds \le C_0 r_0^{-\frac{1}{2}}.$$

Therefore, since $B(b,r_0) \subset B(0,3)$, we use Hölder's inequality to write:

$$\begin{aligned}
\int_{B(b,r_0)} |w_0(y,\tau(s,r_0))|^q dy &\le \left(\int_{B(0,3)} |w_0(y,\tau(s,r_0))|^{p_c+1} dy\right)^{\frac{q}{p_c+1}} |B(b,r_0)|^{1-\frac{q}{p_c+1}} \\
&\le \left(C_0 r_0^{-\frac{1}{2}}\right)^{\frac{q}{p_c+1}} (Cr_0^N
\end{aligned}$$

iii) Moreover, using estimate (34), (31) and the fact that $2(q-1) = p_c+1$, we write

$$\begin{aligned}
\int_{B(b,r_0)} |w_0(y,s)|^q dy &= \int_{B(b,r_0)} |w_0(y,\tau)|^q dy + \int_\tau^s \frac{d}{ds} \int_{B(b,r_0)} |w_0|^q dy ds \\
&\le C_0 r_0^{\frac{N}{2(N+1)}} + q \int_s^{s+r_0^{\frac{1}{2}}} \int_{B(b,r_0)} |\partial_s w_0||w_0|^{q-1} dy ds \\
&\le C_0 r_0^{\frac{N}{2(N+1)}} + q \left(\int_s^{s+r_0^{\frac{1}{2}}} \int_{B(b,r_0)} (\partial_s w_0)^2 dy ds\right)^{\frac{1}{2}} \left(\int_s^{s+r_0^{\frac{1}{2}}} \int_{B(b,r_0)} |w_0|^{2(q-1)} dy ds\right)^{\frac{1}{2}} \\
&\le C_0 r_0^{\frac{N}{2(N+1)}} + C_0 r_0^{\frac{1}{4}} \left(\int_s^{s+1} \int_{B(0,3)} |w_0|^{p_c+1} dy ds\right)^{\frac{1}{2}} \le C_0 r_0^{\frac{1}{4}}.
\end{aligned}$$

This concludes the proof of Proposition 3.1. ■

Step 2: Control of the gradient in $\mathrm{L}^2_{\mathrm{loc,u}}$

We claim the following:

Proposition 3.2 (Uniform control of the $\mathrm{H}^1_{\mathrm{loc,u}}$ norm of $w_a(s)$)
For all $s \geq -\log T + 1$ and $a \in \mathbb{R}^N$,

$$\int_B |\nabla w_a(y,s)|^2 dy \leq C_0.$$

We first recall interpolation estimates in the ball B and then rescale them to get estimates in a ball of radius $r_0 \in (0,1)$. Let $F \in \mathrm{H}^1(B)$. Since (3) yields

$$p_c + 1 = \frac{p_c+3}{2}\frac{2}{N} + 2,$$

we have from the Gagliardo-Nirenberg inequality:

$$\int_B |F|^{p_c+1} \leq C \left(\int_B |F|^{\frac{p_c+3}{2}}\right)^{\frac{2}{N}} \left(\int_B |F|^2 + \int_B |\nabla F|^2\right). \tag{35}$$

Now, if $f \in \mathrm{H}^1\left(B(b,r_0)\right)$, then $F(z) = f(b + r_0 z)$ is defined on the unit ball B. Therefore, it holds from (35) that

$$\int_{B(b,r_0)} |f|^{p_c+1} \leq C \left(\int_{B(b,r_0)} |f|^{\frac{p_c+3}{2}}\right)^{\frac{2}{N}} \left(r_0^{-2}\int_{B(b,r_0)} |f|^2 + \int_{B(b,r_0)} |\nabla f|^2\right). \tag{36}$$

Using this estimate and the non concentration of the L^q norm (33) yields the following:

Lemma 3.3 (Local control of the space L^{p_c+1} norm by the H^1 norm) *For all $r_0 \in (0,1)$, $s \geq -\log T + 1$ and $a \in \mathbb{R}^N$,*

$$\int_B |w_a|^{p_c+1} \leq C r_0^{\frac{1}{2N}} \int_{B(0,1+2r_0)} |\nabla w_a|^2 dy + C r_0^{\frac{1}{2N}-2} \int_{B(0,1+2r_0)} |w_a|^2 dy.$$

Proof : Let us introduce the following covering property for the ball B:

Claim 3.4 *For any $r_0 > 0$, we can cover the unit ball B with a finite number $C(r_0,N)$ of balls $B_i \subset B(0,1+2r_0)$ of radius r_0 such that each point of $B(0,1+2r_0)$ belongs to $C(N)$ balls at most.*

Indeed, we have the following explicit covering

$$B \subset \cup_{b \in \frac{r_0}{\sqrt{N}}\mathbb{Z}^N \cap B(0,1+r_0)} B(b,r_0).$$

Using this claim, we write for any $f \in \mathrm{L}^1\left(B(0,1+2r_0)\right)$,

$$\int_B |f| \leq \sum_i \int_{B_i} |f| \leq C(N) \int_{B(0,1+2r_0)} |f|. \tag{37}$$

Using the Gagliardo-Nirenberg estimate (36) for w_a in the ball $B_i \equiv B(b_i, r_0)$, we get

$$\int_{B_i} |w_a|^{p_c+1} \leq C \left(\int_{B_i} |w_a|^{\frac{p_c+3}{2}}\right)^{\frac{2}{N}} \left(r_0^{-2}\int_{B_i} |w_a|^2 + \int_{B_i} |\nabla w_a|^2\right).$$

Using the estimate of the $L^{\frac{p_c+3}{2}}$ norm of w_a in B_i stated in Proposition 3.1, this yields

$$\int_{B_i} |w_a|^{p_c+1} \le C r_0^{\frac{1}{2N}-2} \int_{B_i} |w_a|^2 + C_0 r_0^{\frac{1}{2N}} \int_{B_i} |\nabla w_a|^2.$$

From the covering property (see (37)) and this inequality, we write

$$\begin{aligned} \int_B |w_a|^{p_c+1} &\le \sum_i \int_{B_i} |w_a|^{p_c+1} \le C_0 r_0^{\frac{1}{2N}} \sum_i \int_{B_i} |\nabla w_a|^2 + C_0 r_0^{\frac{1}{2N}-2} \sum_i \int_{B_i} |w_a|^2 \\ &\le C_0 C(N) r_0^{\frac{1}{2N}} \int_{B(0,1+2r_0)} |\nabla w_a|^2 dy + C_0 C(N) r_0^{\frac{1}{2N}-2} \int_{B(0,1+2r_0)} |w_a|^2 dy. \end{aligned}$$

This concludes the proof of Lemma 3.3. ■

Let us prove Proposition 3.2 now.

Proof of Proposition 3.2: For a given $s \ge -\log T + 1$, there exists $a_0 = a_0(s)$ such that

$$\int_B |\nabla w_{a_0}|^2 (1 - |y|^2) dy \ge \frac{1}{2} \sup_{a \in \mathbb{R}^N} \int_B |\nabla w_a|^2 (1 - |y|^2) dy. \quad (38)$$

i) We claim that a covering argument and the definition of $a_0(s)$ yield

$$\forall a \in \mathbb{R}^N, \quad \int_{B(0,3)} |\nabla w_a|^2 dy \le C \int_B |\nabla w_{a_0}|^2 (1 - |y|^2) dy. \quad (39)$$

Indeed, since we can cover $B(0,3)$ with $k(N)$ balls of radius $1/2$, it is enough to prove that

$$\int_{|y|<\frac{1}{2}} |\nabla w_a(y + y_0, s)|^2 dy \le C \int_B |\nabla w_{a_0}|^2 (1 - |y|^2) dy \quad (40)$$

uniformly for $|y_0| \le 3$ and $a \in \mathbb{R}^N$. Using the relation (29), we see that for all $y \in \mathbb{R}^N$, $\nabla w_a(y + y_0, s) = \nabla w_b(y, s)$ where $b = a + y_0 e^{-s}$. Therefore, since $1 - |y|^2 \ge \frac{3}{4}$ whenever $|y| \le \frac{1}{2}$, we write

$$\begin{aligned} &\int_{|y|<\frac{1}{2}} |\nabla w_a(y + y_0, s)|^2 dy = \int_{|y|<\frac{1}{2}} |\nabla w_{a+y_0 e^{-s}}(y, s)|^2 dy \\ &\le C \int_B |\nabla w_{a+y_0 e^{-s}}(y, s)|^2 (1 - |y|^2) dy \le C \sup_{a \in \mathbb{R}^N} \int_B |\nabla w_a|^2 (1 - |y|^2) dy \\ &\le C \int_B |\nabla w_{a_0}|^2 (1 - |y|^2) dy, \end{aligned}$$

by definition of the supremum (38). This yields (40) and then (39).

ii) From the estimates on the Lyapunov functional E and Lemma 3.3, we have the conclusion. Indeed, using the definition (7) and the bound (9) on E, we see that

$$\int_B |\nabla w_{a_0}|^2 (1 - |y|^2) dy \le C_0 + \frac{2}{p_c + 1} \int_B |w_{a_0}|^{p_c+1} dy. \quad (41)$$

Using (39), (41), the control of the L^{p_c+1} by the H^1 norm of Lemma 3.3 and the bound (31) on the L^2 norm of w_{a_0}, we obtain for all $r_0 \in (0,1)$,

$$\begin{aligned}\int_B |\nabla w_{a_0}|^2 dy &\le C_0 + C_0 r_0^{\frac{1}{2N}} \int_{B(0,1+2r_0)} |\nabla w_{a_0}|^2 dy + C_0 r_0^{\frac{1}{2N}-2} \int_{B(0,1+2r_0)} |w_{a_0}|^2 dy \\ &\le C_0 r_0^{\frac{1}{2N}} \int_{B(0,1+2r_0)} |\nabla w_{a_0}|^2 dy + C_0 \left(1 + r_0^{\frac{1}{2N}-2}\right).\end{aligned}$$

Since $B(0,1+2r_0) \subset B(0,3)$, we use (39) again to write

$$\int_{B(0,1+2r_0)} |\nabla w_{a_0}|^2 dy \le C \int_B |\nabla w_{a_0}|^2 (1-|y|^2) dy \le C \int_B |\nabla w_{a_0}|^2 dy.$$

Therefore, for all $r_0 \in (0,1)$,

$$\int_B |\nabla w_{a_0}|^2 dy \le C_0 r_0^{\frac{1}{2N}} \int_B |\nabla w_{a_0}|^2 + C_0 \left(1 + r_0^{\frac{1}{2N}-2}\right).$$

Taking r_0 small enough such that $C_0 r_0^{\frac{1}{2N}} = \frac{1}{2}$, it follows that for some C_0 independent of s, we have

$$\int_B |\nabla w_{a_0(s)}(y,s)|^2 dy \le C_0.$$

From the property (39) for $a_0(s)$, this yields

$$\text{for all } s \ge -\log T + 1 \text{ and } a \in \mathbb{R}^N, \quad \int_B |\nabla w_a(y,s)|^2 dy \le \int_{B(0,3)} |\nabla w_a(y,s)|^2 dy \le C_0. \tag{42}$$

This concludes the proof of Proposition 3.2. ∎

Step 3: Conclusion of the proof of Theorem 1

The $\mathrm{L}^2(B)$ norms of w_a and ∇w_a have been controlled in Propositions 2.5 and 3.2. Remains to control $\partial_s w_a$.

From the expression and the uniform boundedness of E (see (9)), Lemma 3.3, estimates (31) and (42), we have for all $s \ge -\log T + 1$ and $a \in \mathbb{R}^N$,

$$\begin{aligned}\int_B (\partial_s w_a)^2 \, dy &\le 2E(w) + 2\int_B \left(-\frac{(p_c+1)}{(p_c-1)^2} w_a^2 + \frac{1}{p_c+1}|w_a|^{p_c+1}\right) dy \\ &- \int_B \left(|\nabla w_a|^2 - (y.\nabla w_a)^2\right) dy \le C_0. \end{aligned} \tag{43}$$

This concludes the proof of Theorem 1. ∎

A Properties of the averaging function $P(\sigma, z)$

We first claim that

$$\int_{\partial B(z,1)} \sigma d\sigma = z|\partial B| \text{ and } \int_{\partial B(z,1)} \sigma_i \sigma_j d\sigma = \left(z_i z_j + \frac{\delta_{ij}}{N}\right) |\partial B|. \tag{44}$$

Indeed, the fact that z is the center of mass of $\partial B(z,1)$ gives the first integral. As for the second integral, we make a translation in space and write it as an integral on ∂B:

$$\begin{aligned}
&\int_{\partial B(z,1)} \sigma_i\sigma_j d\sigma = \int_{\partial B} (z_i+\sigma_i)(\sigma_j+z_j)d\sigma \\
= \;& z_i z_j \int_{\partial B} d\sigma + z_i \int_{\partial B} \sigma_j d\sigma + z_j \int_{\partial B} \sigma_i d\sigma + \int_{\partial B} \sigma_i\sigma_j d\sigma = z_i z_j |\partial B| + 0 + 0 + \frac{\delta_{ij}}{N}|\partial B|.
\end{aligned}$$

Now we use (44) and the definition of $P(\sigma,z)$ (13) to prove (14) and (15). We begin with

$$\begin{aligned}
&\int_{\partial B(z,1)} d\sigma P(\sigma,z)\sigma = \int_{\partial B(z,1)} \sigma d\sigma - \sum_{i,j} \frac{z_i e_j}{|z|^2+\frac{1}{N}} \int_{\partial B(z,1)} \sigma_i\sigma_j d\sigma \\
= \;& z|\partial B| - \sum_{i,j} \frac{z_i e_j}{|z|^2+\frac{1}{N}} \left(z_i z_j + \frac{\delta_{ij}}{N}\right)|\partial B| = |\partial B|\left(z - \frac{|z|^2 z + z/N}{|z|^2+\frac{1}{N}}\right) = 0
\end{aligned}$$

which gives (14). As for (15), we first write

$$\int_{\partial B(z,1)} d\sigma P(\sigma,z) = |\partial B| - \frac{z}{|z|^2+\frac{1}{N}}.\int_{\partial B(z,1)} \sigma d\sigma = |\partial B| - \frac{z}{|z|^2+\frac{1}{N}}.z|\partial B| = \frac{|\partial B|}{N|z|^2+1}. \tag{45}$$

Then, we notice that (14) implies that

$$\begin{aligned}
&\int_{\partial B(z,1)} P(\sigma,z)^2 d\sigma = \int_{\partial B(z,1)} P(\sigma,z)\left(1 - \frac{z.\sigma}{|z|^2+\frac{1}{N}}\right)d\sigma \\
= \;& \int_{\partial B(z,1)} P(\sigma,z)d\sigma - \frac{z}{|z|^2+\frac{1}{N}}.\int_{\partial B(z,1)} \sigma P(\sigma,z)d\sigma = \int_{\partial B(z,1)} P(\sigma,z)d\sigma. \qquad (46)
\end{aligned}$$

Thus, (15) follows from (45) and (46).

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