Ergod. Th. & Dynam. Sys. (1983), **3**, 351–385
Printed in Great Britain

Entropy properties of rational endomorphisms of the Riemann sphere

M.Ju. LJUBICH
Department of Mathematics, Kharkov State University, Kharkov, USSR

(*Received* 10 *November* 1981)

Abstract. In this paper the existence of a unique measure of maximal entropy for rational endomorphisms of the Riemann sphere is established. The equidistribution of pre-images and periodic points with respect to this measure is proved.

1. *Introduction*

The present paper deals with the dynamics of the rational function

$$f(z)=\frac{P(z)}{Q(z)} \qquad \text{where } P, Q \text{ are polynomials}$$

of the complex variable, regarded as an analytic endomorphism of the Riemann sphere S^2. The dynamics of such a system depends essentially on the behaviour of trajectories originating from the critical points of the endomorphism f. We shall demonstrate nevertheless that certain ergodic and entropy properties of f are common to all rational functions.

In [**6**] it was shown by potential theory methods that if $f(z)=P(z)$ is a polynomial, then the pre-images $f^{-m}(z)$ have an asymptotically uniform distribution with respect to some invariant mixing measure (for z an arbitrary point of the plane $\mathbb{C}$, except for at most one z). In § 4 we shall extend this result to all rational functions, using the concept of almost periodicity. The measure μ is supported on a set of irregular points of the function f. The dynamical system (f, μ) is exact. We shall consider a more general equation

$$f^m\zeta=\varphi(\zeta),$$

where φ is a rational function, and show that the distribution of its roots is also asymptotically uniform with respect to the measure μ (if φ differs from the two excluded constants). When $\varphi(\zeta)=\zeta$ we have the theorem on distribution of periodic points of f. So we strengthen the classic result on density of periodic points in the set of irregular points.

In § 6 we shall show that the topological entropy $h(f)$ of the rational endomorphism f equals $\ln \deg f$, which proves the conjecture by Bowen ([**5**]). A brief exposition of our proof was given in [**21**]. So rational endomorphisms are the simplest ones in the class of homotopic smooth maps of the sphere S^2. The measure μ is that of maximal entropy for f. We verify that f is asymptotically h-expansive which yields

another proof of existence of a measure of maximal entropy. Finally we shall prove the uniqueness of the measure of maximal entropy.

Rational maps of the Riemann sphere closely resemble expanding endomorphisms. Under certain assumptions they are indeed expanding on the set of irregular points. The proofs of our results are substantially simplified in this case; they can also easily be obtained by means of symbolic dynamics (see [**15**], [**16**], [**17**] & [**13**]). In the general case the restriction of a rational endomorphism to the set of irregular points should, perhaps, be treated as an expanding endomorphism with singularities.

2. *Almost periodic operators in the Banach space*

Let A be a bounded operator in the complex Banach space $\mathscr{B}$.

Definition. The operator A is called *almost periodic* if the orbit

$$\{A^m\varphi\}_{m=1}^{\infty}$$

of any vector $\varphi \in \mathscr{B}$ is strongly conditionally compact.

The Banach–Steinhaus theorem implies that if A is an almost periodic operator, then

$$\|A^m\| \leq C \qquad \text{for some } C \quad m = 1, 2, \ldots.$$

The eigenvalue λ and the related eigenvector are called unitary if $|\lambda| = 1$. The set of unitary eigenvalues of the operator A will be denoted by

$$\operatorname{spec}_u A.$$

Consider the following subspaces:

$\mathscr{B}_u$, the closure of the linear span of the unitary eigenvectors of the operator A and $\mathscr{B}_0 = \{\varphi | A^m\varphi \to 0 \quad (m \to \infty)\}$.

(Here and later the convergence is assumed to be strong.)

The 'weak' version of the theorem that follows is given in [**14**].

Theorem on splitting the unitary discrete spectrum. If $A : \mathscr{B} \to \mathscr{B}$ is an almost periodic operator in the complex Banach space $\mathscr{B}$, then $\mathscr{B}$ can be decomposed into the direct sum

$$\mathscr{B} = \mathscr{B}_u \dotplus \mathscr{B}_0.$$

Proof. We shall follow the ideas of [**19**], [**20**]. Consider the semi-group

$$S = \{A^m\}_{m=1}^{\infty}$$

of operators in $\mathscr{B}$. This is a topological abelian semi-group in the strong operator topology. Almost periodicity of A is equivalent to conditional compactness of the semi-group S. It is easily derived from this that the set

$$G \subset \bar{S}$$

of ω-limit operators of the semi-group S is a compact abelian group. Consider the unit P of the group G. Being idempotent, P is a projection. We show that

$$\mathscr{B}_u = \operatorname{Im} P, \qquad \mathscr{B}_0 = \operatorname{Ker} P$$

which will imply the required statement.

Let us restrict all the operators of the group G to the subspace

$$B = \operatorname{Im} P.$$

We get a strongly continuous representation T of the group G on B. A vector $\varphi \in B$ is called a weight vector if there exists a character $\chi \in \hat{G}$ such that

$$T_g\varphi = \chi(g)\varphi \qquad g \in G.$$

LEMMA. *The set of weight vectors of the representation T is complete in the space B.*

Proof. Let $\chi \in \hat{G}$ be a character of the group G. Consider the operator

$$P_\chi : B \to B, \qquad P_\chi = \int T_g\overline{\chi(g)}\, dg$$

where integration is carried out with the Haar measure on G. We check in the standard way that P_χ is a projection onto the subspace of weight vectors of the group G corresponding to the character χ. Thus we must verify that the system of subspaces

$$\{\operatorname{Im} P_\chi\}_{\chi \in \hat{G}}$$

is complete in B. Let $f \in B^*$ be a functional orthogonal to all subspaces $\operatorname{Im} P_\chi$, $\chi \in \hat{G}$. This means that for any vector $\varphi \in B$ and any character $\chi \in \hat{G}$,

$$\int f(T_g\varphi)\overline{\chi(g)}\, dg = f(P_\chi\varphi) = 0.$$

i.e. all Fourier coefficients of the matrix element $f(T_g\varphi)$ equal 0. By the Peter-Weyl theorem

$$f(T_g\varphi) = 0.$$

Setting $g = P$, we obtain

$$f(\varphi) = 0.$$

Finally, since φ is an arbitrary vector from B, $f = 0$. The lemma is proved. □

We continue the proof of the theorem. Let us show that any weight vector φ of the representation T is a unitary eigenvector of the operator A. Note that if $g \in G$, then $Ag \in G$. Hence, for $\varphi \in B$,

$$A\varphi = (AP)\varphi = \chi(AP)\varphi.$$

Besides

$$|\chi(AP)| = 1,$$

as G is compact. By the lemma we get

$$\operatorname{Im} P = B \subset \mathscr{B}_u.$$

Conversely, let φ be a unitary eigenvector of the operator A. Since $P \in \bar{S}$, φ is a unitary eigenvector of the operator P. Finally, since P is a projection, $\varphi \in \operatorname{Im} P$. Thus,

$$\mathscr{B}_u \subset \operatorname{Im} P.$$

Now, let $\varphi \in \operatorname{Ker} P$. Since P is an ω-limit operator for S, there exists a sequence $m_k \nearrow \infty$ such that

$$A^{m_k}\varphi \to P\varphi = 0.$$

Hence, for $m_k \le m < m_{k+1}$ we obtain

$$\|A^m\varphi\| \le C\|A^{m_k}\varphi\| \to 0 \qquad m \to \infty, \quad \text{where } C = \sup_{1 \le m < \infty} \|A^m\| < \infty.$$

Therefore,

$$\operatorname{Ker} P \subset \mathscr{B}_0.$$

Finally, let $\varphi \in \mathscr{B}_0$. As before, we have

$$P\varphi = \lim_{m_k \to \infty} A^{m_k}\varphi = 0$$

i.e.

$$\mathscr{B}_0 \subset \operatorname{Ker} P.$$

The theorem is proved. □

COROLLARY. *Let $A : \mathscr{B} \to \mathscr{B}$ be an almost periodic operator in the complex Banach space $\mathscr{B}$. Assume that*

$$\operatorname{spec}_u A = \{1\}$$

and the point $\lambda = 1$ is a simple eigenvalue. Let $h \ne 0$ be an invariant vector of the operator A. Then there exists an A^*-invariant functional $\mu \in \mathscr{B}^*$, $\mu(h) = 1$, such that

$$A^m\varphi \to \mu(\varphi)h \qquad m \to \infty.$$

Proof. Under our assumptions $\mathscr{B}_u = \operatorname{Lin}\{h\}$. Then the projection P on $\mathscr{B}_u$, parallel to $\mathscr{B}_0$, has the form

$$P\varphi = \mu(\varphi)h$$

where $\mu \in \mathscr{B}^*$ is a functional orthogonal to $\mathscr{B}_0$ and normalized by the condition $\mu(h) = 1$. The convergence

$$A^m\varphi \to P\varphi \qquad m \to \infty$$

is obvious on $\mathscr{B}_0$ and $\operatorname{Lin}\{h\}$, so it takes place on the entire space $\mathscr{B}$. Finally,

$$\mu(A\varphi)h = \lim_{m \to \infty} A^{m+1}\varphi = \mu(\varphi)h$$

and, therefore, μ is an A^*-invariant functional. □

3. *Preliminary considerations of dynamics of rational endomorphisms*

The results given in this section, except proposition 4, are identical or similar to those reported in [**10**], [**18**], (see also [**24**], [**6**]).

Let S^2 be a unit sphere in $\mathbb{R}^3$ with the induced metric d. By means of the stereographic projection the sphere S^2 is identified with the complete complex plane $\bar{\mathbb{C}}$. The meromorphic functions become maps

$$f : U \to S^2$$

where U is some domain in S^2.

Definition 1. The family of meromorphic functions $\{f_\alpha\}_\alpha$ defined on the domain $U \subset S^2$ is called *normal*, if it is conditionally compact in the topology of uniform convergence on compact subsets of U.

Definition 2. The point $a \in S^2$ is called an *excluded value* for the family $\{f_\alpha\}_\alpha$ if

$$\forall \alpha \quad \forall z \in U \qquad f_\alpha(z) \neq a.$$

The Montel theorem. If a family of meromorphic functions has three excluded values, then it is normal. Further, let

$$f(z) = \frac{P(z)}{Q(z)}$$

be a rational function (the polynomials P and Q are relatively prime). We shall denote by n the degree of the related endomorphism $f: S^2 \to S^2$; $\deg f = \max(\deg P, \deg Q)$. *The equation* $f\zeta = z$, *for any z has n roots (taking account of multiplicities).*

For $n = 1$ the dynamics of f is trivial, so later we assume $n \geq 2$.

Definition. The point $z \in S^2$ is *regular* for f if the family $\{f^m\}_{m=1}^\infty$ is normal in some neighbourhood of the point z; otherwise z is irregular.

It follows from the Montel theorem that if U is a neighbourhood of an irregular point, then the complement of $\bigcup_{m=1}^\infty f^m U$ in the sphere is at most two points.

Remark. The point z is regular if and only if the trajectory $\{f^m z\}_{m=1}^\infty$ is stable in the sense of Liapunov.

The set of irregular points is denoted by F. It follows directly from the definition that F is compact and invariant with respect to f and f^{-1}, the latter symbol denoting the operation of finding the inverse image. A periodic point $z \neq \infty$ with a period p is called a *sink* (resp., a *source*) if

$$|(f^p)'(z)| < 1 \qquad (>1).$$

For $z = \infty$, one should substitute $1/z$ for z. Sinks are obviously regular. Let z be a source with a period p (assume $z \neq \infty$). Then

$$|(f^p)'(z)| > 1.$$

That implies

$$|(f^{pm})'(z)| \to \infty \qquad m \to \infty.$$

Hence a source is irregular.

The density theorem. The set of sources is dense in F.

If φ is a linear-fractional transformation of the Riemann sphere, then $\varphi^{-1} f \varphi$ is called a *Möbius transformation* of f. A rational function f is (up to Möbius transformation) a polynomial if and only if there exists a point a such that

$$f^{-1} a = \{a\}.$$

Indeed, if $\varphi(\infty) = a$, then the rational function $\varphi^{-1} f \varphi$ has no finite poles, i.e. it is polynomial.

Now, let $z \in F$. Assume that the family $\{f^m\}_{m=1}^\infty$ in any sufficiently small neighbourhood of the point z has only one excluded value a. Then $f^{-1} a = \{a\}$, since the set of excluded values of the family $\{f^m\}_{m=1}^\infty$ is obviously invariant with respect to f^{-1}.

So the situation is as just described. Let the family $\{f^m\}_{m=1}^{\infty}$ possess in the neighbourhood of an irregular point z two excluded values a and b. Then there are two alternatives:

(1) a and b are fixed points; then, up to Möbius transformation, $f(z)=z^n$;

(2) a and b form the second order cycle; then, up to Möbius transformation, $f(z)=z^{-n}$.

In any case:

(i) excluded values of the family $\{f^m\}_{m=1}^{\infty}$ in a small neighbourhood of an irregular point z do not depend on z (later they will be called excluded points of f);

(ii) excluded points are regular;

(iii) the set of excluded points is attracting and, therefore, has a base of invariant neighbourhoods;

(iv) the sets of excluded points of the functions f and f^m coincide, $(m=2,3,\ldots)$.

PROPOSITION 1. *Let U be a domain containing an irregular point, and K be a compact set containing no excluded points. Then there is an N such that*

$$m \geq N \Rightarrow f^m U \supset K.$$

Proof. Expand K to a compact set L containing no excluded points such that $fL \supset L$ (which is possible since the set of excluded points has a base of invariant neighbourhoods). The density theorem implies that U contains a source ζ. Let p be its period. Then there exists a neighbourhood $V \subset U$ of the point ζ such that

$$f^p V \supset V.$$

Since ζ is an irregular point of the function f, Montel's theorem implies

$$\bigcup_{m=0}^{\infty} f^{pm} V \supset L.$$

Since L is compact and $\{f^{pm}\}_{m=0}^{\infty}$ is an increasing sequence of open sets, we have, for some l:

$$f^{pl} V \supset L.$$

Finally, as $fL \supset L$, then

$$m \geq pl \Rightarrow f^m U \supset f^m V \supset L \supset K. \qquad \square$$

COROLLARY. *For any non-excluded point z and any irregular point ζ there exists a sequence $\{z_m\}_{m=1}^{\infty}$, such that*

$$z_m \in f^{-m} z \quad \text{and} \quad z_m \to \zeta \qquad m \to \infty.$$

To show this just set $K=\{z\}$ in proposition 1.

PROPOSITION 2. *If there exists a family $\{f_\lambda^{-m}\}_{m,\lambda}$ of single-valued analytic branches of the inverse functions f^{-m} in the domain U, then this family is normal.*

Proof. Normality is a local property. Hence it suffices to check it in the proper neighbourhood of each point $z \in U$. Let ζ be a periodic point with period $p \geq 3$ such that z does not lie in its orbit

$$O(\zeta)=\{f^m \zeta\}_{m=0}^{p-1}.$$

Then there exists a neighbourhood $V \subset U$ of the point z such that

$$V \cap O(\zeta) = \varnothing.$$

Hence, the set $O(\zeta)$ is excluded for the family $\{f_\lambda^{-m}\}_{m,\lambda}$ in the neighbourhood V. Now we use the Montel theorem. □

V is said to be *compactly contained* in the domain U ($V \Subset U$) if $\bar{V} \subset U$.

PROPOSITION 3. *Let U contain an irregular point and let $\{f_\lambda^{-m}\}_{m,\lambda}$ be a family of branches of f^{-m} in the domain U. If $V \Subset U$, then*

$$\operatorname{diam}(f_\lambda^{-m} V) \to 0 \qquad \text{as } m \to \infty.$$

Proof. Consider a domain U' such that

$$V \Subset U' \subset U,$$

U' contains an irregular point b, and

$$|S^2 \backslash U'| > 2,$$

where $|X|$ denotes the cardinality of the set X. Let $f_{\lambda_k}^{-m_k} \to \varphi$ uniformly, as $k \to \infty$, on compact subsets of U. Assume $\varphi \neq$ const. Then, according to the Hurwitz theorem, there exists a neighbourhood Y of the point $a = \varphi(b)$ such that

$$f_{\lambda_k}^{-m_k} U' \supset Y$$

for sufficiently large k. So $f^{m_k} Y \subset U'$. But

$$a = \lim_{k \to \infty} f_{\lambda_k}^{-m_k} b \in F.$$

By proposition 1, for any non-excluded point z and sufficiently large m, $z \in f^m Y$. Since $|S^2 \backslash U'| > 2$, then outside U' there are non-excluded points and, therefore,

$$f^m Y \not\subset U'$$

for large enough m. The contradiction we have arrived at shows that $\varphi =$ const. Therefore, for any convergent subsequence $\{f_{\lambda_k}^{-m_k}\}_{k=1}^{\infty}$ we have

$$\operatorname{diam}(f_{\lambda_k}^{-m_k} V) \to 0 \qquad \text{as } k \to \infty.$$

Since the family $\{f_\lambda^{-m}\}_{m,\lambda}$ is normal, then

$$\operatorname{diam}(f_\lambda^{-m} V) \to 0 \qquad \text{as } m \to \infty.$$ □

Denote by Z the set of critical points of the function f (i.e. points in whose neighbourhood f is not a local diffeomorphism). Let

$$Z_m = \bigcup_{i=1}^{m} f^i Z$$

be the set of critical values of the function f^m, for $1 \leq m < \infty$,

$$\tau = |Z_1| \leq 2(n-1).$$

If a simply-connected domain U does not intersect with the set Z_m, then n^m single-valued analytic branches f_λ^{-m}, $1 \leq \lambda \leq n^m$, of the function f^{-m} exist in U. In the general case denote the number of all branches in U by $\sigma_m(U)$.

PROPOSITION 4. *Let U be a simply connected domain and $U \cap Z_l = \varnothing$. Then*

$$\sigma_m(U) \geq n^m(1 - 2\tau n^{-l}).$$

Remark. Thus, for large l, single-valued analytic branches of the function f^{-m} are connected with the overwhelming majority of roots of the equation $f^m\zeta = z$, $z \in U$.

Proof. We show by induction that

$$n^m - \sigma_m \leq \tau \sum_{i=1}^{m-l} n^i \tag{1}$$

where the sum on the right-handside is 0 for $m \leq l$. For $1 \leq m \leq l$, n^m branches of the function f^{-m} exist in U, and this estimate is trivial. To make the inductive step $m \rightsquigarrow m+1$ consider σ_m simply-connected domains $f_\lambda^{-m} U$, $\lambda = 1, \ldots, \sigma_m$. Since these domains do not intersect, then at most τ of them contain critical values of the function f^{-1}. There exist n single-valued analytical branches of f^{-1} on each remaining domain. Composing these branches with corresponding ones of the function f^{-m}, we obtain at least

$$n(\sigma_m - \tau) \geq n^{m+1} - \tau \sum_{i=1}^{m+1-l} n^i$$

branches of the function $f^{-(m+1)}$, so the estimate (1) is valid. Taking a less severe estimate, we obtain:

$$n^m - \sigma_m \leq 2\tau n^{m-l}. \qquad \square$$

4. *Distribution of roots of the equation $f^m\zeta = \varphi(\zeta)$*

Consider the operator $A: C(S^2) \to C(S^2)$ in the space of continuous functions on S^2; the operator is defined as follows:

$$(A\varphi)(z) = \frac{1}{n} \sum_{\zeta \in f^{-1}z} \varphi(\zeta) \qquad \varphi \in C(S^2), \quad z \in S^2,$$

the roots of the equation $f(\zeta) = z$ are counted with their multiplicity. The operator A is well defined, as the roots of the latter equation depend continuously on z.

Denote the function that is identically equal to 1 by $\mathbb{1}$. If $K \subset S^2$, then we shall denote $\sup_{z \in K} |\varphi(z)|$ by $\|\varphi\|_K$. By 'measure' we mean the complex regular Borel measure.

THEOREM 1. *There exists an A^*-invariant probability measure μ on the sphere S^2 such that*

$$\left\| A^m\varphi - \left(\int \varphi \, d\mu \right) \mathbb{1} \right\|_K \to 0 \qquad \text{as } m \to \infty$$

for any compact $K \subset S^2$ containing no excluded values of the function f. The measure μ is supported on the set F of irregular points.

Proof. First assume $f^{-1}K \subset K$. Then the factor-operator

$$A_K : C(K) \to C(K),$$

$$(A_K\varphi)(z) = \frac{1}{n} \sum_{\zeta \in f^{-1}z} \varphi(\zeta) \qquad \varphi \in C(K), \quad z \in K,$$

is well defined.

LEMMA 1. *A_K is an almost periodic operator.*

Proof. Let $\varepsilon > 0$. Find $\delta > 0$ such that if $z, \zeta \in K$ and $d(z, \zeta) < \delta$, then

$$|\varphi(z) - \varphi(\zeta)| < \varepsilon/2:$$

Set

$$M = \|\varphi\|_K.$$

We are going to check the equicontinuity of the family $\{A^m\varphi\}_{m=1}^{\infty}$ in the neighbourhood of an arbitrary point $z \in K$.

First suppose that $z \notin Z_\infty$. Find l such that

$$8M\tau n^{-l} < \varepsilon,$$

and a simply-connected neighbourhood U of the point z, such that

$$U \cap Z_l = \varnothing.$$

By proposition 4 there exist

$$\sigma_m \geq n^m(1 - 2\tau n^{-l}) > n^m\left(1 - \frac{\varepsilon}{4M}\right)$$

branches f_λ^{-m} of the function f^{-m} in U. By proposition 2 this family of branches is normal. Hence, there exists $R > 0$ such that

$$d(z, \zeta) < R \Rightarrow d(f_\lambda^{-m}z, f_\lambda^{-m}\zeta) < \delta$$

which implies

$$|\varphi(f_\lambda^{-m}z) - \varphi(f_\lambda^{-m}\zeta)| < \varepsilon/2 \qquad m = 1, 2, \ldots, \quad \lambda = 1, \ldots, \sigma_m.$$

Thus for $d(z, \zeta) < R$ we have:

$$\begin{aligned}
&|A_K^m\varphi(z) - A_K^m\varphi(\zeta)| \\
&\quad = \frac{1}{n^m}\left|\sum_{u \in f^{-m}z} \varphi(u) - \sum_{v \in f^{-m}\zeta} \varphi(v)\right| \\
&\quad \leq \frac{1}{n^m}\left[\sum_{\lambda=1}^{\sigma_m} |\varphi(f_\lambda^{-m}z) - \varphi(f_\lambda^{-m}\zeta)| + 2M(n^m - \sigma_m)\right] \\
&\quad < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\end{aligned}$$

Suppose now that $z \in K \cap Z_\infty$ and z is not a periodic point. If $f^l u = z$, then $f^m u \neq z$ for $m > l$. Hence, there exists l, such that

$$f^{-l}z \cap Z_\infty = \varnothing.$$

Let $f^{-l}z = \{z_i\}_{i=1}^{n^l}$ where the points z_i are counted with multiplicity. We have already proved that there exists $R > 0$, such that

$$d(z_i, \zeta) < R \Rightarrow |A_K^m\varphi(z_i) - A_K^m\varphi(\zeta)| < \varepsilon \qquad i = 1, \ldots, n^l, \quad m = 1, 2, \ldots.$$

Making use of the continuous dependence of the roots of an algebraic equation on its coefficients, find $\rho > 0$, such that if $d(z, \zeta) < \rho$, then the points of $f^{-l}\zeta$ can be numbered, counting multiplicity, $\zeta_1, \ldots, \zeta_{n^l}$, and we have $d(z_i, \zeta_i) < R$, $i = 1, \ldots, n^l$. Note also that

$$A_K^{l+m}\varphi(v) = A_K^l(A_K^m\varphi)(v) = \frac{1}{n^l}\sum_{u \in f^{-l}v} (A_K^m\varphi)(u).$$

https://doi.org/10.1017/S0143385700002030 Published online by Cambridge University Press

Therefore, for $d(z,\zeta)<\rho$ we obtain

$$|A_K^{l+m}\varphi(z)-A_K^{l+m}\varphi(\zeta)|\leq\frac{1}{n^l}\sum_{i=1}^{n^l}|A_K^m\varphi(z_i)-A_K^m\varphi(\zeta_i)|<\varepsilon$$

for $m=1,2,\ldots$.

Suppose, finally, that $z\in K$ is a periodic point, $f^p z=z$, and that t is the multiplicity of the root $\zeta=z$ in the equation $f^p\zeta=z$. Then $t<n^p$, for otherwise z would be an excluded point for f^p and, therefore, for f. The multiplicity of the root $\zeta=z$ in the equation $f^{pl}\zeta=z$ is t^l. Find l, such that

$$\frac{t^l}{n^{pl}}<\frac{\varepsilon}{4M}.$$

Let z_i be the roots of the equation

$$f^{pl}\zeta=z \qquad i=1,\ldots,n^{pl}-t^l,$$

different from z. Since all z_i are non-periodic points, we can find $R>0$, such that

$$d(z_i,\zeta)<R\Rightarrow|A_K^m\varphi(z_i)-A_K^m\varphi(\zeta)|<\frac{\varepsilon}{2}$$

for $i=1,\ldots,n^{p^l}-t^l$; $m=1,2,\ldots$. As before we find $\rho>0$, such that

$$d(z,\zeta)<\rho\Rightarrow d(z_i,\zeta_i)<R$$

where ζ_i are appropriately numbered points from $f^{-pl}\zeta$, for $i=1,\ldots,n^{pl}-t^l$. Finally for $d(z,\zeta)<\rho$ we obtain

$$\begin{aligned}&|A_K^{pl+m}\varphi(z)-A_K^{pl+m}\varphi(\zeta)|\\&\leq\frac{1}{n^{pl}}\left[\sum_{i=1}^{n^{pl}-t^l}|A_K^m\varphi(z_i)-A_K^m\varphi(\zeta_i)|+2Mt^l\right]\\&<\varepsilon.\end{aligned}$$

(We used the fact that A_K is a contraction and that is why $\|A_K^m\varphi\|\leq M$.)

So the family $\{A_K^m\varphi\}_{m=1}^\infty$ is equicontinuous in the neighbourhood of any point $z\in K$. It has been mentioned that $\|A_K^m\varphi\|\leq M$. By the Arzela theorem $\{A_K^m\varphi\}_{m=1}^\infty$ is conditionally compact in $C(K)$ and lemma 1 is proved. □

LEMMA 2. $\operatorname{spec}_u A_K=\{1\}$, $\qquad\mathscr{B}_u=\operatorname{Lin}\{\mathbb{1}\}$.

Proof. Let $\lambda\in\operatorname{spec}_u A_K$ and $\varphi\neq 0$ be the related eigenfunction:

$$\frac{1}{n}\sum_{\zeta\in f^{-1}z}\varphi(\zeta)=\lambda\varphi(z).$$

Let z be the maximum point of the function $|\varphi(\zeta)|$. It is easy to see that if $\zeta\in f^{-1}z$ then

$$\varphi(\zeta)=\lambda\varphi(z).$$

Iteration shows that if $\zeta\in f^{-m}z$, then

$$\varphi(\zeta)=\lambda^m\varphi(z).$$

By the corollary to proposition 1 for any irregular point x there is a sequence $\{z_m\}_{m=1}^{\infty}$, such that

$$z_m \in f^{-m}z \quad \text{and} \quad z_m \to x, \qquad m \to \infty.$$

Hence,

$$\lambda^m \varphi(z) \to \varphi(x),$$

and that implies $\lambda = 1$.

Assume temporarily that φ is real. Let z be the maximum point of the function φ. By a similar argument we come to the conclusion that for any irregular point x,

$$\varphi(x) = \varphi(z).$$

In the same way we show that

$$\varphi(x) = \varphi(\zeta),$$

where ζ is the minimum point of the function φ. So

$$\min \varphi = \max \varphi,$$

hence $\varphi = \text{const}$. If φ is not real, then one must treat the real and imaginary parts separately. □

Proof of theorem 1. Lemmas 1 and 2 have led us to the conditions of the corollary to the theorem on splitting the unitary discrete spectrum. There exists a complex A_K^*-invariant measure μ_K on the compact K such that

$$A_K^m \varphi \to \left(\int \varphi \, d\mu_K\right) \mathbb{1}, \qquad \varphi \in C(K).$$

We regard μ_K as a measure on the entire sphere S^2. Let us show that μ_K is independent of K. Notice that $F \subset K$. Indeed, let $z \in K$; z is not an excluded point for f, so

$$F \subset \overline{\bigcup_{m=1}^{\infty} f^{-m} z} \subset K,$$

(we still assume that $f^{-1}K \subset K$).

Let us fix some irregular point $\zeta \in F$. Let $\varphi \in C(S^2)$ and φ_K be the restriction of φ to K. Then

$$\int \varphi \, d\mu_K = \int \varphi_K \, d\mu_K = \lim_{m \to \infty} (A_K^m \varphi_K)(\zeta) = \lim_{m \to \infty} A^m \varphi(\zeta).$$

The expression on the right is independent of K, so μ_K too is independent of K. Further,

$$\int (A\varphi) \, d\mu = \int (A_K \varphi_K) \, d\mu_K = \int \varphi_K \, d\mu_K = \int \varphi \, d\mu.$$

Therefore, μ is A^*-invariant.

Let us show that $\operatorname{supp} \mu = F$. Since $f^{-1}F = F$ and F contains no excluded points of the function f, we have $\mu = \mu_F$. Hence,

$$\operatorname{supp} \mu = \operatorname{supp} \mu_F \subset F.$$

Conversely, let $z \in F$; $\varphi \in C(F)$, $\varphi \geq 0$ and $\varphi(z) > 0$. We assert that

$$\int \varphi \, d\mu > 0.$$

Consider the neighbourhood

$$U = \{\zeta \in F \mid \varphi(\zeta) > 0\}$$

of the point z in the set F. By proposition 1, for some N we have

$$f^{-N}\zeta \cap U = \varnothing, \qquad \text{for all } \zeta \in F.$$

Hence $A_F^N \varphi > 0$. But then $A_F^N \varphi \geq c\mathbb{1}$ for some $c > 0$; therefore, for $m \geq N$ we have

$$A_F^m \varphi \geq A_F^{m-N}(c\mathbb{1}) \geq c\mathbb{1}.$$

This implies that

$$\int \varphi \, d\mu = \lim_{m \to \infty} \int (A_F^m \varphi) \, d\mu \geq c.$$

Finally, let K be an arbitrary compact set containing no excluded points. Since the set of excluded points has an invariant neighbourhood base, there exists a compact set $L \supset K$ containing no excluded points and such that $f^{-1}L \subset L$. But then

$$\left\| A^m \varphi - \int \varphi \, d\mu \right\|_K \leq \left\| A^m \varphi - \int \varphi \, d\mu \right\|_L = \left\| A_L^m \varphi_L - \int_L \varphi_L \, d\mu \right\| \to 0.$$

The theorem is proved. □

The operator A is similar to that of Ruelle, and the above theorem is similar to the Ruelle version of the Perron-Frobenius theorem ([**26**], see also [**4**], [**27**]). Nevertheless the theorem stated above is not contained in the framework of the referred theorems. On the other hand, the Ruelle version can be proved from the standpoint of almost periodicity.

The next theorem states that if z is a non-excluded point, then the roots of the equation $f^m\zeta = z$ (counted with or without their multiplicity) have asymptotically uniform distribution with respect to the measure μ.

We shall denote by δ_ζ a unit mass concentrated at the point ζ. Introduce the following measures:

$$\mu_{m,z} = \frac{1}{n^m} \sum_{\zeta \in f^{-m}z} \delta_\zeta$$

where the roots of the equation $f^m\zeta = z$ are counted with multiplicity;

$$\mu'_{m,z} = \frac{1}{\chi_{m,z}} \sum_{\zeta \in f^{-m}z} \delta_\zeta$$

where the roots of the equation $f^m\zeta = z$ are counted once regardless of their multiplicity and $\chi_{m,z}$ is their total number.

THEOREM 2. *For all points $z \in S^2$, except at most two, the measures $\mu_{m,z}$ and $\mu'_{m,z}$ converge weakly to a probability measure μ independent of z. The measure μ is supported on the set of irregular points.*

Proof. Set $K=\{z\}$ in theorem 1 where z is a non-excluded point. We obtain

$$\int \varphi\, d\mu_{m,z} = A^m\varphi(z) \to \int \varphi\, d\mu,$$

i.e. the sequence $\{\mu_{m,z}\}_{m=1}^{\infty}$ converges weakly to the measure μ. We intend to show that

$$\mu'_{m,z} \to \mu \qquad \text{as } m \to \infty.$$

As in the proof of theorem 1, consider the following three cases:

(a) $z \notin Z_\infty$; then all the roots of the equation $f^m\zeta = z$, for $m=1, 2, \ldots$, are simple, i.e.

$$\mu'_{m,z} = \mu_{m,z} \to \mu \qquad \text{as } m \to \infty.$$

(b) $z \in Z_\infty$ and z is an aperiodic point. Then for some N we have

$$f^{-N}z \cap Z_\infty = \varnothing.$$

Let $f^{-N}z = \{z_1, \ldots, z_{\chi_N}\}$ where z_i are pairwise different and $\chi_N \equiv \chi_{N,z}$. Then

$$f^{-(N+m)}z = \bigcup_i f^{-m}z_i$$

and, therefore, by (a),

$$\mu'_{m+N,z} = \frac{1}{\chi_N n^m}\sum_i \sum_{\zeta \in f^{-m}z_i} \delta_\zeta = \frac{1}{\chi_N}\sum_i \mu'_{m,z_i} \to \mu$$

as $m \to \infty$.

(c) z is a periodic point with period p. Let

$$\mu' = \lim_{m_i \to \infty} \mu'_{m_i,z}$$

be a limiting measure for the family $\{\mu_{m,z}\}_{m=1}^{\infty}$. We show that $\mu' = \mu$, and this, due to the compactness of the space of probability measures, will imply the required result.

Check that for all ε, there exists N such that

$$\frac{\chi_m}{\chi_{m+N}} < \varepsilon$$

for $m = 1, 2, \ldots$. Indeed there exists l, such that in $f^{-m}z$ at most l points belong to Z_∞, $m = 1, 2, \ldots$. It follows that

$$\chi_{m+N} \geq (\chi_m - l)n^N,$$

i.e.

$$\frac{\chi_m}{\chi_{m+N}} \leq \frac{1}{n^N} + \frac{l}{\chi_{m+N}} < \varepsilon$$

for sufficiently large N; ($\chi_{m+N} \to \infty$ as $N \to \infty$ since z is a nonexcluded point).

Now, find k such that

$$\frac{\chi_m}{\chi_{m-kp}} < \varepsilon$$

for $m \geq kp+1$. We have

$$f^{-m}z = f^{-(m-kp)}z \cup \left[\bigcup_{i=1}^{\chi_{kp}-1} f^{-(m-kp)}z_i\right]$$

where z_i are those roots of the equation $f^{kp}\zeta = z$ that differ from z. Hence

$$\mu'_{m,z} = \frac{1}{\chi_m} \sum_{\zeta \in f^{-(m-kp)}z} \delta_\zeta + \frac{1}{\chi_m} \sum_{i=1}^{\chi_{kp}-1} \sum_{\zeta \in f^{-(m-kp)}z_i} \delta_\zeta$$
$$= \frac{\chi_{m-kp}}{\chi_m} \mu'_{m-kp,z} + \frac{1}{\chi_m} \sum_{i=1}^{\chi_{kp}-1} \chi_{m-kp} \mu'_{m-kp,z_i}.$$

Passing to the limit of a subsequence $\{m_i\}$, we obtain

$$\mu' = \eta + \mu \qquad \text{where } \|\eta\| < 2\varepsilon.$$

Since ε is arbitrary, the theorem is proved. □

Let $\varphi(z)$ be a rational function of degree k. Then the equation $f^m\zeta = \varphi(\zeta)$ has $n^m + k$ roots counting multiplicity. Placing into each root a mass proportional to its multiplicity, we obtain a probability measure

$$\mu_{m,\varphi} = \frac{1}{n^m + k} \sum_{f^m\zeta = \varphi(\zeta)} \delta_\zeta.$$

Consider another measure

$$\mu'_{m,\varphi} = \frac{1}{\chi_{m,\varphi}} \sum_{f^m\zeta = \varphi(\zeta)} \delta_\zeta$$

where only different roots are counted, $\chi_{m,\varphi}$ being their total number. In general, we shall denote by $\psi_{m,\varphi}(U)$ and $\chi_{m,\varphi}(U)$ the number of roots of the equation $f^m\zeta = \varphi(\zeta)$ counted with or without their multiplicity respectively, provided they are contained in the set U;

$$\psi_{m,\varphi}(S^2) = n^m + k, \qquad \chi_{m,\varphi}(S^2) \equiv \chi_{m,\varphi}.$$

THEOREM 3. *If a rational function φ differs from the excluded constants, then*

$$\mu_{m,\varphi} \to \mu \quad \textit{and} \quad \mu'_{m,\varphi} \to \mu \qquad \textit{as } m \to \infty$$

(*the convergence is weak*).

Proof. We assume that $\varphi \neq \text{const}$. Let Ω be an open set on the sphere. Fix a point $c \in \varphi\Omega \backslash Z_\infty$, a natural number l and an $\varepsilon > 0$. Cutting the sphere in the proper way, we find a simply-connected domain

$$\tilde{X} \subset S^2 \backslash Z_l$$

which contains c and some irregular point, and $\mu(\varphi^{-1}\tilde{X}) = 1$. Set

$$X = \varphi^{-1}\tilde{X} \cap \Omega$$

and find open sets U, V, W such that

$$X \Supset U \Supset V \Supset W, \qquad c \in \varphi U$$

and

$$\mu(\Omega \backslash W) = \mu(X \backslash W) < \varepsilon.$$

Let $\{f_\lambda^{-m}\}_{\lambda=1}^{\sigma_m}$ be the family of all branches of the function f^{-m} defined in the domain $\tilde{X}$. We know that

$$0 \le n^m - \sigma_m \le 2\tau n^{m-l}$$

by proposition 4. Introduce the notation

$$\tilde{U} = \varphi U \Subset \tilde{X}, \qquad r = d(V, \partial U) > 0.$$

By proposition 3 find N such that

$$m \ge N \Rightarrow \operatorname{diam} f_\lambda^{-m} \tilde{U} < r \qquad \lambda = 1, \ldots, \sigma_m.$$

Suppose that $a \equiv f_\lambda^{-m} c \in V$ for some m, λ. Then

$$f_\lambda^{-m} \tilde{U} \Subset B(a; r)$$

where

$$B(a; r) = \{z \colon d(a, z) < r\},$$

is a circle of radius r with the centre at a. Since $\varphi B(a; r) \subset \tilde{U}$,

$$(f_\lambda^{-m} \varphi) B(a; r) \Subset B(a; r).$$

By the fixed point theorem there exists $z = z(m, \lambda)$, such that $(f_\lambda^{-m}\varphi)(z) = z$, which implies that z is a root of the equation $f^m \zeta = \varphi(\zeta)$. Since $z(m, \lambda) \in \operatorname{Im} f_\lambda^{-m}$, $z(m, \lambda) \ne z(m, \alpha)$ for $\alpha \ne \lambda$. Hence, for $m \ge N$ the number $\chi_{m,\varphi}(U)$ of roots of the equation $f^m \zeta = \varphi(\zeta)$ contained in U is not less than the number $t_m(V)$ of points of the form $f_\lambda^{-m} c$, $\lambda = 1, \ldots, \sigma_m$, contained in V. But

$$\psi_{m,c}(V) - t_m(V) \le n^m - \sigma_m \le 2\tau n^{m-l}.$$

Hence,

$$\mu_{m,\varphi}(U) = \frac{1}{n^m + k} \psi_{m,\varphi}(U) \ge \frac{n^m}{n^m + k} [\mu_{m,c}(V) - 2\tau n^{-l}]. \tag{1}$$

Now let $\nu = \lim_{m_k \to \infty} \mu_{m_k,\varphi}$ be a limit point for the family $\{\mu_{m,\varphi}\}_m$ of measures. Then

$$\begin{aligned} \nu(\Omega) &\ge \overline{\lim_{m_k \to \infty}} \mu_{m_k,\varphi}(U) \\ &\ge \underline{\lim_{m_k \to \infty}} [\mu_{m_k,c}(V) - 2\tau n^{-l}] \ge \mu(W) - 2\tau n^{-l} \\ &\ge \mu(\Omega) - 2\tau n^{-l} - \varepsilon. \end{aligned}$$

As l and ε are arbitrary, $\nu(\Omega) \ge \mu(\Omega)$. So $\nu \ge \mu$ and since ν and μ are probability measures, $\nu = \mu$. Finally, ν is an arbitrary limiting point for the precompact family $\{\mu_{m,\varphi}\}_m$, so $\mu_{m,\varphi} \to \mu$ as $m \to \infty$.

We have shown above that

$$\chi_{m,\varphi}(U) \ge t_m(V)$$

(though we only used the inequality $\psi_{m,\varphi}(U) \ge t_m(V)$). Besides,

$$\chi_{m,\varphi} \le n^m + k.$$

So the main estimate (1) is valid for the measure $\mu'_{m,\varphi}$:

$$\mu'_{m,\varphi}(U)=\frac{\chi_{m,\varphi}(U)}{\chi_{m,\varphi}}\geq\frac{n^m}{n^m+k}[\mu_{m,c}(V)-2\tau n^{-l}].$$

This estimate implies the rest automatically. □

COROLLARY. *Periodic points of a rational endomorphism have an asymptotically uniform distribution with respect to the measure* μ.

Remark. The measure μ was constructed in [**6**] when f is a polynomial. In this paper μ appears as the Robin measure on the set F. This property does not hold in general for a rational function. A similar measure for expanding mappings was constructed in [**1**].

5. *Ergodic properties of the dynamical system* (f, μ)

For the concepts and facts used below see, for example, [**25**].

PROPOSITION 5. *The measure* μ *is* f*-invariant.*

Proof. Let $\varphi\in C(S^2)$. Since the measure μ is A^*-invariant,

$$\int(\varphi\circ f)\,d\mu=\int A(\varphi\circ f)\,d\mu.$$

But

$$A(\varphi\circ f)=\varphi.$$

Let ε denote, as usual, the partition of the sphere into one-point subsets; let ν_z denote the conditional measure on the element of the partition $f^{-1}\varepsilon$ containing the point z; and let

$$P\colon L_2(\mu)\to L_2(\mu)$$

denote the operator of conditional expectation:

$$P\varphi(z)=\int\varphi\,d\nu_z.$$

Finally, let U be the isometry of $L_2(\mu)$ induced by the endomorphism f, and let U^* be its adjoint operator. P is orthogonal projection onto $\operatorname{Im} U$. Hence $UU^*=P$, i.e.

$$U^*\varphi(fz)=\int\varphi\,d\nu_z.$$

LEMMA.

$$\nu_z=\mu_{1,fz}=\frac{1}{n}\sum_{\zeta:\,f\zeta=fz}\delta_\zeta.$$

The operator A *is the restriction of* U^* *onto the space* $C(S^2)$.

More precisely, $U^*j=jA$ where the homomorphism $j\colon C(S^2)\to L_2(\mu)$ corresponds in a natural way to the continuous function element of the space $L_2(\mu)$. In the following j will be omitted, to simplify notation.

Proof. For $\varphi \in C(S^2)$ we have

$$(UA)\varphi(z) = A\varphi(fz) = \frac{1}{n} \sum_{\zeta:\, f\zeta = fz} \varphi(\zeta) = \int \varphi \, d\mu_{1,fz}.$$

Hence

$$\int \varphi \, d\mu = \int (UA)\varphi \, d\mu = \int \left(\int \varphi \, d\mu_{1,fz} \right) d\mu.$$

Since the conditional measures are uniquely defined by the property

$$\int \varphi \, d\mu = \int \left(\int \varphi \, d\nu_z \right) d\mu \qquad \varphi \in C(S^2),$$

the first part of the lemma is established. The second part follows from the first:

$$A\varphi(fz) = \int \varphi \, d\mu_{1,fz} = \int \varphi \, d\nu_z = U\varphi(fz) \qquad \varphi \in C(S^2). \qquad \square$$

THEOREM 4. (a) *The dynamical system* (f, μ) *is exact.*

(b) *The entropy* $h_\mu(f)$ *equals* $\ln n$.

Proof. (a) It follows from theorem 1 and the above lemma that

$$(U^*)^m \varphi \to \int \varphi \, d\mu$$

in the space $L_2(\mu)$. But this property is equivalent to the exactness of f.

(b) It follows from the above lemma that the conditional entropy $H(f^{-1}\varepsilon|\varepsilon)$ equals $\ln n$. But

$$h_\mu(f) \geq H(f^{-1}\varepsilon|\varepsilon).$$

We shall prove the reverse inequality in the next section. $\square$

Remark. It is possible that the system (f, μ) is isomorphic to the one-sided Bernoulli shift with n equiprobable states.

We shall formulate the following results without proof.

THEOREM 5. *The measure* μ *is mutually singular with the Lebesgue measure on the sphere.*

We shall say that $z \in S^2$ is a *point of the first kind* if:

(a) z is irregular, or

(b) z is regular and all limit functions of the family $\{f^m\}_{m=1}^{\infty}$ in a neighbourhood of z are constants.

Otherwise z is a *point of the second kind.*

PROPOSITION 6. *Let* z *be a point of the second kind. Then there exists a component* U *of the set of regular points such that*

(a) $f^k U = U$, *for some* k; $\varphi(f^k\zeta) = \lambda\varphi(\zeta)$, $\zeta \in U$, *where* φ *is a conformal map of* U *onto the unit circle or onto the ring,* $|\lambda| = 1$, *and* $\arg \lambda$ *is irrational*;

https://doi.org/10.1017/S0143385700002030 Published online by Cambridge University Press

(b) $f^m z \in U$ *for some m.*

Let $X = \bigcup_{c \in Z} \omega(c)$.

THEOREM 6. *Assume that $F \neq S^2$. Then for almost all (with respect to the Lebesgue measure) points of the first kind we have*

$$f^m z \to X, \qquad as\ m \to \infty.$$

COROLLARY. *Assume that $F \neq S^2$ and X is finite and does not contain neutral periodic points. Then the Lebesgue measure of F equals* 0.

The last result strengthens the results of Fatou [**10**] and Brolin [**6**]. It seems plausible that, if $F \neq S^2$, then the Lebesgue measure of F always equals 0.

6. *Entropy of the rational endomorphism of the Riemann sphere*

A brief proof of the main result of this section is given in [**21**].

Let M be a metric space with distance d; $Z = \{z_1, \ldots, z_t\}$ be a finite subset of M; $|Z| = t$ be the number of elements in Z. Let

$$d(\zeta, Z) = \min_{z \in Z} d(\zeta, z) \qquad \zeta \in M,$$

$$B(Z; r) = \{\zeta \in M \mid d(\zeta, Z) < r\}.$$

Let $0 < \varepsilon < \delta$, $0 < k < 1$. Set

$$\rho_Z(\zeta) \equiv \rho_Z(\zeta; \delta, k) = \min\{kd(\zeta, Z), \delta\}.$$

Denote by $\gamma(\delta, k, Z, \varepsilon)$ the least number of points $u_1, \ldots, u_\gamma$ such that

$$\bigcup_{\mu=1}^{\gamma} B(u_\mu; \rho_Z(u_\mu)) \supset M \backslash B(Z; \varepsilon);$$

systems with this property will be called *feasible.*

LEMMA. *Let $M = B(Z; \delta) \subset \mathbb{R}^2$, where $\mathbb{R}^2$ is the Euclidian plane. Then*

$$\gamma(\delta, k, Z, \varepsilon) \leq C(k)|Z|^2[\ln(\delta/\varepsilon) + A(k)].$$

Proof. Consider the r-neighbourhood $B(Z; r)$ of the set Z. Its boundary is a collection of arcs L_{ij} of circumferences

$$S_i = \{\zeta \mid |\zeta - z_i| = r\}$$

left after deleting arcs, contained in the circles $B(z_j; r)$, $j \neq i$, from S_i. Each S_i contains at most $t-1$ such arcs. Consider some of these arcs

$$L_{ij} = [e, f] \subset S_i.$$

Let φ_{ij} be its angular size. Let $u_1 (= e), u_2, \ldots, u_s$ be a sequence of points on L_{ij} such that the arcs $[u_\mu, u_{\mu+1}]$, $1 \leq \mu \leq s-1$, have their angular size equal to k, and $0 < [u_s, f] \leq k$. Then

$$s < \frac{\varphi_{ij}}{k} + 1.$$

Having applied this procedure to all arcs L_{ij} of the circumference S_i, we obtain at most

$$\frac{2\pi}{k}+t-1\leq\frac{2\pi t}{k}$$

points. Having processed all the circumferences we obtain at most $2\pi t^2/k$ points $u_1,\ldots,u_l$. It is important that this estimate is independent of r. It follows by similarity that there exists $\alpha=\alpha(k)>1$, also independent of r, such that

$$\bigcup_{\mu=1}^{l} B(u_\mu;kr)\supset B(Z;\alpha r)\backslash B(Z;r).$$

Perform the above construction for all values of $r=\alpha^i\varepsilon<\delta$. The number of such values does not exceed

$$[\ln(\delta/\varepsilon)/\ln\alpha]+1.$$

Therefore, we shall obtain not more than $[2\pi t^2/(k\ln\alpha(k))](\ln(\delta/\varepsilon)+\ln\alpha(k))$ points $u_1,\ldots,u_\gamma$. The system constructed has the following property:

$$\bigcup_{\mu=1}^{\gamma} B(u_\mu;kd(u_\mu;Z))\supset B(Z;\delta)\backslash B(Z;\varepsilon)=M\backslash B(Z;\varepsilon).$$

Since

$$kd(u_\mu,Z)=\rho_Z(u_\mu),$$

the system $\{u_\mu\}_{\mu=1}^{\gamma}$ is feasible. Hence, $\gamma(\delta,k,Z,\varepsilon)\leq\gamma$, and we have obtained the required estimate with

$$C(k)=\frac{2\pi}{k\ln\alpha(k)},\qquad A(k)=\ln\alpha(k).\qquad\square$$

PROPOSITION 7. *Let M be a smooth compact two-dimensional manifold with the Riemannian metric d. Then*

$$\gamma(\delta,k,Z,\varepsilon)\leq C(k)|Z|^2(\ln(\delta/\varepsilon)+A(k))+R(\delta,k).$$

If Ω is a dense subset of M, then the related system $\{u_\mu\}_{\mu=1}^{\gamma}$ can be chosen from Ω.

Remark. The logarithmic dependence of the estimate on ε is most essential. The form of dependence on δ and k is of no importance, since these parameters do not vary with 'time'.

Proof. The proof consists of applying the lemma locally on the manifold, combined with the use of the local equivalence of the manifold Riemannian metric and the plane Euclidian metric. Some difficulties arise close to the map boundary. The complete proof is rather cumbersome, but we shall bear with it to the end.

Denote by $\tilde{d}$ the Euclidian metric on the plane $\mathbb{R}^2$, and by $D(z;r)$ the Euclidian circles (to avoid confusion with similar objects on the manifold M).

Consider the atlas

$$\varphi_s\colon V^s\to\mathbb{R}^2\qquad 1\leq s\leq l$$

on the manifold M. There exist open sets $U^s\Subset V^s$ such that

$$\bigcup_{s=1}^{l} U^s=M.$$

https://doi.org/10.1017/S0143385700002030 Published online by Cambridge University Press

For some $\mathscr{C}_1$, $\mathscr{C}_2 > 0$ the following estimates are valid:

$$\mathscr{C}_1 d(z, \zeta) \leq \tilde{d}(\varphi_s(z), \varphi_s(\zeta)) \leq \mathscr{C}_2 d(z, \zeta)$$

$z, \zeta \in U^s$; $s = 1, \ldots, l$.

If $X \subset M$, then denote the complement of X by CX. Fix σ and κ, such that

$$0 < \sigma < \kappa < \inf_{Z \in M} \max_{s:\, Z \in U^s} d(z, CU^s)$$

which is the Lebesgue number of the atlas $\{U^s\}_{s=1}^l$. Denote $\{z \in Z | d(z, CU^s) > \alpha\}$ by Z_α^s. Then

$$Z = \bigcup_{s=1}^{l} Z_\kappa^s.$$

Set

$$\delta_0 = \min\{(\kappa - \sigma)\mathscr{C}_1 / 4\mathscr{C}_2; \sigma\}.$$

Let us prove the required estimate for $\delta < \delta_0$.
The relation $\delta < (\kappa - \sigma)/2$ immediately implies that

$$B(Z_\kappa^s; \delta) \cap B(Z \backslash Z_\sigma^s; \varepsilon) = \varnothing.$$

It follows from here that

$$B(Z_\kappa^s; \delta) \backslash B(Z; \varepsilon) = B(Z_\kappa^s; \delta) \backslash B(Z_\sigma^s; \varepsilon) \tag{1}$$

We shall apply the lemma for the parameter values

$$\tilde{\delta} = \mathscr{C}_2 \delta, \quad \tilde{\varepsilon} = \mathscr{C}_1 \varepsilon, \quad \tilde{k} = (\mathscr{C}_1 / \mathscr{C}_2) k.$$

If $X \subset U^s$, then we denote $\varphi_s X \subset \mathbb{R}^2$ by $\tilde{X}$.

Since $\delta < \sigma$, the sets $B(Z_\kappa^s; \delta)$ and $B(Z_\sigma^s; \varepsilon)$ lie within the set U^s. Therefore, we have

$$\varphi_s B(Z_\kappa^s; \delta) \backslash \varphi_s B(Z_\sigma^s; \varepsilon) \subset D(\tilde{Z}_\kappa^s; \tilde{\delta}) \backslash D(\tilde{Z}_\sigma^s; \tilde{\varepsilon}).$$

Combining the latter inclusion with (1), we obtain

$$B(Z_\kappa^s; \delta) \backslash B(Z; \varepsilon) \subset \varphi_s^{-1}[D(\tilde{Z}_\kappa^s; \tilde{\delta}) \backslash D(\tilde{Z}_\sigma^s; \tilde{\varepsilon})], \tag{2}$$

where φ_s^{-1} is the operation of taking the inverse image; (actually, as we shall see later, when $\delta < \delta_0$ the set $D(\tilde{Z}_\kappa^s; \tilde{\delta})$ lies within $\tilde{U}^s$).

Further we use the lemma, with the data $(\tilde{\delta}, \tilde{k}, \tilde{Z}_\sigma^s, \tilde{\varepsilon})$ to construct the minimal feasible system $\{\tilde{u}_\mu\}$, and choose those of them which satisfy the condition

$$D(\tilde{u}_\mu^s; \tilde{k}\tilde{d}(\tilde{u}_\mu, \tilde{Z}_\sigma^s)) \cap D(\tilde{Z}_\kappa^s; \tilde{\delta}) \neq \varnothing.$$

Then our system $\{\tilde{u}_\mu^s\}_{\mu=1}^{\gamma_s}$ has the following properties:

(ã) $d(\tilde{u}_\mu^s; \tilde{Z}_\kappa^s) < 2\tilde{\delta}$;

(b̃) $\bigcup_{\mu=1}^{\gamma_s} D(\tilde{u}_\mu^s; \tilde{k}\tilde{d}(\tilde{u}_\mu^s, \tilde{Z}_\sigma^s)) \supset D(\tilde{Z}_\kappa^s; \tilde{\delta}) \backslash D(\tilde{Z}_\sigma^s; \tilde{\varepsilon})$;

(c̃) $\gamma_s \leq \tilde{C}(\tilde{k}) |Z_\sigma^s|^2 (\ln(\tilde{\delta}/\tilde{\varepsilon}) + \tilde{A}(\tilde{k}))$

$\leq \tilde{C}(\tilde{K}) t^2 (\ln(\delta/\varepsilon) + A_1(k))$

where $A_1(k) = \tilde{A}(\tilde{k}) + \ln(\mathscr{C}_2/\mathscr{C}_1)$.

We check that $\tilde{u}^s_\mu \in \tilde{U}^s$. Indeed,

$$\tilde{d}(\tilde{Z}^s_\kappa, C\tilde{U}^s) \geq \mathscr{C}_1 d(Z^s_\kappa, CU^s) > \mathscr{C}_1\kappa > 2\tilde{\delta} > \tilde{d}(\tilde{u}^s_\mu, \tilde{Z}^s_\kappa),$$

and we obtain the required relation. Thus we can consider the system

$$u^s_\mu = \varphi_s^{-1}\tilde{u}^s_\mu \qquad 1 \leq s \leq l, \quad 1 \leq \mu \leq \gamma_s.$$

Now check the relation

$$d(u^s_\mu, Z) = d(u^s_\mu, Z^s_\sigma). \tag{3}$$

By ($\tilde{\text{a}}$) we have

$$d(u^s_\mu, Z^s_\kappa) \leq \frac{1}{\mathscr{C}_1}\tilde{d}(\tilde{u}^s_\mu, \tilde{Z}^s_\kappa) \leq \frac{2\mathscr{C}_2}{\mathscr{C}_1}\delta.$$

Hence,

$$\begin{aligned} d(u^s_\mu, Z\backslash Z^s_\sigma) &\geq -d(u^s_\mu, Z^s_\kappa) + d(Z^s_\kappa, CU^s) - d(CU^s, Z\backslash Z^s_\sigma) \\ &> -\frac{2\mathscr{C}_2}{\mathscr{C}_1}\delta + \kappa - \sigma > \frac{2\mathscr{C}_2}{\mathscr{C}_1}\delta \geq d(u^s_\mu, Z^s_\kappa). \end{aligned}$$

(3) follows immediately. Further,

$$\begin{aligned} \varphi_s^{-1}D(\tilde{u}^s_\mu; \tilde{k}\tilde{d}(\tilde{u}^s_\mu, \tilde{Z}^s_\sigma)) &\subset B\left(u^s_\mu; \frac{\tilde{k}}{\mathscr{C}_1}\tilde{d}(\tilde{u}^s_\mu, \tilde{Z}^s_\sigma)\right) \\ &\subset B\left(u^s_\mu; \frac{\tilde{k}\mathscr{C}_2}{\mathscr{C}_1}d(u^s_\mu, Z^s_\sigma)\right) \\ &= B(u^s_\mu; kd(u^s_\mu, Z)). \end{aligned}$$

From the inclusion (2) and property ($\tilde{\text{b}}$) of the system $\{\tilde{u}^s_\mu\}_{\mu=1}^{\gamma_s}$ it follows that

$$B(Z^s_\kappa; \delta)\backslash B(Z; \varepsilon) \subset \bigcup_{\mu=1}^{\gamma_s} B(u^s_\mu; kd(u^s_\mu, Z^s_\sigma)).$$

The latter two inclusions imply

$$B(Z^s_\kappa; \delta)\backslash B(Z; \varepsilon) \subset \bigcup_{\mu=1}^{\gamma_s} B(u^s_\mu; kd(u^s_\mu, Z^s_\sigma)).$$

Finally, taking the union over all the maps of the atlas, we obtain the system of points $\{u_\mu\}_{\mu=1}^{\beta}$ such that

(a) $\bigcup_\mu B(u_\mu; kd(u_\mu, Z)) \supset B(Z; \delta)\backslash B(Z; \varepsilon)$

and their number β is estimated as follows

(b) $\beta = \sum_{s=1}^{l} \gamma_s \leq C_1(k)t^2(\ln(\delta/\varepsilon) + A_1(K))$

where $C_1(k) = l\tilde{C}(\tilde{k})$.

As before we may assume

(c) $B(u_\mu; kd(u_\mu, Z)) \cap B(Z; \delta) \neq \varnothing$.

Suppose temporarily that $k \leq \frac{1}{2}$. Then (c) implies

$$d(u_\mu, Z) \leq \delta + kd(u_\mu, Z).$$

From here we obtain

$$kd(u_\mu, Z) \leq (k/1-k)\delta \leq \delta.$$

https://doi.org/10.1017/S0143385700002030 Published online by Cambridge University Press

Hence, $\rho_Z(u_\mu) = kd(u_\mu, Z)$. Thus the system $\{u_\mu\}_{\mu=1}^{\beta}$ is feasible for the set $B(Z; \delta)$. Now we must make it complete on the whole manifold M. For this purpose we consider the minimal $(k\delta)$-net $\{u_\mu\}_{\mu=\beta+1}^{\gamma}$ of the set $M \backslash B(Z; \delta)$ (belonging to this set). The number of elements $\gamma - \beta$ in this net does not exceed the number $R_1(\delta, k)$ (independent of Z and ε) of elements in the minimal $k\delta/2$-net of the space M. Thus

$$\gamma \leq C_1(k)t^2(\ln(\delta/\varepsilon) + A_1(k)) + R_1(k, \delta).$$

Besides, since $d(u_\mu, Z) \geq \delta$, $\rho_Z(u_\mu) \geq k\delta$, $\mu = \beta+1, \ldots, \gamma$. Hence,

$$\bigcup_{\mu=\beta+1}^{\gamma} B(u_\mu; \rho_Z(u_\mu)) \supset \bigcup_{\mu=\beta+1}^{\gamma} B(u_\mu; k\delta) \supset M \backslash B(Z; \delta),$$

and we have obtained the required estimate for $\delta < \delta_0$, $k \leq \frac{1}{2}$.

Now we shall get rid of the last constraints. To do this, set

$$k' = \min(k, \tfrac{1}{2}), \quad \delta' = \min(\delta, \delta_0/2) \quad \text{and} \quad \varepsilon' = \min(\varepsilon, \delta'/2),$$

the latter is needed to guarantee the relation $\varepsilon' < \delta'$. Note that, with Z fixed, the function $\gamma(\delta, k, Z, \varepsilon)$ increases when δ, k, ε decrease. Therefore,

$$\gamma(\delta, k, Z, \varepsilon) \leq \gamma(\delta', k', Z, \varepsilon') \leq C(k)t^2(\ln(\delta/\varepsilon) + A(k)) + R(\delta, k)$$

where $C(k) = C_1(k')$, $A(k) = A_1(k') + \ln 2$, $R(\delta, k) = R_1(\delta', k')$. We have obtained the required estimate.

Finally, to prove uniqueness of the measure of maximal entropy (see § 7) we need $u_\mu \in \Omega$, $1 \leq \mu \leq \gamma$, where Ω is some dense subset of the manifold M. It suffices to construct a minimal feasible system $\{u'_\mu\}$ with parameters $(\delta/2, k/2, Z, \varepsilon)$ and approximate u'_μ by the points $u_\mu \in \Omega$ in such a way that

$$B(u_\mu; \rho_Z(u_\mu; k, \delta)) \supset B(u'_\mu; \rho_Z(u'_\mu; k/2, \delta/2))$$

for $1 \leq \mu \leq \gamma(\delta/2, k/2, Z, \varepsilon)$. Only the constants C, A, R will change in the estimate. □

As before, we shall denote Euclidian circles in the plane $\mathbb{C}$ by $D(u; \rho)$–to be distinguished from circles $B(u; \rho)$ in the spherical metric d. Let $g(z)$ be a holomorphic function in the circle $D(0; \rho)$. By $M(g, R)$ (resp. $m(g, R)$) we denote the radius of the circle with the centre at $g(0)$, circumscribed (resp. inscribed) over (resp. in) the domain $gD(0; R)$. We shall need

The Köebe distortion theorem. Let $0 < r < 1$. Let the function $g(z)$ be holomorphic and univalent in the circle $D(0; \rho)$. Then

$$\frac{M(g, r\rho)}{m(g, r\rho)} \leq \left(\frac{1+r}{1-r}\right)^2.$$

The proof is given in [7].

PROPOSITION 8. *Let $0 < \eta < \pi/2$, $\delta > 0$. There exists $k = k(\eta, \delta) \in (0, 1)$ possessing the following property: if*

$$h: B(u; \rho) \to S^2 \qquad 0 < \rho < \pi$$

is a univalent meromorphic function, taking no values from some η-net X on the

sphere, then

$$hB(u;\, k\rho) \subset B(hu;\, \delta).$$

Proof. Let w be the point diametrically opposite to the point $hu = v$, i.e. $d(v, w) = \pi$. Let $\xi \in B(w;\, \eta) \cap X$. Then $d(v, \xi) > \pi - \eta$.

Consider the two isometries of the Riemann sphere

$$\varphi(z) = \frac{\bar{\xi} z + 1}{-z + \xi}, \qquad \psi(z) = \frac{z - u}{\bar{u} z + 1}.$$

We have $\varphi(\xi) = \infty$, $\psi(u) = 0$. Consider the function

$$g = \varphi h \psi^{-1}.$$

It is meromorphic and univalent in the circle

$$B(0;\, \rho) = \psi B(u;\, \rho).$$

Since ξ is an excluded value of the function h, $\varphi(\xi) = \infty$ is an excluded value of the function g, i.e. g is holomorphic in the circle $B(0;\, \rho)$.

Set $\omega = g(0)$. As φ is isometric, then

$$d(\omega, \infty) = d(v, \xi) > \pi - \eta.$$

Hence,

$$d(\omega, 0) = \pi - d(\omega, \infty) < \eta.$$

The function h takes no values from the η-net φX. Hence, there exists an h-exclusive value

$$\zeta \in B(\omega;\, \eta) \subset B(0;\, 2\eta).$$

The estimates

$$2\cos^2 \eta |z - \zeta| \le d(z, \zeta) \le 2|z - \zeta|$$

are valid in the circle $B(0;\, 2\eta) = D(0;\, \tan \eta)$. Therefore,

$$|\zeta - \omega| \le \frac{\eta}{2\cos^2 \eta}.$$

Thus

$$m\left(g, \tan \frac{\rho}{2}\right) \le \frac{\eta}{2\cos^2 \eta}.$$

By the distortion theorem

$$M\left(g, r \tan \frac{\rho}{2}\right) \le \left(\frac{1+r}{1-r}\right)^2 \frac{\eta}{2\cos^2 \eta},$$

(r is fixed once and for ever). Let

$$k < \min\left\{r, \frac{\delta r \cos^2 \eta}{\eta}\left(\frac{1-r}{1+r}\right)^2\right\}.$$

By the Schwartz lemma

$$M\left(g, k \tan \frac{\rho}{2}\right) = M\left(g, \frac{k}{r}\left(r \tan \frac{\rho}{2}\right)\right) \le \frac{k}{r} M\left(g, r \tan \frac{\rho}{2}\right) < \delta/2.$$

So

$$gD\left(0;\, k\tan\frac{\rho}{2}\right) \subset D(\omega;\, \delta/2).$$

Return now to the spherical metric. Since $d(z,\zeta) \le 2|z-\zeta|$, it follows that $D(\omega;\, \delta/2) \subset B(\omega;\, \delta)$. Since

$$\tan\frac{k\rho}{2} \le k\tan\frac{\rho}{2} \qquad 0<\rho<\pi,\, 0<k<1,$$

we have

$$B(0;\, k\rho) = D\left(0;\, \tan\frac{\rho}{2}\right) \subset D\left(0;\, k\tan\frac{\rho}{2}\right).$$

Hence, $gB(0;\, k\rho) \subset B(\omega;\, \delta)$. Returning to the function h, we shall obtain the required result. □

We shall make use of the definition of the topological entropy in terms of (m, δ)-nets (see [**2**], [**9**]). Let $f: K \to K$ be a continuous endomorphism of a compact metric space K with distance d; $X \subset K$. Define the metric d_m as

$$d_m(x, y) = \max_{0 \le i \le m} d(f^i x, f^i y).$$

Set

$$B_m(x;\, \delta) = \{y \in K \mid d(f^i x, f^i y) < \delta, \quad i = 0, \ldots, m\},$$

B_m is a ball in the metric d_m. Denote the minimal number of elements in (m, δ)-nets of the set X (i.e. in δ-nets of the metric d_m) by $r_m(X, \delta)$. Then

$$h_f(X) = \lim_{\delta \to 0} \overline{\lim_{m \to \infty}} \frac{1}{m} \ln r_m(X, \delta).$$

If $X = K$, then we shall write $r_m(X, \delta) \equiv r_m(\delta)$, and $h_f(X) \equiv h(f)$ is the topological entropy of the endomorphism f.
Write

$$\phi_\delta(x) = \{y \in K \mid d(f^i x, f^i y) < \delta, \quad i = 0, 1, \ldots\}.$$

The following entropy-type characteristic of f is defined in [**2**]:

$$h_f^*(\delta) = \sup_{x \in K} h_f(\phi_\delta(x)).$$

The endomorphism f is called *h-expansive* if there exists $\delta > 0$, such that $h_f^*(\delta) = 0$, [**2**].
The endomorphism f is called *asymptotically h-expansive* if $\lim_{\delta \to 0} h_f^*(\delta) = 0$, [**22**].
In [**9**], [**11**], [**12**] it was shown that

$$h(f) = \sup_{\mu \in M_f(K)} h_\mu(f)$$

where $M_f(K)$ is the space of f-invariant measures. If $h(f) = h_\mu(f)$, $\mu \in M_f(K)$, then μ is called a *measure of maximal entropy.*
We now give a construction of an (m, δ)-net for a rational endomorphism of the sphere. Fix $\eta \in (0, \pi/2)$ and find $k = k(\eta, \delta) \in (0, 1)$ by proposition 8. Consider a

finite η-net X of the Riemann sphere containing all the critical points of the endomorphism f. Set

$$X_m = \bigcup_{i=1}^{m} f^i X$$

for $m = 1, 2, \ldots$. Finally, set

$$\varepsilon_m = \frac{\delta}{2L^m}$$

where L is the Lipschitz constant of f in the spherical metric. Let $\{u_\mu\}_{\mu=1}^{\gamma_m}$ be a minimal feasible point system constructed according to the data $(\delta/2, k, X_m, \varepsilon_m)$ (thus $\gamma_m = \gamma(\delta/2, k, X_m, \varepsilon_m)$).

Let $z \in S^2$. Consider the trajectory $\{f^i z\}_{i=0}^{m}$ of length m. Suppose that for some $j \in [0, m-1]$

$$f^j z \in S^2 \backslash B(X_m; \varepsilon_m) \quad \text{and} \quad f^{j+i} z \in B(X_m; \varepsilon_m).$$

Then we can find a point u_μ in the feasible system and $z_\nu \in Z_m$ such that

$$f^j z \in B(u_\mu; \rho_{m,\mu}) \quad \text{and} \quad f^{j+1} z \in B(z_\nu; \varepsilon_m)$$

(where $\rho_{m,\mu} = \rho_{X_m}(u_\mu)$). Furthermore, $\rho_{m,\mu} \le kd(u_\mu, X_m)$, so in the circle $B(u_\mu; (1/k)\rho_{m,\mu})$ there exist all branches f_λ^{-i}, $1 \le i \le m$, $1 \le \lambda \le n^i$, of the function f^{-m}. Being inverse to a single-valued function, all f_λ^{-i} are univalent. f_λ^{-i} take no values from the η-net X. Each branch is uniquely determined by its value at some point. We shall single out $f_{\lambda_i}^{-i}$ in the following manner:

$$f_{\lambda_i}^{-i} \xi = f^{j-i} z$$

where $\xi = f^j z$, $1 \le i \le j$. By proposition 8

$$d(f_{\lambda_i}^{-i} \xi, f_{\lambda_i}^{-i} u) < \delta/2 \qquad 1 \le i \le j. \tag{1}$$

Set $v_{\mu,\lambda_j} = f_{\lambda_j}^{-i} u_\mu$. It is easy to check that

$$f_{\lambda_i}^{-i} u_\mu = f^{j-i} v_{\mu,\lambda_j}.$$

Hence, (1) can be rewritten as

$$d(f^{j-i} v_{\mu,\lambda_j}, f^{j-i} z) < \delta/2, \qquad 1 \le i \le j$$

or

$$d(f^i v_{\mu,\lambda_j}, f^i z) < \delta/2 \qquad i = 0, \ldots, j-1. \tag{a}$$

Furthermore, since $\rho_{m,\mu} < \delta/2$,

$$d(f^j z, f^j v_{\mu,\lambda_j}) = d(f^j z, u_\mu) < \delta/2. \tag{b}$$

Finally, $f^{j+1} z \in B(z_\nu; \varepsilon_m)$ implies that

$$d(f^i z, f^{i-j-1} z_\nu) \le L^{i-j-1} d(f^{j+1} z, z_\nu) < \delta/2 \qquad i = j+1, \ldots, m. \tag{c}$$

If, for all $i = 0, 1, \ldots, m$, we have $f^i z \notin B(X_m; \varepsilon_m)$, then, (set $j = m$), there exists a point $v_{\mu,\lambda_m} = f_{\lambda_m}^{-m} u_\mu$ such that

$$d(f^i z, f^i v_{\mu,\lambda_m}) < \delta/2 \qquad i = 0, \ldots, m.$$

If, on the contrary, $f^i z \in B(X_m; \varepsilon_m)$ for $i = 0, \ldots, m$, we can find (set $j = -1$) a point $z_\nu \in X_m$, such that

$$d(f^i z, f^i z_\nu) < \delta/2 \qquad i = 0, \ldots, m.$$

So to each point $z \in S^2$ we assign a moment of time $-1 \leq j \leq m$ and a pair of points $v_{\mu,\lambda}$, z_ν, such that the 'start' $\{f^i z\}_{i=0}^j$ of the orbit of the point z is $\delta/2$-close to the orbit of the point $v_{\mu,\lambda}$ (see (a) and (b)), while the 'finish' $\{f^i z\}_{i=j+1}^m$ is $\delta/2$-close to the orbit of the point z_ν (see (c)).

For each triple $j \in [-1, m]$; $v_{\mu,\lambda} = f_\lambda^{-j} u_\mu (\mu = 1, \ldots, \gamma_m; \lambda = 1, \ldots, n^j)$ and $z_\nu \in X_m$, find the point $x = x(j, \mu, \lambda, \nu)$, such that the triple $(j, v_{\mu,\lambda}, z_\nu)$ corresponds to the point x in the sense just described (if such a point x exists). Then for any point $z \in S^2$ there exists a point $x = x(j, \mu, \lambda, \nu)$, such that

$$d(f^i z, f^i x) < \delta \qquad i = 0, \ldots, m.$$

Therefore, the points $x(j, \mu, \lambda, \nu)$ form an (m, δ)-net of the sphere S^2.

The following theorem provides the answer to the question formulated in [**5**]. Some partial results for endomorphisms of degree 2 were obtained in [**8**].

THEOREM 7. *Let $f: S^2 \to S^2$ be a rational endomorphism of the Riemann sphere. Then*

$$h(f) = \ln n.$$

Proof. By the Goodwin theorem [**12**],

$$h(f) \geq h_\mu(f) \geq \ln n$$

(see theorem 4(b)). The estimate $h(f) \geq \ln n$ follows also from the Misiurevich-Przytycki theorem [**23**].

To prove the estimate from above we make use of the constructed (m, δ)-net $\{x(j, \mu, \lambda, \nu)\}$. Here $-1 \leq j \leq m$;

$$1 \leq \mu \leq \gamma_m \leq C(k)|X|^2 m^2 (m \ln L + A(k)) + R(\delta, k) = O(m^3)$$

(by proposition 7); $1 \leq \lambda \leq n^j \leq n^m$; $1 \leq \nu \leq |X_m| \leq |X| m$. Therefore, the number of elements in our (m, δ)-net is equal to $O(m^5 n^m)$. Hence

$$r_m(\delta) = O(m^5 n^m).$$

From here it follows immediately that $h(f) \leq \ln n$. □

Remarks. (1). This calculation shows that only the parameter λ brings about an exponential contribution. Later we shall use this fact several times.

(2). The estimate $h(f) \leq \ln n$ we have just proved fills the gap in theorem 4(b) as $h_\mu(f) \leq h(f)$. Theorem 4(b) and theorem 7 show that the measure μ constructed in § 4 is of maximal entropy.

(3). The smooth endomorphism $f: S^2 \to S^2$ of the compact manifold can be naturally regarded as the 'simplest' if $h(f) \leq h(g)$ for any smooth endomorphism g homotopic to f. The Misiurewicz-Pryzytycki theorem and our theorem 7 show that a rational endomorphism of the sphere S^2 is the simplest.

THEOREM 8. *The rational endomorphism f of the Riemann sphere is asymptotically h-expansive.*

Proof. Fix $\varepsilon > 0$. Let $\rho(\varepsilon)$ be the lower bound of those $\rho > 0$ for which the function f is univalent in the circle $B(z; 2\varepsilon)$ for $d(z, Z) > \rho$. Set

$$\sigma(\varepsilon) = \sup_{z \in B(Z;\, \rho + 2\varepsilon)} \|f' z\|$$

where

$$\|f'(z)\| = \frac{|f'(z)|(1+|z|^2)}{1+|f(z)|^2}$$

is the spherical norm of the derivative. Finally, find $\nu = \nu(\varepsilon)$ from the equation

$$\sigma^{\nu} L^{1-\nu} = 1.$$

Obviously,

$$\nu(\varepsilon) \to 0 \qquad \text{as } \varepsilon \to 0.$$

Let $z \in S^2$. Denote the number of points from the trajectory $\{f^i z\}_{i=0}^{m-1}$ contained in $B(Z;\rho)$ by $\nu_m(z)$, $(\nu_0(z) \equiv 0)$. Let $E_m(z)$ be the set of roots of the equation $f^m \zeta = C$ contained in $B_m(z; 2\varepsilon)$. We show that

$$|E_m(z)| \le n^{\nu_m(z)}.$$

For $m = 0$ we have

$$|E_0(z)| \le 1 = n^{\nu_0(z)}.$$

Suppose by induction that

$$|E_{m-1}(fz)| \le n^{\nu_{m-1}(fz)}.$$

Evidently,

$$fE_m(z) \subset E_{m-1}(fz).$$

Now consider two cases:

(1) $d(z, Z) \le \rho$. We have

$$|E_m(z)| \le n|E_{m-1}(fz)| \le n^{\nu_{m-1}(fz)+1}.$$

But

$$\nu_{m-1}(fz) + 1 = \nu_m(z).$$

(2) $d(z, Z) > \rho$. Then the function f is univalent on $B(z; 2\varepsilon)$. Since

$$E_m(z) \subset B(z; 2\varepsilon),$$

f is an injective map of $E_m(z)$ into $E_{m-1}(fz)$. Hence,

$$|E_m(z)| \le |E_{m-1}(fz)| \le n^{\nu_{m-1}(fz)}.$$

But

$$\nu_m(z) = \nu_{m-1}(fz).$$

Now set

$$l = l_m(z) = \max\{i \mid 1 \le i \le m,\ \nu_i(z) \le \nu i\}.$$

Then $\nu_i(f^l z) > \nu i$, $1 \le i \le m - l$. Indeed, otherwise we get

$$\nu_{l+i}(z) = \nu_l(z) + \nu_i(f^l z) \le (l+i)\nu.$$

Since $l < l + i \le m$, we have a contradiction with the choice of l.

Fix $0 < \delta < \varepsilon$. We intend to show that

$$r_m(\phi_\varepsilon(z), \delta) = r_l(\phi_\varepsilon(z), \delta).$$

Let $\{x_k\}_{k=1}^r$ be an (l, δ)-net of the set $\phi_\varepsilon(z)$. Consider $\zeta \in \phi_\varepsilon(z)$ and find k such that $\zeta \in B_l(x_k; \delta)$. Show that

$$d(f^{l+i}x_k, f^{l+i}\zeta) \leq \delta\sigma^{\nu_i(f^l z)}L^{i-\nu_i(f^l z)} \qquad (\leq \delta(\sigma^\nu L^{1-\nu})^i = \delta)$$

for $0 \leq i \leq m-l$. For $i=0$ this leads to a valid estimate

$$d(f^l x_k, f^l\zeta) \leq \delta.$$

Consider two cases again:

(1) $d(f^{l+i}z, Z) \leq \rho$. Then $f^{l+i}x_k \in B(Z; \rho+2\varepsilon)$ and, therefore,

$$d(f^{l+i+1}x_k, f^{l+i+1}\zeta) \leq d(f^{l+i}x_k, f^{l+i}\zeta)\sigma.$$

But

$$\nu_{i+1}(f^l z) = \nu_i(f^l z)+1.$$

(2) $d(f^{l+i}z, Z) > \rho$. Then in any case

$$d(f^{l+i+1}z, f^{l+i+1}\zeta) \leq Ld(f^{l+i}z, f^{l+i}\zeta),$$

and we get the required result, since

$$\nu_{i+1}(f^l z) = \nu_i(f^l z).$$

Therefore, for all $0 \leq i \leq m-l$ the estimate

$$d(f^{l+i}x_k, f^{l+i}\zeta) \leq \delta\sigma^{\nu_i(f^l z)}L^{i-\nu_i(f^l z)} \leq \delta\sigma^{\nu i}L^{(1-\nu)i} = \delta$$

is valid. Hence, $\zeta \in B_m(x_k; \delta)$. Thus the set $\{x_k\}_{k=1}^r$ is an (m, δ)-net for $\phi_\varepsilon(z)$ and, therefore,

$$r_m(\phi_\varepsilon(z), \delta) \leq r_l(\phi_\varepsilon(z), \delta).$$

Since the opposite estimate is evident, we have proved the required equality.

Further, consider the minimal (l, δ)-net of the set $\phi_\varepsilon(z)$, chosen from the points of the (l, δ)-net constructed above. Each element of this (l, δ)-net is uniquely determined by the data $(j, v_{\mu,\lambda}, z_\nu)$. But in the situation considered λ can take on at most $n^{\nu m}$ different values (with j, μ fixed). Really,

$$v_{\mu,\lambda} \in B_j(\zeta; \delta/2)$$

for some $\zeta \in \phi_\varepsilon(z)$, since otherwise the (l, δ)-net could be decreased. Hence,

$$v_{\mu,\lambda} \in B(z; 2\varepsilon).$$

Besides, $v_{\mu,\lambda}$ satisfies the equation $f^j\zeta = u_\mu$. So, $v_{\mu,\lambda} \in E_j(z)$, with $c = u_\mu$. But

$$|E_j(z)| \leq n^{\nu_j(z)} \leq n^{\nu_l(z)} \leq n^{\nu l} \leq n^{\nu m}.$$

Therefore, there are $O(m^5 n^{\nu m})$ triples of the indicated form. Thus,

$$r_m(\phi_\varepsilon(z), \delta) = O(m^5 n^{\nu m}),$$

and we conclude from here that

$$h_f(\phi_\varepsilon(z)) = \lim_{\delta\to 0}\overline{\lim_{m\to\infty}}\frac{1}{m}\ln r_m(\phi_\varepsilon(z), \delta) \leq \nu \ln n.$$

The point z is arbitrary, so

$$h_f^*(\varepsilon) = \sup_{z\in S^2} h_f(\phi_\varepsilon(z)) \leq \nu \ln n.$$

Finally,

$$\nu = \nu(\varepsilon) \to 0 \qquad \text{as } \varepsilon \to 0$$

implies asymptotic h-expansiveness. □

COROLLARY 1. *If f is a rational spherical endomorphism, then the functional $h_\mu(f)$ on the space of invariant probability measures is semi-continuous from above.*

COROLLARY 2. *A rational spherical endomorphism possesses a measure of maximal entropy.*

Proof. In [**22**] it was shown that any asymptotically h-expansive endomorphism possesses the property of the first corollary. The second property follows from the first one.

Remarks. (1) Actually the second property contains no new information, since the measure μ constructed in § 4 is of maximal entropy.

(2) A rational endomorphism f (and even its restriction to the set F of irregular points) does not have to be h-expansive. For instance, if fz is a source for some critical point z, and the trajectories of other critical points converge to attracting cycles, then $f|F$ is not h-expansive. As seen from the proof of theorem 8, however, if F contains no critical points, then $f|F$ is h-expansive.

7. *Uniqueness of the measure of maximal entropy*

The following lemma is a variation of the Goodwin theorem, suitable for our purposes. In fact, it is contained in [**3**]. For the sake of being self-contained we give its proof here.

LEMMA 1. *Let $f: X \to X$ be a continuous endomorphism of the compact metric space X, μ be an f-invariant ergodic probability measure on X. Let $Y \subset X$ and $\mu(Y) > 0$. Then*

$$h_\mu(f) \leq h_f(Y).$$

Proof. We restrict ourselves to the case of finite-dimensional X. Then there exists a Borel partition

$$\mathscr{D} = \{\mathscr{D}_1, \ldots, \mathscr{D}_k\}$$

into arbitrarily small subsets, such that the covering

$$\bar{\mathscr{D}} = \{\bar{\mathscr{D}}_1, \ldots, \bar{\mathscr{D}}_k\}$$

has multiplicity that does not exceed $\dim X + 1 \equiv M$. Consider the covering X consisting of open sets

$$X \setminus \bigcup_{s=1}^{k-M} \bar{\mathscr{D}}_{i_s} \qquad 1 \leq i_1 < i_2 < \cdots < i_{k-M} \leq k.$$

Let ε_0 be the Lebesgue number of this covering. Then for $\varepsilon < \varepsilon_0$ any ball $B(x; \varepsilon)$, $x \in X$, intersects at most M elements of the partition $\mathscr{D}$. It follows that any ball $B_{m-1}(x; \varepsilon)$ intersects at most M^m elements of the partition $\mathscr{D}$.

Denote the element of the partition $\mathscr{D}^m$ containing the point x by $\mathscr{D}^m(x)$. By the Shannon–McMillan–Breiman theorem

$$h_\mu(f) = -\lim \frac{1}{m} \ln \mu(\mathscr{D}^m(x))$$

for almost all points x. In particular, this is true for almost all $x \in Y$. By the Yegorov theorem there exists a set $U \subset Y$ such that $\mu(U) > 0$ and

$$-\frac{1}{m} \ln \mu(\mathscr{D}^m(x)) \to h_\mu(f, \mathscr{D})$$

uniformly on U. Hence, for any $\delta > 0$ we can find an N, such that for $m \geq N$ the estimate

$$\mu(\mathscr{D}^m(x)) \leq \exp[-(h_\mu(f, \mathscr{D}) - \delta)m] \qquad x \in U$$

is valid. Let $\mathscr{D}_1^m, \ldots, \mathscr{D}_s^m$ be elements of the partition $\mathscr{D}^m$ intersecting U. The above argument implies that

$$\mu(U) \leq \sum_{i=1}^{s} \mu(\mathscr{D}_i^m) \leq s \exp[-(h_\mu(f, \mathscr{D}) - \delta)m].$$

Further, let $\{x_1, \ldots, x_r\}$ be a minimal (m, ε)-net of the set Y, $r = r_m(Y, \varepsilon)$. Consider the map $\varphi: \{1, \ldots, s\} \to \{1, \ldots, r\}$, $\varphi(i) = j$ provided that $\mathscr{D}_i^m \cap B_{m-1}(x_j; \varepsilon) \neq \varnothing$, (if there are several such j's, we choose any of them). $\varphi^{-1}j$ consists of at most M^m elements. Therefore,

$$s \leq M^m r_m(Y, \varepsilon).$$

We have

$$0 < \mu(U) \leq r_m(Y, \varepsilon) M^m \exp[-(h_\mu(f, \mathscr{D}) - \delta)m].$$

Hence,

$$h_f(Y) \geq \overline{\lim_{m \to \infty}} \frac{1}{m} \ln r_m(Y, \varepsilon) \geq h_\mu(f, \mathscr{D}) - \delta - \ln M.$$

As δ is arbitrary, and the partition $\mathscr{D}$ is arbitrarily fine, then

$$h_f(Y) \geq h_\mu(f) - \ln M.$$

Now substitute f^m for f:

$$h_f(Y) \geq h_\mu(f) - \frac{1}{m} \ln M.$$

We obtain the required result when we let $m \to \infty$. □

Let $E = \{1, \ldots, n\}$; $0 \leq \kappa \leq 1$. Consider the set $G_m(\kappa) \subset E^m$ of sequences with length m where the element 1 occurs at least κm times.

LEMMA 2. *Let $\kappa > 1/n$. Then there exists $K > 0$ and $0 < \theta < n$, such that*

$$|G_m(\kappa)| \leq K\theta^m.$$

Proof. We start from

$$G_m(\kappa) = \sum_{\kappa m \le i \le m} \binom{m}{i} (n-1)^{m-i}.$$

Using the Stirling formula, we get

$$\binom{m}{i} \le C \frac{m^m}{i^i (m-i)^{m-i}} \qquad r \le i \le m-r.$$

We have

$$\frac{m^m (n-1)^{m-i}}{i^i (m-i)^{m-i}} = \exp\left[\varphi\left(\frac{i}{m}\right) m\right]$$

where

$$\varphi(x) = (1-x)\ln(n-1) - x\ln x - (1-x)\ln(1-x) \qquad 0 \le x \le 1.$$

The maximum of the function $\varphi(x)$ on the segment $[0, 1]$ is $\ln n$ and is attained at the single point $x = 1/n$. Hence,

$$\max_{\kappa \le x \le 1} \varphi(x) = \ln \theta < \ln n.$$

It follows from here that

$$\sum_{\kappa m \le i \le m-r} \binom{m}{i} (n-1)^{m-i} \le m\theta^m.$$

Besides,

$$\sum_{m-r \le i \le m} \binom{m}{i} (n-1)^{m-i} \le C_1 m^r.$$

Combining the last two estimates and slightly increasing θ, we get the required inequality. □

A finite partition is said to *distinguish* the measures μ and ν, if

$$\nu(A) \ne \mu(A)$$

where A is an atom of the partition.

LEMMA 3. *Let the invariant measure ν be mutually singular with the measure μ. Then we can find an r and a system Γ of sphere cuts connecting the critical values of the function f^r, such that*

(1) *the domain $U \equiv S^2 \backslash \Gamma$ is simply-connected;*
(2) $(\mu + \nu)(\Gamma) = 0$;
(3) *the partition* mod 0 $\xi = \{f_\lambda^{-r} U\}_{\lambda=1}^{n^r}$ *distinguishes the measures μ and ν.*

Proof. Fix ε and l, such that

$$1 - 2\tau n^{-l} - 4\varepsilon > 0,$$

where τ is the number of critical values of the function f. Since the measures μ and ν are mutually singular, one can find compact sets K_1 and K_2 such that

$$K_1 \cap K_2 = \varnothing;$$
$$\mu(K_1) > 1 - \varepsilon \quad \text{and} \quad \nu(K_2) > 1 - \varepsilon.$$

Construct a system Γ' of cuts connecting the points from Z_l (l has been chosen above) so as to satisfy the following conditions:

(1) the domain $U' = S^2 \backslash \Gamma'$ should be simply-connected,

(2) $(\mu + \nu)(\Gamma') = 0$.

Afterwards find a neighbourhood $\Delta \supset \Gamma'$ such that $(\mu + \nu)(\Delta) < \varepsilon$ and write

$$V = S^2 \backslash \Delta \Subset U'.$$

Further, set $\rho = d(K_1, K_2) > 0$. Find M such that if $m \geq M$, then for any branch f_λ^{-m} of the function f^{-m} in U' we have

$$\operatorname{diam}(f_\lambda^{-m} V) < \rho.$$

Since the number of such branches is greater than $n^m(1 - 2\tau n^{-l})$ we have

$$\mu\left(\bigcup_\lambda f_\lambda^{-m} U'\right) \geq 1 - 2\tau n^{-l};$$

$\mu(f_\lambda^{-m} U) = 1/n^m$, since conditional measures of partition $f^{-1}\varepsilon$ are uniform (see § 5). Further,

$$\bigcup_\lambda f_\lambda^{-m} V = \bigcup_\lambda f_\lambda^{-m}(U' \backslash \Delta) = \bigcup_\lambda [f_\lambda^{-m} U' \backslash f_\lambda^{-m}(\Delta \backslash \Gamma)]$$

$$= \left[\bigcup_\lambda f_\lambda^{-m} U'\right] \backslash \left[\bigcup_\lambda f_\lambda^{-m}(\Delta \backslash \Gamma)\right] \supset \left[\bigcup_\lambda f_\lambda^{-m} U'\right] \backslash f^{-m}\Delta.$$

Hence,

$$\mu\left(\bigcup_\lambda f_\lambda^{-m} V\right) \geq 1 - 2\tau n^{-l} - \varepsilon.$$

Set $I = \{\lambda \mid f_\lambda^{-m} V \cap K_1 \neq \varnothing\}$. Consider the set $C = \bigcup_{\lambda \in I} f_\lambda^{-m} V$. We get

$$\mu(C) \geq \mu(C \cap K_1) = \mu\left[\left(\bigcup_{\lambda \in I} f_\lambda^{-m} V\right) \cap K_1\right]$$

$$\geq 1 - 2\tau n^{-l} - 2\varepsilon.$$

By definition of ρ and m, we have $C \cap K_2 = \varnothing$. Hence, $\nu(C) < \varepsilon$. Finally, let

$$C \subset \mathscr{D} = \bigcup_{\lambda \in I} f_\lambda^{-m} U' \subset C \cup f^{-m}\Delta.$$

Therefore,

$$\mu(\mathscr{D}) \geq \mu(C) \geq 1 - 2\tau n^{-l} - 2\varepsilon$$

and

$$\nu(\mathscr{D}) \leq \nu(C) + \nu(\Delta) \leq 2\varepsilon.$$

By the choice of l and ε, we get $\mu(\mathscr{D}) > \nu(\mathscr{D})$.

Complete the cut system Γ' to the cut system Γ, connecting the points of Z_m and satisfying the conditions (1), (2) as well as Γ'. Denote the simply-connected domain

$S^2\backslash\Gamma$ by U. There exist all branches of the function f^{-m} in U. If f_λ^{-m} is an analytic single-valued branch in U', then $f_\lambda^{-m}|U$ is an analytic single-valued branch in U. Besides

$$f_\lambda^{-m}(U') = f_\lambda^{-m}(U) \cup f_\lambda^{-m}(\Gamma\backslash\Gamma')$$

and, therefore,

$$\mu(f_\lambda^{-m}U) = \mu(f_\lambda^{-m}U') \quad \text{and} \quad \nu(f_\lambda^{-m}U) = \nu(f_\lambda^{-m}U').$$

Hence,

$$\mu\left(\bigcup_{\lambda\in I} f_\lambda^{-m}U\right) > \nu\left(\bigcup_{\lambda\in I} f_\lambda^{-m}U\right).$$

We have obtained what we needed: the partition $\{f_\lambda^{-m}U\}_{\lambda=1}^{n^m}$ distinguishes the measures μ and ν. □

THEOREM 9. *A rational endomorphism $f: S^2 \to S^2$ of the Riemann sphere has a unique measure of maximal entropy.*

Proof. $h_\mu(f)$ is a concave functional semi-continuous from above on the space $M_f(S^2)$ of invariant probability measures (corollary 1 to theorem 8). Therefore, the set W of measures of maximal entropy is a convex compact set. The extreme points of W are the ergodic measures of maximal entropy. Hence, if $|W| > 1$, then there exists an ergodic measure of maximal entropy ν, other than μ constructed in theorem 1. (This may be shown without theorem 8, making use of the Rochlin decomposition ([**25**]) of an arbitrary measure of maximal entropy into ergodic components.) As μ is also ergodic, then μ and ν are mutually singular.

By lemma 3 we find a simply-connected domain $U = S^2\backslash\Gamma$ and an integer r. Set $g = f^r$, $U_k = (g_k^{-1}U)\backslash\Gamma$, $k = 1, \ldots, N$, where $N = n^r = \deg g$. As the partition mod 0 $\xi = \{U_k\}_{k=1}^N$ distinguishes the measures μ and ν, then one can find $k \in [1, N]$ such that

$$\nu(U_k) > \mu(U_k).$$

To be definite, set k to 1. There exists a compact $V \subset U_1$, such that

$$\nu(V) > \mu(U_1).$$

Fix κ such that

$$\nu(V) > \kappa > \mu(U_1) = 1/N.$$

For any measurable set X, point $z \in S^2$, and natural integer t denote by $S_{X,t}(z)$ the number of points of the trajectory $\{g^i z\}_{i=0}^{t-1}$ contained in X. Set

$$H_t = \left\{ z \in S^2 \middle| \frac{1}{i} S_{V,i}(z) \geq \kappa \quad i = t, t+1, \ldots \right\}.$$

From the individual ergodic theorem it follows that $\nu(H_t) > 0$ for a sufficiently large t.

Let us estimate $h_g(H_t)$. Let $0 < \delta < d(V, \partial U_1)$. Consider the (m, δ)-net on the sphere S^2, constructed in § 6, points u_μ being chosen on the dense set

$$\Omega = S^2\backslash\bigcup_{i=0}^{\infty} g^i\Gamma.$$

Choose from this net a minimal set E which is the (m, δ)-net of H_t. Each element of E is uniquely determined by the triple $(j, v_{\mu,\lambda}, z_\nu)$. Besides, there exists $z \in H_t$, such that $v_{\mu,\lambda} \in B_j(z; \delta)$ (otherwise the net E could be decreased). It follows from the choice of δ that if $g^i z \in V$, then $f^i v_{\mu,\lambda} \in U_1$. Hence, if $j \geq t$, then

$$S_{U_1,j}(v_{\mu,\lambda}) \geq S_{V,j}(z) \geq \kappa j.$$

Since $u_\mu \in \Omega$, $g^i v_{\mu,\lambda} \in U$, $i = 0, 1, \ldots, j$. Therefore,

$$g^i v_{\mu,\lambda} \in U \cap g^{-1} U = \bigcup_{k=1}^{N} U_k \qquad i = 0, \ldots, j-1,$$

i.e. there exists a sequence $(\alpha_0, \ldots, \alpha_{j-1})$, $1 \leq \alpha_i \leq N$ such that

$$v_{\mu,\lambda} \in \bigcap_{i=0}^{j-1} g^{-i} U_{\alpha_i} \equiv U_{\alpha_0 \ldots \alpha_{j-1}}.$$

By induction on j we check that for the given μ the index λ is uniquely reconstructed from the sequence $(\alpha_0, \ldots, \alpha_{j-1})$. Let the points $v_{\mu,\lambda}$ and $v_{\mu,\lambda'}$ be related to the same sequence. It follows from the inductive hypothesis that $g v_{\mu,\lambda} = g v_{\mu,\lambda'}$. But $v_{\mu,\lambda}$ and $v_{\mu,\lambda'} \in U_{\alpha_0}$, hence

$$v_{\mu,\lambda} = g_{\alpha_0}^{-1}(g v_{\mu,\lambda}) = g_{\alpha_0}^{-1}(g v_{\mu,\lambda'}) = v_{\mu,\lambda'}.$$

Furthermore, since $S_{U_1,j}(v_{\mu,\lambda}) \geq \kappa j$, $(j \geq t)$, $(\alpha_0, \ldots, \alpha_{j-1}) \in G_j(\kappa)$. Because $v_{\mu,\lambda}$ for the fixed μ is uniquely reconstructed from the sequence $(\alpha_0, \ldots, \alpha_{j-1})$, for $j \geq t$, the index λ may take on at most

$$|G_j(\kappa)| \leq K \theta^j$$

different values (lemma 2). This means that the triple $(j, v_{\mu,\lambda}, z_\nu)$ may take at most

$$\left[\sum_{j=0}^{t-1} n^j + K \sum_{j=t}^{m} \theta^j\right] |X_m| \gamma_m = O(m^5 \theta^m)$$

different values. So

$$r_m(H_t, \delta) = O(m^5 \theta^m)$$

for all sufficiently small δ, therefore

$$h_g(H_t) \leq \ln \theta < \ln N.$$

Since $\nu(H_t) > 0$, $h_\nu(g) < \ln N$ by lemma 1. Returning to f, we obtain a contradiction. □

Having finished the present paper the author learned about a preprint by M. Gromov where a similar formula has been obtained for the topological entropy of an analytic endomorphism of the complex projective space $\mathbb{CP}^K$. Our approach differs essentially from that of M. Gromov.

The author wishes to thank B. A. Rubštein for a simplification of the proof of theorem 4(a).

REFERENCES

[1] A. Avez. Propriétés ergodiques des endomorphisms dilatants de variétés compactes. *C.R. Acad. Paris*, ser A, **266** (1968), 610–612.

[2] R. Bowen, Entropy expansive mappings. *Trans. Amer. Math. Soc.* **164** (1972), 323–332.
[3] R. Bowen. Topological entropy for noncompact sets. *Trans. Amer. Math. Soc.* **184** (1973), 125–136.
[4] R. Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Lecture Notes Math. 470. Springer-Verlag: Berlin–Heidelberg–New York, 1975.
[5] R. Bowen. *On axiom A diffeomorphisms.* Conf. Board Math. Sci. **35** (1978).
[6] H. Brolin. Invariant sets under iteration of rational functions. *Arkiv Math.* **6** (1965), 103–144.
[7] R. Courant. *Geometrische Funktionen.* Springer-Verlag: Berlin–Göttingen–Heidelberg–New York, 1964.
[8] M. Denker & T. Keane. Eine Bemerkung zur topologischen Entropie. *Monatsh. Math.* **85** (1978), 177–183.
[9] E. J. Dinaburg. On the relations among various entropy characteristics of dynamical systems. *Math. USSR Izvestija* **5** (1971), #2, 337–378.
[10] P. Fatou. Sur les équations fonctionneles. *Bull. Soc. Math. France* **47** (1919), 161–271; **48** (1920), 33–94 & 208–314.
[11] T. Goodman. Relating topological entropy and measure entropy. *Bull. London Math. Soc.* **3** (1971), 176–180.
[12] L. Goodwin. Topological entropy bounds measure-theoretic entropy. *Proc. Amer. Math. Soc.* **23** (1969), 679–688.
[13] J. Guckenheimer. Endomorphisms of the Riemann sphere. *Proc. Symp. Pure Math.* **14** (1970), 95–123.
[14] K. Jacobs. Fastperiodizitätseigenschaften allgemeiner Halbgruppen in Banachräumen. *Math. Z.* **67** (1957), 83–92.
[15] M. V. Jakobson. Structure of polynomial mappings on a singular set. *Math. USSR Sbornik* **6** (1968), #1, 97–114.
[16] M. V. Jakobson. On the problem of classification of polynomial endomorphisms of the plane. *Math. USSR Sbornik* **9** (1969), #3, 345–369.
[17] M. V. Jakobson. On the problem of classification of rational maps of the Riemann sphere. *Uspehi Math. Nauk.* **28** (1973), 247–248.
[18] G. Julia. Mémoire sur l'iteration des fonctions rationneles. *J. Math. Pure Appl.* **8** (1918), 47–245.
[19] Yu. I. Lyubich. Almost periodic functions in the spectral analysis of operators. *Soviet Math. Doklady* **1** (1960), 593–595.
[20] Yu. I. Lyubich. On the boundary spectrum of contractions in Minkowski spaces. *Siberian Math. J.* **11** #2, (1970), 271–279.
[21] M. Yu. Lyubich. Entropy of analytic endomorphisms of the Riemannian sphere. *Funct. Anal. Appl.* **15** #4, (1981), 300–302.
[22] M. Misiurewicz. Diffeomorphisms without any measure with maximal entropy. *Bull. Acad. Pol. Sci.* **21** (1973), 903–910.
[23] M. Misiurewicz & F. Przytycki. Topological entropy and degree of smooth mappings. *Bull. Acad. Pol. Sci.* **25** (1977), 573–574.
[24] P. Montel. *Leçons sur les familles normales de fonctions analytique et leurs applications.* Gauthier-Villars: Paris, 1927.
[25] V. A. Rokhlin. Lectures on the entropy theory of measure-preserving transformations. *Russian Math. Surveys* **22** (1967), #5, 1–52.
[26] D. Ruelle. Statistical mechanics of a one-dimensional lattice gas. *Commun. Math. Phys.* **9** (1968), 267–278.
[27] P. Walters. Invariant measures and equilibrium states for some mappings which expand distances. *Trans. Amer. Math. Soc.* **236** (1978), 121–153.