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ORIGINAL RESEARCH

Global Stability of Complex-Valued Neural Networks on Time Scales

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Abstract In this paper, activation dynamics of complex-valued neural networks are studied on general time scales. Besides presenting conditions guaranteeing the existence of a unique equilibrium pattern, its global exponential stability is discussed. Some numerical examples for different time scales are given in order to highlight the results.

Keywords Time scales · Complex-valued neural networks · ψ-globally exponentially stable

Mathematics Subject Classification (2000) 34N05 · 93C70 · 92B20

Introduction

The study of complex-valued neural networks (CVNNs for short) is a fast growing area of research in recent times as is apparent from a large number of publications (see e.g., [6–8,10,11] and the references therein). The main focus of these networks is aimed at exploring new capabilities and higher performance, making it possible to solve problems which cannot be solved with their real-valued counterparts. For example [9], the XOR problem

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and the detection of symmetry problem cannot be solved with a single real-valued neuron (i.e., a two-layered real-valued neural network), but they can be solved with a single complex-valued neuron (i.e., a two-layered complex-valued neural network) with the orthogonal decision boundaries, which reveals the potent computational power of complex-valued neurons. CVNNs have been found highly useful in extending the scope of applications in optoelectronics, filtering, imaging, speech synthesis, computer vision, remote sensing, quantum devices, spatio-temporal analysis of physiological neural devices and systems, and artificial neural information processing [6,11]. Usually, information flow is represented by waves such as acoustic, light, electromagnetic etc. In recent years optical flow has been represented as two-dimensional vector fields consisting of two dimensional vectors. Since waves and two-dimensional vectors can be represented by complex numbers, complex-valued networks are well suited to handle these applications. It is known that in the human brain regular activities are routinized (which simplifies the thinking process) and are performed subconsciously. On the other hand the conscious "focus of attentions" or "short-term working memory" attends to the important aspects through the activations. Thus, the study of activation dynamics plays a significant role in the modeling of brain activities. In a recent paper by Rao and Murthy [10], the authors have studied global activation dynamics of a discrete CVNN and have obtained easily verifiable sufficient conditions for global exponential stability of the unique equilibrium pattern. It is known that discrete CVNNs follow a specific time scale. This fact makes the study incomplete in the sense that in real-life situations it is not always the case that the time scales match with the commonly known integer-valued discrete time scales. Thus it is important to study global dynamics on general time scales, and this is the starting point of the present investigation. For an introduction to the theory of time scales, we refer the readers to [3,4]. Stability analysis of systems on time scales has been studied in [1,2,5].

Let $\mathbb{T}$ be a time scale and $\mathbb{C}$ be the set of complex numbers. We consider the generalized CVNN described by the equation

$$z^{\Delta}(t) = Cz(t) + Af(z(t)) + r, \tag{1}$$

where $z : \mathbb{T} \to \mathbb{C}^n$, C and the connection weight matrix A are $n \times n$-matrices with complex entries, the activation functions are given by $f : \mathbb{C}^n \to \mathbb{C}^n$, and the inputs are given by $r \in \mathbb{C}^n$. System (1) represents complex-valued activations of the associative memory as a complex-valued dynamical system.

The organization of this paper is as follows. In Section 2 we present some preliminary results on time scales that are needed in the remainder of this paper. Section 3 discusses the existence of a unique equilibrium pattern to the CVNN (1). In Section 4, we derive new conditions that guarantee the global exponential stability of the equilibrium pattern for (1) on any arbitrary time scale. These conditions are new for the continuous case and improve known results even in the discrete case [10]. In Section 5, we present numerical examples for different choices of time scales. Finally a discussion follows in Section 6.

Essentials of Time Scales

An arbitrary nonempty closed subset $\mathbb{T}$ of the set of real numbers $\mathbb{R}$ is called a *time scale*. In this paper, we only consider time scales that are unbounded above. Examples of such time scales are the reals $\mathbb{R}$ (continuous calculus), the integers $\mathbb{Z}$ (discrete calculus), $h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ with $h > 0$, and $q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$ (quantum calculus). We define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ (and similarly the *backward jump operator* ρ) by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$. A point $t \in \mathbb{T}$ is called *right-scattered*, *right-dense*, *left-scattered*, *left-dense*, if $\sigma(t) > t,\ \sigma(t) = t,\ \rho(t) < t,\ \rho(t) = t$ holds, respectively. The *graininess* $\mu : \mathbb{T} \to [0]$ is defined by $\mu(t) = \sigma(t) - t$. For $\mathbb{T} = \mathbb{R},\ \mathbb{Z},\ h\mathbb{Z},\ q^{\mathbb{N}_0}$ we have $\mu(t) = 0,\ 1,\ h$, and $(q-1)t$, respectively.

For $f : \mathbb{T} \to \mathbb{C}^n$ we note that the real and imaginary parts of f are real valued and one can use the time scales results below for the real-valued entries of $\operatorname{Re} f$ and $\operatorname{Im} f$. We say that $f : \mathbb{T} \to \mathbb{R}$ is *delta differentiable* at $t \in \mathbb{T}$ provided there exists an α such that for all $\varepsilon > 0$ there is a neighborhood $\mathcal{N}$ of t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \le \varepsilon\, |\sigma(t) - s| \quad \text{for all} \quad s \in \mathcal{N}.$$

In this case we denote α by $f^{\Delta}(t)$, and if f is differentiable for every $t \in \mathbb{T}$, then f is said to be *differentiable* on $\mathbb{T}$ and f^{Δ} is a new function defined on $\mathbb{T}$. Then it is easy to see that

$$f^{\Delta}(t) = \begin{cases} \lim_{s \to t, s \in \mathbb{T}} \frac{f(t) - f(s)}{t - s} & \text{if} \quad \mu(t) = 0 \\ \frac{f(\sigma(t)) - f(t)}{\mu(t)} & \text{if} \quad \mu(t) > 0. \end{cases}$$

Other useful results are the product rule

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} = f g^{\Delta} + f^{\Delta} g^{\sigma}, \tag{2}$$

where we put $f^{\sigma} = f \circ \sigma$, and the simple useful formula

$$f^{\sigma} = f + \mu f^{\Delta}. \tag{3}$$

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by C_{rd}. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *regressive* provided $1 + \mu(t) f(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions is denoted by $\mathcal{R}$. The set $\mathcal{R}^{+}$ of all positively regressive function consists of those $p \in \mathcal{R}$ that satisfy $1 + \mu(t) p(t) > 0$ for all $t \in \mathbb{T}$. It is known that if $p \in \mathcal{R}$ and $t_0 \in \mathbb{T}$, then the initial value problem $y^{\Delta} = p(t) y,\ y(t_0) = 1$ possesses a unique solution. This solution is called the exponential function on the time scale and is denoted by $e_p(\cdot, t_0)$. The following properties of the exponential function are known [3].

Lemma 1 *Let $p, q \in \mathcal{R}$ and $t, s, r \in \mathbb{T}$. Then*

(i) $e_0(t, t_0) \equiv 1$ *and* $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t) p(t)) e_p(t, s)$;
(iii) $\frac{1}{e_p(t, t_0)} = e_{\ominus p}(t, t_0)$, *where* $\ominus p = -\frac{p}{1 + \mu p}$;
(iv) $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;
(v) $e_p(t, s) e_p(s, r) = e_p(t, r)$;
(vi) $e_p(t, s) e_q(t, s) = e_{p \oplus q}(t, s)$, *where* $p \oplus q = p + q + \mu p q$;
(vii) $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s)$, *where* $p \ominus q = \frac{p - q}{1 + \mu q}$;
(viii) *if* $\mathbb{T} = \mathbb{R}$, *then* $e_p(t, t_0) = e^{\int_{t_0}^{t} p(\tau) \mathrm{d}\tau}$;
(ix) *if* $\mathbb{T} = \mathbb{R}$ *and* $p(t) \equiv \alpha$, *then* $e_p(t, s) = e^{\alpha(t - s)}$;
(x) *if* $\mathbb{T} = \mathbb{Z}$ *and* $s < t$, *then* $e_p(t, s) = \prod_{\tau = s}^{t-1} (1 + p(\tau))$;
(xi) *if* $\mathbb{T} = h\mathbb{Z}$, $h > 0$, *and* $p(t) \equiv \alpha$, *then* $e_p(t, s) = (1 + h\alpha)^{(t - s)/h}$;
(xii) *if* $p \in \mathcal{R}^{+}$, *then* $e_p(t, s) > 0$ *for all* $t > s$.

The following inequality result [3, Theorem 6.1] will be used.

Lemma 2 *Let $y, f \in C_{rd}$ and $p \in \mathcal{R}^+$. If y is differentiable on $[t_0, \infty) \cap \mathbb{T}$ such that*

$$y^{\Delta}(t) \leq p(t)y(t) + f(t) \quad \text{for all} \quad t \in [t_0, \infty) \cap \mathbb{T},$$

then

$$y(t) \leq y(t_0)e_p(t, t_0) + \int_{t_0}^{t} e_p(t, \sigma(\tau)) f(\tau)\Delta\tau \quad \text{for all} \quad t \in [t_0, \infty) \cap \mathbb{T}.$$

For a more detailed study of time scales we refer to [3,4].

Existence of a Unique Equilibrium Pattern

In this section, we consider the model equation (1) and obtain conditions that guarantee the existence of a unique equilibrium to the system (1). We consider the space $\mathbb{C}^n$ of all n-vectors of complex numbers and let $|z| = \sqrt{z^* z}$ denote the absolute value of $z \in \mathbb{C}^n$, where * indicates the conjugate transpose.

Theorem 3 *Suppose $f : \mathbb{C}^n \to \mathbb{C}$ is continuous with $f(0) = 0$ and there exists $L > 0$ such that $|f(z) - f(\hat{z})| \leq L|z - \hat{z}|$ for all $z, \hat{z} \in \mathbb{C}^n$. Let $\|\cdot\|$ be the operator norm induced by the absolute value in $\mathbb{C}^n$ and define*

$$\gamma := \|I + C\| + L\|A\|.$$

If $\gamma \in (0, 1)$, then the system (1) possesses a unique equilibrium pattern.

Proof Clearly, the assumptions of the theorem imply $|f(z)| \leq L|z|$ for all $z \in \mathbb{C}^n$. Now define the operator $T : \mathbb{C}^n \to \mathbb{C}^n$ by

$$T(z) = (I + C)z + Af(z) + r.$$

For $z, \hat{z} \in \mathbb{C}^n$, we have

$$\begin{aligned}
|T(z) - T(\hat{z})| &= |(I + C)(z - \hat{z}) + A(f(z) - f(\hat{z}))| \\
&\leq \|I + C\||z - \hat{z}| + \|A\||f(z) - f(\hat{z})| \\
&\leq \|I + C\||z - \hat{z}| + L\|A\||z - \hat{z}| \\
&\leq (\|I + C\| + L\|A\|)|z - \hat{z}| \\
&= \gamma|z - \hat{z}|.
\end{aligned}$$

Thus the mapping T is a contraction on $\mathbb{C}^n$. Therefore, T has a unique fixed point in $\mathbb{C}^n$ since $\mathbb{C}^n$ is complete. This unique fixed point is the required equilibrium pattern for the CVNN (1). □

Remark 4 In the proof of Theorem 3, we can obtain a localization result for the unique fixed point as follows: If we choose $\alpha > 0$ so that

$$\alpha \geq \frac{|r|}{1 - \gamma}$$

and let $S(\alpha) = \{z \in \mathbb{C}^n : |z| \le \alpha\}$, then

$$\begin{aligned} |T(z)| &\le |(I + C)z| + |Af(z)| + |r| \\ &\le (\|I + C\| + L\|A\|)\,|z| + |r| \\ &\le (\|I + C\| + L\|A\|)\,\alpha + \alpha(1 - \gamma) \\ &= \alpha. \end{aligned}$$

This shows that $T(S(\alpha)) \subseteq S(\alpha)$. Thus the mapping T is a contraction on $S(\alpha)$. From Banach's contraction mapping principle, there exists a unique $\tilde{z} \in S(\alpha)$ satisfying $T(\tilde{z}) = \tilde{z}$.

Global Exponential Stability

We assume that the network (1) possesses a unique equilibrium pattern $\tilde{z}$. Using the transformation $\hat{z}(t) = z(t) - \tilde{z}$ in Eq. (1), we get

$$\hat{z}^{\Delta}(t) = C\hat{z}(t) + Ag\left(\hat{z}(t)\right), \quad \text{where} \quad g(x) = f(x + \tilde{z}) - f(\tilde{z}).$$

Redesignating $\hat{z}(t)$ as $z(t)$, we obtain

$$z^{\Delta}(t) = Cz(t) + Ag(z(t)). \tag{4}$$

Clearly the stability of $\tilde{z}$ for the system (1) is equivalent to the stability of the trivial solution for the system (4). We use the following concept of global exponential stability.

Definition 5 Suppose $\psi \in \mathrm{C_{rd}}$ is such that $e_{\psi}(t, t_0) \to 0$ as $t \to \infty$. Then the trivial solution of (4) is said to be ψ-*globally exponentially stable* if for any solution z of (4) there exists a constant $M > 0$ such that

$$z^*(t)z(t) \le Me_{\psi}(t, t_0) \quad \text{for all} \quad t \ge t_0.$$

In order to prove our main result about global exponential stability of the trivial solution of (4), we require the following two lemmas.

Lemma 6 *If z is a solution of* (4), *then $w = z^*z$ satisfies*

$$\begin{aligned} w^{\Delta} &= z^*(C^* + C + \mu C^*C)z + \mu g^*(z)A^*Ag(z) \\ &\quad + g^*(z)A^*(I + \mu C)z + z^*(I + \mu C)^*Ag(z). \end{aligned}$$

Proof We use the product rule (2) and the simple useful formula (3) to calculate

$$\begin{aligned} w^{\Delta} &= (z^*)^{\Delta}z^{\sigma} + z^*z^{\Delta} \\ &= (z^{\Delta})^*(z + \mu z^{\Delta}) + z^*z^{\Delta} \\ &= (z^{\Delta})^*z + z^*z^{\Delta} + \mu(z^{\Delta})^*z^{\Delta} \\ &= (z^*C^* + g^*(z)A^*)z + z^*(Cz + Ag(z)) \\ &\quad + \mu(z^*C^* + g^*(z)A^*)(Cz + Ag(z)) \\ &= z^*C^*z + g^*(z)A^*z + z^*Cz + z^*Ag(z) \\ &\quad + \mu z^*C^*Cz + \mu z^*C^*Ag(z) + \mu g^*(z)A^*Cz + \mu g^*(z)A^*Ag(z) \\ &= z^*(C^* + C + \mu C^*C)z + \mu g^*(z)A^*Ag(z) \\ &\quad + g^*(z)A^*(I + \mu C)z + z^*(I + \mu C)^*Ag(z). \end{aligned}$$

This completes the proof. □

We can use the equality from Lemma 6 to prove an estimate of w^{Δ} as follows.

Lemma 7 *Let $\beta > 0$. If z is a solution of (4), then $w = z^* z$ satisfies*

$$
\begin{aligned}
w^{\Delta} \leq z^* &\left(C^* + C + \mu C^* C + \frac{1}{\beta}(I + \mu C^*)(I + \mu C) \right) z \\
&+ (\mu + \beta) g^*(z) A^* A g(z).
\end{aligned}
$$

Proof First notice that for $\beta > 0$ we have

$$
\begin{aligned}
0 &\leq \left(\sqrt{\beta} A g(z) - \frac{1}{\sqrt{\beta}}(I + \mu C) z \right)^* \left(\sqrt{\beta} A g(z) - \frac{1}{\sqrt{\beta}}(I + \mu C) z \right) \\
&= \beta g^*(z) A^* A g(z) - g^*(z) A^* (I + \mu C) z \\
&\quad - z^* (I + \mu C)^* A g(z) + \frac{1}{\beta} z^* (I + \mu C)^* (I + \mu C) z.
\end{aligned}
$$

Using this in Lemma 6 results in

$$
\begin{aligned}
w^{\Delta} &= z^*(C^* + C + \mu C^* C) z + \mu g^*(z) A^* A g(z) \\
&\quad + g^*(z) A^* (I + \mu C) z + z^* (I + \mu C)^* A g(z) \\
&\leq z^*(C^* + C + \mu C^* C) z + \mu g^*(z) A^* A g(z) \\
&\quad + \beta g^*(z) A^* A g(z) + \frac{1}{\beta} z^* (I + \mu C)^* (I + \mu C) z \\
&= z^* \left(C^* + C + \mu C^* C + \frac{1}{\beta}(I + \mu C^*)(I + \mu C) \right) z \\
&\quad + (\mu + \beta) g^*(z) A^* A g(z),
\end{aligned}
$$

which completes the proof. □

The following stability result is the main theorem of this paper.

Theorem 8 *Suppose g satisfies a Lipschitz condition with Lipschitz constant L. Assume that $C = \operatorname{diag}(c_1, c_2, \ldots, c_n)$ is a diagonal matrix. If there exists $\beta > 0$ such that*

$$
\psi := \tilde{c} + (\mu + \beta) L^2 \lambda \quad \text{satisfies} \quad \psi \in \mathcal{R}^+ \quad \text{and} \quad \lim_{t \to \infty} e_{\psi}(t, t_0) = 0,
$$

where λ is the maximal eigenvalue of $A^* A$ and

$$
\tilde{c} = \max_{1 \leq i \leq n} \left\{ 2 \operatorname{Re} c_i + \mu |c_i|^2 + \frac{1}{\beta} |1 + \mu c_i|^2 \right\},
$$

then the trivial solution of (4) is ψ-globally exponentially stable.

Proof Let z be any solution of (4) and define $w = z^* z$. Then

$$
g^*(z) A^* A g(z) \leq \lambda g^*(z) g(z) = \lambda |g(z)|^2 \leq \lambda (L|z|)^2 = L^2 \lambda |z|^2 = L^2 \lambda w
$$

and

$$\begin{aligned}
z^* &\left(C^* + C + \mu C^* C + \frac{1}{\beta}(I + \mu C^*)(I + \mu C) \right) z \\
&= \sum_{i=1}^{n} \overline{z_i} \left\{ 2\operatorname{Re} c_i + \mu |c_i|^2 + \frac{1}{\beta}|1 + \mu c_i|^2 \right\} z_i \\
&= \sum_{i=1}^{n} \left\{ 2\operatorname{Re} c_i + \mu |c_i|^2 + \frac{1}{\beta}|1 + \mu c_i|^2 \right\} |z_i|^2 \\
&\leq \tilde{c} \sum_{i=1}^{n} |z_i|^2 = \tilde{c}|z|^2 = \tilde{c} w.
\end{aligned}$$

Thus, by Lemma 7, we have

$$\begin{aligned}
w^{\Delta} &\leq z^* \left(C^* + C + \mu C^* C + \frac{1}{\beta}(I + \mu C^*)(I + \mu C) \right) z + (\mu + \beta) g^*(z) A^* A g(z) \\
&\leq \tilde{c} w + (\mu + \beta) L^2 \lambda w = \psi w.
\end{aligned}$$

Now Lemma 2 yields

$$w(t) \leq w(t_0) e_{\psi}(t, t_0) \to 0 \quad \text{as} \quad t \to \infty,$$

so that the claim follows. □

Remark 9 Note that ψ in general is a function of t, namely whenever the graininess of the time scale is not constant. E.g., $\mathbb{R}$ and $h\mathbb{Z}$ with $h > 0$ have constant graininess, while, e.g., $q^{\mathbb{N}_0}$ with $q > 1$ does not. For that reason, we allow β to be a function of t as well, but of course $\beta(t) > 0$ is required for all $t \in \mathbb{T}$.

Examples and Applications

Theorem 8 ensures sufficient conditions for global exponential stability of CVNN (4) for any arbitrary time scale $\mathbb{T}$, which obviously includes the well-known time scales $\mathbb{R}$ and $\mathbb{Z}$. We start with formulating Theorem 8 for these two cases. Our results are new additions to the literature even for these two classical cases.

Example 10 For $\mathbb{T} = \mathbb{R},\ \mu(t) \equiv 0$, and if there exists a constant $\beta > 0$ such that

$$\alpha := \tilde{c} + \beta L^2 \lambda < 0,$$

where λ is the maximal eigenvalue of $A^* A$ and

$$\tilde{c} = \max_{1 \leq i \leq n} \left\{ 2\operatorname{Re} c_i + \frac{1}{\beta} \right\},$$

then the trivial solution of (4) is α-globally exponentially stable.

Example 11 When $\mathbb{T} = \mathbb{Z},\ \mu(t) \equiv 1$, and if there exists a constant $\beta > 0$ such that

$$\alpha := \tilde{c} + (1 + \beta) L^2 \lambda \in (-1, 0),$$

where λ is the maximal eigenvalue of A^*A and

$$\tilde{c} = \max_{1\le i\le n}\left\{2\,\mathrm{Re}\,c_i + |c_i|^2 + \frac{1}{\beta}|1+c_i|^2\right\},$$

then the trivial solution of (4) is α-globally exponentially stable.

We now consider some numerical examples. The following example is from [10]. We have applied this example to obtain results for a few well-known time scales.

Example 12 Consider the network

$$z^\Delta(t) = \begin{bmatrix} -0.99 & 0 \\ 0 & -0.988 \end{bmatrix} z(t) + \begin{bmatrix} 0.025+0.025\mathrm{i} & -0.05+0.025\mathrm{i} \\ 0.075-0.05\mathrm{i} & -0.025+0.025\mathrm{i} \end{bmatrix} g(z(t)). \tag{5}$$

If in (5) we choose the function g as a Lipschitz function with Lipschitz constant equal to 1 and $\beta = 1$, then

$$\begin{aligned} c_1 &= -0.99, \quad 2c_1 + c_1^2 + (1+c_1)^2 = -0.9998, \\ c_2 &= -0.988, \quad 2c_2 + c_2^2 + (1+c_2)^2 = -0.999712 \end{aligned}$$

so that $\tilde{c} = -0.999712$. The two eigenvalues of A^*A can be computed as $(11 \pm \sqrt{68})0.025^2$ so that $\lambda = (11+\sqrt{68})0.025^2$. Hence

$$\psi = -0.999712 + (\mu+\beta)(11+\sqrt{68})0.025^2$$

is constant if μ and β are constant, in which case we write $\alpha = \psi$.

1. For $\mathbb{T} = \mathbb{R}$ we have $\mu(t) \equiv 0$. We find $\alpha \approx -0.9876831$ and thus $e_\alpha(t,0) = e^{\alpha t} \to 0$ as $t \to \infty$ so that the trivial solution of (5) is α-globally exponentially stable.
2. For $\mathbb{T} = \mathbb{Z}$ we have $\mu(t) \equiv 1$. We find $\alpha \approx -0.97565$ and thus $e_\alpha(t,0) = (1+\alpha)^t \to 0$ as $t \to \infty$ so that the trivial solution of (5) is α-globally exponentially stable.
3. For $\mathbb{T} = h\mathbb{Z}$ with $h > 0$ we have $\mu(t) \equiv h$. With $\beta = 1$, we find $\tilde{c} = 2\,\mathrm{Re}\,c_2 + h|c_2^2| + |1+hc_2|^2 = 0.99960004h^2 - 0.99999996h - 0.9996$. Thus $\alpha \approx 0.99960004h^2 - 0.987971078h - 0.987571118$ and $e_\alpha(t,0) = (1+h\alpha)^{t/h} \to 0$ as $t \to \infty$ if $\alpha < 0$ or h should be less than 1.60422071, and then the trivial solution of (5) is α-globally exponentially stable.

The following example illustrates the improvement of Theorem 4 over [10, Theorem 3.2], especially in the case when $\mathbb{T} = \mathbb{Z}$.

Example 13 For the time scale $\mathbb{T} = \mathbb{Z}$, let us consider the network

$$z^\Delta(t) = \begin{bmatrix} -0.75 & 0 \\ 0 & -0.65 \end{bmatrix} z(t) + \begin{bmatrix} 0.025+0.025\mathrm{i} & -0.05+0.025\mathrm{i} \\ 0.075-0.05\mathrm{i} & -0.025+0.025\mathrm{i} \end{bmatrix} g(z(t)). \tag{6}$$

We observe (using the notation from [10, Theorem 3.2]) $\rho = 2.9894 > 1$, and as such [10, Theorem 3.2] fails to guarantee the global exponential stability of the network (6). On the other hand, Theorem 8 with $\mu(t) \equiv 1$ and $\beta = 1$ gives us $\alpha = -0.7525 + 0.02406 = -0.7284 \in (-1, 0)$, implying that the trivial solution of (6) is α-globally exponentially stable.

Discussion

In this paper we have studied the activation dynamics of a complex-valued neural network on a general time scale. Sufficient conditions for the existence of a unique equilibrium solution are derived. Further, we have introduced the notion of ψ-global exponential stability of the equilibrium pattern, which generalizes the notion of global exponential stability. The global exponential stability conditions derived in this paper are fairly general and offer greater flexibility in handling time scales of practical importance. Examples of a few time scales are presented.

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