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Research Article

Hermite-Hadamard and Simpson-Like Type Inequalities for Differentiable Harmonically Convex Functions

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A new identity for differentiable functions is derived. A consequence of the identity is that the author establishes some new general inequalities containing all of the Hermite-Hadamard and Simpson-like types for functions whose derivatives in absolute value at certain power are harmonically convex. Some applications to special means of real numbers are also given.

1. Introduction

Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds. This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. Note that some of the classical inequalities for means can be derived from (1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

The following inequality is well known in the literature as Simpson inequality.

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\|f^{(4)}\|_\infty = \sup_{x\in(a,b)}|f^{(4)}(x)| < \infty$. Then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \\ &\quad\leq \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^4.\end{aligned} \tag{2}$$

For some results which generalize, improve, and extend the Hermite-Hadamard and Simpson inequalities, one refers the reader to the recent papers (see [1–8]).

In [9], the author introduced the concept of harmonically convex functions and established some results connected with the right-hand side of new inequalities similar to inequality (1) for these classes of functions. Some applications to special means of positive real numbers were also given.

Definition 2. Let $I \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$. If inequality in (3) is reversed, then f is said to be harmonically concave.

The following result of the Hermite-Hadamard type holds.

Theorem 3. *Let $f : I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

The above inequalities are sharp.

Some results connected with the right part of (4) were given in [9] as follows.

Theorem 4. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q \geq 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\lambda_1^{1-1/q}\left[\lambda_2\left|f'(a)\right|^q + \lambda_3\left|f'(b)\right|^q\right]^{1/q}, \tag{5}$$

where

$$\lambda_1 = \frac{1}{ab} - \frac{2}{(b-a)^2}\ln\left(\frac{(a+b)^2}{4ab}\right),$$
$$\lambda_2 = \frac{-1}{b(b-a)} + \frac{3a+b}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right), \tag{6}$$
$$\lambda_3 = \frac{1}{a(b-a)} - \frac{3b+a}{(b-a)^3}\ln\left(\frac{(a+b)^2}{4ab}\right) = \lambda_1 - \lambda_2.$$

Theorem 5. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$, $1/p + 1/q = 1$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left(\frac{1}{p+1}\right)^{1/p}\left(\mu_1\left|f'(a)\right|^q + \mu_2\left|f'(b)\right|^q\right)^{1/q}, \tag{7}$$

where

$$\mu_1 = \frac{\left[a^{2-2q} + b^{1-2q}\left[(b-a)(1-2q) - a\right]\right]}{2(b-a)^2(1-q)(1-2q)},$$
$$\mu_2 = \frac{\left[b^{2-2q} - a^{1-2q}\left[(b-a)(1-2q) + b\right]\right]}{2(b-a)^2(1-q)(1-2q)}. \tag{8}$$

In this paper, one gives some general integral inequalities connected with the left and right parts of (4); as a result of this, one obtains some new midpoint, trapezoid, and Simpson-like type inequalities for differentiable harmonically convex functions.

2. Main Results

In order to prove our main results we need the following lemma.

Lemma 6. *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ then for $\lambda \in [0,1]$ one has the equality*

$$(1-\lambda)f\left(\frac{2ab}{a+b}\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx = \frac{ab(b-a)}{2}\left[\int_0^{1/2}\frac{\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt + \int_{1/2}^1\frac{2-\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt\right], \tag{9}$$

where $A_t = tb + (1-t)a$.

Proof. It suffices to note that

$$\begin{aligned} I_1 &= ab(b-a)\int_0^{1/2}\frac{\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt \\ &= (2t-\lambda)f\left(\frac{ab}{A_t}\right)\Big|_0^{1/2} - 2\int_0^{1/2}f\left(\frac{ab}{A_t}\right)dt \\ &= (1-\lambda)f\left(\frac{2ab}{a+b}\right) + \lambda f(b) - 2\int_0^{1/2}f\left(\frac{ab}{A_t}\right)dt. \end{aligned} \tag{10}$$

Set $x = ab/A_t$ and $dx = (-ab(b-a)/A_t^2)dt$, which gives

$$I_1 = (1-\lambda)f\left(\frac{2ab}{a+b}\right) + \lambda f(b) - \frac{2ab}{b-a}\int_{2ab/(a+b)}^b\frac{f(x)}{x^2}dx. \tag{11}$$

Similarly, we can show that

$$\begin{aligned} I_2 &= ab(b-a)\int_{1/2}^1\frac{2-\lambda-2t}{A_t^2}f'\left(\frac{ab}{A_t}\right)dt \\ &= \lambda f(a) + (1-\lambda)f\left(\frac{2ab}{a+b}\right) - \frac{2ab}{b-a}\int_a^{2ab/(a+b)}\frac{f(x)}{x^2}dx. \end{aligned} \tag{12}$$

Thus,

$$\frac{I_1+I_2}{2} = (1-\lambda)f\left(\frac{2ab}{a+b}\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx \tag{13}$$

which is required. □

Theorem 7. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is* *harmonically convex on $[a,b]$ for $q \geq 1$ and then one has the following inequality for $\lambda \in [0,1]$:*

$$\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2} \times\left\{C_1^{1-1/q}(\lambda;a,b) \times\left[C_2(\lambda;a,b)\left|f'(a)\right|^q + C_3(\lambda;a,b)\left|f'(b)\right|^q\right]^{1/q} + C_1^{1-1/q}(\lambda;b,a) \times\left[C_3(\lambda;b,a)\left|f'(a)\right|^q + C_2(\lambda;b,a)\left|f'(b)\right|^q\right]^{1/q}\right\}, \tag{14}$$

where

$$C_1(\lambda;u,\vartheta) = \frac{1}{(\vartheta-u)^2} \times\left[-4+\frac{[\lambda(\vartheta-u)+2u](3u+\vartheta)}{u(u+\vartheta)} + 2\ln\left(\frac{2u(u+\vartheta)}{(2u+\lambda(\vartheta-u))^2}\right)\right],$$
$$C_2(\lambda;u,\vartheta) = \frac{1}{(\vartheta-u)^3} \times\left\{[\lambda(\vartheta-u)+4u]\ln\left(\frac{[\lambda(\vartheta-u)+2u]^2}{2u(u+\vartheta)}\right) - \frac{[\lambda(\vartheta-u)+2u](5u+3\vartheta)}{u+\vartheta}+7u+\vartheta\right\},$$
$$C_3(\lambda;u,\vartheta) = C_1(\lambda;u,\vartheta) - C_2(\lambda;u,\vartheta), \quad u,\vartheta>0. \tag{15}$$

Proof. Let $A_t = tb + (1-t)a$. From Lemma 6 and using the Hölder inequality, we have

$$\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \leq \frac{ab(b-a)}{2}\left[\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt + \int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right]$$
$$\leq \frac{ab(b-a)}{2} \times\left\{\left(\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}dt\right)^{1-1/q} \times\left(\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q} + \left(\int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}dt\right)^{1-1/q} \times\left(\int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right\}. \tag{16}$$

Hence, by harmonically convexity of $|f'|^q$ on $[a,b]$, we have

$$\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right|$$
$$\leq \frac{ab(b-a)}{2} \times\left\{\left(\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}dt\right)^{1-1/q} \times\left(\int_0^{1/2}\left(|\lambda-2t|\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]\right) \times\left(A_t^2\right)^{-1}dt\right)^{1/q} + \left(\int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}dt\right)^{1-1/q} \times\left(\int_{1/2}^1\left(|2-\lambda-2t|\left[t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q\right]\right) \times\left(A_t^2\right)^{-1}dt\right)^{1/q}\right\}$$
$$\leq \frac{ab(b-a)}{2}C_1^{1-1/q}(\lambda;a,b) \times\left\{\left[C_2(\lambda;a,b)\left|f'(a)\right|^q + C_3(\lambda;a,b)\left|f'(b)\right|^q\right]^{1/q} + \left[C_3(\lambda;b,a)\left|f'(a)\right|^q + C_2(\lambda;b,a)\left|f'(b)\right|^q\right]^{1/q}\right\}. \tag{17}$$

It is easily to check that

$$\begin{aligned}
&\int_0^{1/2} \frac{|\lambda - 2t|}{A_t^2} dt \\
&= C_1 (\lambda; a, b) \\
&= \frac{1}{(b-a)^2} \times \left[-4 + \frac{[\lambda (b-a) + 2a] (3a+b)}{a(a+b)} \right. \\
&\qquad \left. + 2\ln\left(\frac{2a(a+b)}{(2a + \lambda (b-a))^2} \right) \right], \\
&\int_0^{1/2} \frac{|\lambda - 2t| t}{A_t^2} dt \\
&= C_2 (\lambda; a, b) \\
&= \frac{1}{(b-a)^3} \\
&\quad \times \left\{ [\lambda (b-a) + 4a] \ln\left(\frac{[\lambda (b-a) + 2a]^2}{2a(a+b)} \right) \right. \\
&\qquad \left. - \frac{[\lambda (b-a) + 2a] (5a+3b)}{a+b} + 7a + b \right\}, \\
&\int_0^{1/2} \frac{|\lambda - 2t| (1-t)}{A_t^2} dt \\
&= C_3 (\lambda; a, b) = C_1 (\lambda; a, b) - C_2 (\lambda; a, b).
\end{aligned} \tag{18}$$

This concludes the proof. □

Corollary 8. *Under the assumptions of Theorem 7 with $\lambda = 0$, one has*

$$\begin{aligned}
&\left| f\left(\frac{2ab}{a+b} \right) - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
&\leq \frac{ab(b-a)}{2} \\
&\quad \times \left\{ C_1^{1-1/q} (0; a, b) \right. \\
&\qquad \times \left[C_2 (0; a, b) \left| f'(a) \right|^q + C_3 (0; a, b) \left| f'(b) \right|^q \right]^{1/q} \\
&\qquad + C_1^{1-1/q} (0; b, a) \\
&\qquad \left. \times \left[C_3 (0; b, a) \left| f'(a) \right|^q + C_2 (0; b, a) \left| f'(b) \right|^q \right]^{1/q} \right\},
\end{aligned} \tag{19}$$

where

$$\begin{aligned}
C_1 (0; u, \vartheta) &= \frac{2}{(\vartheta - u)^2} \left[\ln\left(\frac{u+\vartheta}{2u} \right) - \frac{\vartheta - u}{u + \vartheta} \right], \\
C_2 (0; u, \vartheta) &= \frac{1}{(\vartheta - u)^3} \left[\frac{(3u+\vartheta)(\vartheta - u)}{u+\vartheta} + 4u \ln\left(\frac{2u}{u+\vartheta} \right) \right], \\
C_3 (0; u, \vartheta) &= \frac{1}{(\vartheta - u)^2} \\
&\quad \times \left[\frac{2(u+\vartheta)}{\vartheta - u} \ln\left(\frac{u+\vartheta}{2u} \right) - \frac{u + 3\vartheta}{u+\vartheta} \right], \\
&\qquad u, \vartheta > 0.
\end{aligned} \tag{20}$$

Corollary 9. *Under the assumptions of Theorem 7 with $\lambda = 1$, one has*

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
&\leq \frac{ab(b-a)}{2} \\
&\quad \times \left\{ C_1^{1-1/q} (1; a, b) \right. \\
&\qquad \times \left[C_2 (1; a, b) \left| f'(a) \right|^q + C_3 (1; a, b) \left| f'(b) \right|^q \right]^{1/q} \\
&\qquad + C_1^{1-1/q} (1; b, a) \\
&\qquad \left. \times \left[C_3 (1; b, a) \left| f'(a) \right|^q + C_2 (1; b, a) \left| f'(b) \right|^q \right]^{1/q} \right\},
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
C_1 (1; u, \vartheta) &= \frac{1}{(\vartheta - u)^2} \left[\frac{\vartheta - u}{u} + 2\ln\left(\frac{2u}{u+\vartheta} \right) \right], \\
C_2 (1; u, \vartheta) &= \frac{1}{(\vartheta - u)^3} \left[(3u+\vartheta) \ln\left(\frac{u+\vartheta}{2u} \right) - 2(\vartheta - u) \right], \\
C_3 (1; u, \vartheta) &= \frac{1}{(\vartheta - u)^2} \left[\frac{u+\vartheta}{u} - \frac{u+3\vartheta}{\vartheta - u} \ln\left(\frac{u+\vartheta}{2u} \right) \right], \\
&\qquad u, \vartheta > 0.
\end{aligned} \tag{22}$$

Corollary 10. *Under the assumptions of Theorem 7 with $\lambda = 1/3$, one has*

$$\begin{aligned}
&\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{2ab}{a+b} \right) \right] - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\
&\leq \frac{ab(b-a)}{2} \\
&\quad \times \left\{ C_1^{1-1/q} \left(\frac{1}{3}; a, b \right) \right. \\
&\qquad \times \left[C_2 \left(\frac{1}{3}; a, b \right) \left| f'(a) \right|^q + C_3 \left(\frac{1}{3}; a, b \right) \left| f'(b) \right|^q \right]^{1/q} \\
&\qquad + C_1^{1-1/q} \left(\frac{1}{3}; b, a \right) \\
&\qquad \left. \times \left[C_3 \left(\frac{1}{3}; b, a \right) \left| f'(a) \right|^q + C_2 \left(\frac{1}{3}; b, a \right) \left| f'(b) \right|^q \right]^{1/q} \right\},
\end{aligned} \tag{23}$$

where

$$C_1\left(\frac{1}{3};u,\vartheta\right) = \frac{1}{(\vartheta-u)^2}\left[\frac{(\vartheta-u)(\vartheta-3u)}{3u(u+\vartheta)} + 2\ln\left(\frac{18u(u+\vartheta)}{(5u+\vartheta)^2}\right)\right],$$

$$C_2\left(\frac{1}{3};u,\vartheta\right) = \frac{1}{(\vartheta-u)^3}\left[\left(\frac{11u+\vartheta}{3}\right)\ln\left(\frac{(5u+\vartheta)^2}{18u(u+\vartheta)}\right) + \frac{4u(\vartheta-u)}{3(u+\vartheta)}\right],$$

$$C_3\left(\frac{1}{3};u,\vartheta\right) = \frac{1}{(\vartheta-u)^2}\left[\frac{\vartheta^2-4u\vartheta-u^2}{3u(u+\vartheta)} + \frac{5u+7\vartheta}{3(\vartheta-u)}\ln\left(\frac{18u(u+\vartheta)}{(5u+\vartheta)^2}\right)\right],$$

$$u,\vartheta > 0. \tag{24}$$

Theorem 11. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$ and then one has the following inequality for $\lambda \in [0,1]$:*

$$\begin{aligned}&\left|(1-\lambda)f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\ &\leq \frac{ab(b-a)}{4} \times \frac{C_4^{1/p}(\lambda,p)}{\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}} \\ &\times\left\{\left(C_5(q;a,b)\left|f'(a)\right|^q + C_6(q;a,b)\left|f'(b)\right|^q\right)^{1/q} + \left(C_6(q;b,a)\left|f'(a)\right|^q + C_5(q;b,a)\left|f'(b)\right|^q\right)^{1/q}\right\},\end{aligned} \tag{25}$$

where

$$\begin{aligned}C_4(\lambda,p) &= \frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{p+1}, \\ C_5(q;u,\vartheta) &= \left[\left(\frac{u+\vartheta}{2}\right)^{1-2q}\left[\frac{\vartheta-3u}{2} - q(\vartheta-u)\right] + u^{2-2q}\right], \\ C_6(q;u,\vartheta) &= \left[\left(\frac{u+\vartheta}{2}\right)^{1-2q}\left[\frac{3\vartheta-u}{2} - q(\vartheta-u)\right] + u^{1-2q}\left[u-2\vartheta+2q(\vartheta-u)\right]\right], \\ &\qquad u,\vartheta > 0\end{aligned} \tag{26}$$

and $1/p + 1/q = 1$.

Proof. Let $A_t = tb + (1-t)a$. Using Lemma 6 and Hölder's integral inequality, we deduce

$$\begin{aligned}&\left|(1-\lambda)f\left(\frac{2ab}{a+b}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\ &\leq \frac{ab(b-a)}{2}\left[\int_0^{1/2}\frac{|\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt + \int_{1/2}^1\frac{|2-\lambda-2t|}{A_t^2}\left|f'\left(\frac{ab}{A_t}\right)\right|dt\right] \\ &\leq \frac{ab(b-a)}{2} \times\left\{\left(\int_0^{1/2}|\lambda-2t|^p dt\right)^{1/p}\times\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right. \\ &\quad\left. + \left(\int_{1/2}^1|2-\lambda-2t|^p dt\right)^{1/p}\times\left(\int_{1/2}^1\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right\}.\end{aligned} \tag{27}$$

Using the harmonically convexity of $|f'|^q$, we obtain

$$\begin{aligned}&\int_0^{1/2}\frac{1}{A_t^{2q}}\left|f'\left(\frac{ab}{A_t}\right)\right|^q dt \leq \int_0^{1/2}\frac{t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q}{A_t^{2q}}dt \\ &= \frac{1}{2(1-q)(1-2q)(b-a)^2} \times\left\{\left[\left(\frac{a+b}{2}\right)^{1-2q}\left[\frac{b-3a}{2}-q(b-a)\right]+a^{2-2q}\right]\left|f'(a)\right|^q\right. \\ &\quad\left.+\left[\left(\frac{a+b}{2}\right)^{1-2q}\left[\frac{3b-a}{2}-q(b-a)\right] + a^{1-2q}\left[a-2b+2q(b-a)\right]\right]\left|f'(b)\right|^q\right\},\end{aligned}$$

$$\int_0^{1/2} \frac{1}{A_t^{2q}} \left| f'\left(\frac{ab}{A_t}\right)\right|^q dt$$
$$\leq \int_{1/2}^1 \frac{t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q}{A_t^{2q}} dt$$
$$= \frac{1}{2(1-q)(1-2q)(b-a)^2}$$
$$\times \left\{ \left[b^{1-2q}\left[b-2a-2q(b-a)\right] + \left(\frac{a+b}{2}\right)^{1-2q}\left[\frac{3a-b}{2}+q(b-a)\right]\right]\left|f'(a)\right|^q + \left[\left(\frac{a+b}{2}\right)^{1-2q}\left[\frac{a-3b}{2}+q(b-a)\right] + b^{2-2q}\right] \times \left|f'(b)\right|^q \right\}. \tag{28}$$

Further, we have

$$\int_0^{1/2} |\lambda - 2t|^p dt = \int_{1/2}^1 |2-\lambda-2t|^p dt = \frac{\lambda^{p+1}+(1-\lambda)^{p+1}}{2(p+1)}. \tag{29}$$

A combination of (27)–(29) gives the required inequality (25). □

Corollary 12. *Under the assumptions of Theorem 11 with $\lambda = 0$, one has*

$$\left| f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)}{4(p+1)^{1/p}} \times \frac{1}{\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}}$$
$$\times \left\{ \left(C_5(q;a,b)\left|f'(a)\right|^q + C_6(q;a,b)\left|f'(b)\right|^q\right)^{1/q} + \left(C_6(q;b,a)\left|f'(a)\right|^q + C_5(q;b,a)\left|f'(b)\right|^q\right)^{1/q}\right\}. \tag{30}$$

Corollary 13. *Under the assumptions of Theorem 11 with $\lambda = 1$, one has*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)}{4(p+1)^{1/p}} \times \frac{1}{\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}}$$
$$\times \left\{ \left(C_5(q;a,b)\left|f'(a)\right|^q + C_6(q;a,b)\left|f'(b)\right|^q\right)^{1/q} + \left(C_6(q;b,a)\left|f'(a)\right|^q + C_5(q;b,a)\left|f'(b)\right|^q\right)^{1/q}\right\}. \tag{31}$$

Corollary 14. *Under the assumptions of Theorem 11 with $\lambda = 1/3$, one has*

$$\left| \frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{2ab}{a+b}\right)\right] - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)}{4(3^{p+1}(p+1))^{1/p}} \times \frac{1+2^{p+1}}{\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}}$$
$$\times \left\{ \left(C_5(q;a,b)\left|f'(a)\right|^q + C_6(q;a,b)\left|f'(b)\right|^q\right)^{1/q} + \left(C_6(q;b,a)\left|f'(a)\right|^q + C_5(q;b,a)\left|f'(b)\right|^q\right)^{1/q}\right\}. \tag{32}$$

Theorem 15. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically convex on $[a,b]$ for $q > 1$ and then one has the following inequality for $\lambda \in [0,1]$:*

$$\left| (1-\lambda) f\left(\frac{2ab}{a+b}\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \right| \leq \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q}$$
$$\times \left\{ C_7^{1/p}(a,b;p) \times \left[C_9(\lambda,q)\left|f'(a)\right|^q + C_{10}(\lambda,q)\left|f'(b)\right|^q\right]^{1/q} + C_8^{1/p}(a,b;p) \times \left[C_{10}(\lambda,q)\left|f'(a)\right|^q + C_9(\lambda,q)\left|f'(b)\right|^q\right]^{1/q}\right\}, \tag{33}$$

where

$$\begin{aligned} C_7(a,b;p) &= a^{1-2p} - \left(\frac{a+b}{2}\right)^{1-2p}, \\ C_8(a,b;p) &= \left(\frac{a+b}{2}\right)^{1-2p} - b^{1-2p}, \\ C_9(\lambda,q) &= \lambda^{q+2} + (1-\lambda)^{q+1}(q+1+\lambda), \\ C_{10}(\lambda,q) &= \lambda^{q+1}(4+2q-\lambda) \\ &\quad + (1-\lambda)^{q+1}(3+q-\lambda), \end{aligned} \tag{34}$$

and $1/p + 1/q = 1$.

Proof. Let $A_t = tb + (1-t)a$. Using Lemma 6 and Hölder's integral inequality, we deduce

$$\begin{aligned} &\left|(1-\lambda) f\left(\frac{2ab}{a+b}\right) + \lambda\left(\frac{f(a)+f(b)}{2}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ &\leq \frac{ab(b-a)}{2} \\ &\quad \times \left\{\left(\int_0^{1/2} \frac{1}{A_t^{2p}}dt\right)^{1/p} \right. \\ &\quad \times \left(\int_0^{1/2} |\lambda - 2t|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q} \\ &\quad + \left(\int_{1/2}^1 \frac{1}{A_t^{2p}}dt\right)^{1/p} \\ &\quad \left. \times \left(\int_{1/2}^1 |2-\lambda-2t|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt\right)^{1/q}\right\}. \end{aligned} \tag{35}$$

Using the harmonically convexity of $|f'|^q$, we obtain

$$\begin{aligned} &\int_0^{1/2} |\lambda - 2t|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt \\ &\leq \int_0^{1/2} |\lambda - 2t|^q \left[t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q\right] dt \\ &= \frac{1}{4(q+1)(q+2)} \\ &\quad \times \left\{\left[\lambda^{q+2} + (1-\lambda)^{q+1}(q+1+\lambda)\right]\left|f'(a)\right|^q \right. \\ &\quad + \left[\lambda^{q+1}(4+2q-\lambda) + (1-\lambda)^{q+1}(3+q-\lambda)\right] \\ &\quad \left. \times \left|f'(b)\right|^q\right\}, \\ &\int_{1/2}^1 |2-\lambda-2t|^q \left|f'\left(\frac{ab}{A_t}\right)\right|^q dt \\ &\leq \int_{1/2}^1 |2-\lambda-2t|^q \left[t\left|f'(a)\right|^q + (1-t)\left|f'(b)\right|^q\right] dt \\ &= \frac{1}{4(q+1)(q+2)} \\ &\quad \times \left\{\left[\lambda^{q+1}(4+2q-\lambda) + (1-\lambda)^{q+1}(3+q-\lambda)\right] \right. \\ &\quad \times \left|f'(a)\right|^q \\ &\quad \left. + \left[\lambda^{q+2} + (1-\lambda)^{q+1}(q+1+\lambda)\right]\left|f'(b)\right|^q\right\}. \end{aligned} \tag{36}$$

Further, we have

$$\begin{aligned} \int_0^{1/2} \frac{1}{A_t^{2p}}dt &= \frac{1}{(b-a)(2p-1)}\left[a^{1-2p} - \left(\frac{a+b}{2}\right)^{1-2p}\right], \\ \int_{1/2}^1 \frac{1}{A_t^{2p}}dt &= \frac{1}{(b-a)(2p-1)}\left[\left(\frac{a+b}{2}\right)^{1-2p} - b^{1-2p}\right]. \end{aligned} \tag{37}$$

A combination of (35)–(37) gives the required inequality (33). □

Corollary 16. *Under the assumptions of Theorem 15 with* $\lambda = 0$, *one has*

$$\begin{aligned} &\left|f\left(\frac{2ab}{a+b}\right) - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ &\leq \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \\ &\quad \times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q} \\ &\quad \times \left\{C_7^{1/p}(a,b;p)\left[(q+1)\left|f'(a)\right|^q + (q+3)\left|f'(b)\right|^q\right]^{1/q} \right. \\ &\qquad + C_8^{1/p}(a,b;p) \\ &\qquad \left. \times\left[(q+3)\left|f'(a)\right|^q + (q+1)\left|f'(b)\right|^q\right]^{1/q}\right\}. \end{aligned} \tag{38}$$

Corollary 17. *Under the assumptions of Theorem 15 with* $\lambda = 1$, *one has*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx\right| \\ &\leq \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \end{aligned}$$

$$\times\left[\frac{1}{4(q+1)(q+2)}\right]^{1/q}$$
$$\times\left\{C_7^{1/p}(a,b;p)\left[\left|f'(a)\right|^q+(2q+3)\left|f'(b)\right|^q\right]^{1/q}\right.$$
$$\left.+C_8^{1/p}(a,b;p)\left[(2q+3)\left|f'(a)\right|^q+\left|f'(b)\right|^q\right]^{1/q}\right\}. \quad (39)$$

Corollary 18. *Under the assumptions of Theorem 15 with $\lambda = 1/3$, one has*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{2ab}{a+b}\right)\right]\right.$$
$$\left.-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right|$$
$$\leq\frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}}$$
$$\times\left[\frac{1}{4(q+1)(q+2)}\right]^{1/q}$$
$$\times\left\{C_7^{1/p}(a,b;p)\right.$$
$$\times\left[C_9\left(\frac{1}{3},q\right)\left|f'(a)\right|^q+C_{10}\left(\frac{1}{3},q\right)\left|f'(b)\right|^q\right]^{1/q}$$
$$+C_8^{1/p}(a,b;p)\left[C_{10}\left(\frac{1}{3},q\right)\left|f'(a)\right|^q\right.$$
$$\left.\left.+C_9\left(\frac{1}{3},q\right)\left|f'(b)\right|^q\right]^{1/q}\right\}, \quad (40)$$

where

$$C_9\left(\frac{1}{3},q\right)=\frac{1}{3^{q+2}}\left(1+2^{q+1}(3q+4)\right),$$
$$C_{10}\left(\frac{1}{3},q\right)=\frac{1}{3^{q+2}}\left(11+6q+2^{q+1}(8+3q)\right). \quad (41)$$

3. Some Applications for Special Means

Let us recall the following special means of two nonnegative number a, b with $b > a$.

(1) The arithmetic mean

$$A=A(a,b):=\frac{a+b}{2}. \quad (42)$$

(2) The geometric mean

$$G=G(a,b):=\sqrt{ab}. \quad (43)$$

(3) The harmonic mean

$$H=H(a,b):=\frac{2ab}{a+b}. \quad (44)$$

(4) The logarithmic mean

$$L=L(a,b):=\frac{b-a}{\ln b-\ln a}. \quad (45)$$

(5) The p-logarithmic mean

$$L_p=L_p(a,b):=\left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{1/p},\quad p\in\mathbb{R}\setminus\{-1,0\}. \quad (46)$$

(6) The identric mean

$$I=I(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}. \quad (47)$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H\leq G\leq L\leq I\leq A. \quad (48)$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

Proposition 19. *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\left|(1-\lambda)H+\lambda A-\frac{G^2}{L}\right|$$
$$\leq\frac{ab(b-a)}{2}\left\{C_1(\lambda;a,b)+C_1(\lambda;b,a)\right\}, \quad (49)$$

where C_1 is defined as in Theorem 7.

Proof. The assertion follows from inequality (14) in Theorem 7, for $f:(0,\infty)\to\mathbb{R}$, $f(x)=x$. □

Proposition 20. *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\left|(1-\lambda)H+\lambda A-\frac{G^2}{L}\right|\leq\frac{ab(b-a)C_4^{1/p}(\lambda,p)}{4\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}}$$
$$\times\left\{\left(C_5(q;a,b)+C_6(q;a,b)\right)^{1/q}\right.$$
$$\left.+\left(C_6(q;b,a)+C_5(q;b,a)\right)^{1/q}\right\}, \quad (50)$$

where $q > 1$, $1/p + 1/q = 1$, and C_4, C_5, and C_6 are defined as in Theorem 11.

Proof. The assertion follows from inequality (25) in Theorem 11, for $f:(0,\infty)\to\mathbb{R}$, $f(x)=x$. □

Proposition 21. *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\left|(1-\lambda)H + \lambda A - \frac{G^2}{L}\right| \leq \frac{ab(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \times \left[\frac{\lambda^{q+1}+(1-\lambda)^{q+1}}{2(q+1)}\right]^{1/q} \times \left[C_7^{1/p}(a,b;p) + C_8^{1/p}(a,b;p)\right], \quad (51)$$

where $q > 1$, $1/p + 1/q = 1$, and C_7 and C_8 are defined as in Theorem 15.

Proof. The assertion follows from inequality (33) in Theorem 15, for $f:(0,\infty) \to \mathbb{R}$, $f(x) = x$. □

Proposition 22. *Let $0 < a < b$, $\lambda \in [0,1]$ and $q \geq 1$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^2 + \lambda A\left(a^2,b^2\right) - G^2\right| \\ &\leq ab(b-a) \\ &\times \left\{C_1^{1-1/q}(\lambda;a,b)\left[C_2(\lambda;a,b)a^q + C_3(\lambda;a,b)b^q\right]^{1/q}\right. \\ &+ C_1^{1-1/q}(\lambda;b,a) \\ &\left.\times\left[C_3(\lambda;b,a)a^q + C_2(\lambda;b,a)b^q\right]^{1/q}\right\},\end{aligned} \quad (52)$$

where C_1, C_2, and C_3 are defined as in Theorem 7.

Proof. The assertion follows from inequality (14) in Theorem 7, for $f:(0,\infty) \to \mathbb{R}$, $f(x) = x^2$. □

Proposition 23. *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^2 + \lambda A\left(a^2,b^2\right) - G^2\right| \\ &\leq \frac{ab(b-a)C_4^{1/p}(\lambda,p)}{2\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}} \\ &\times \left\{\left(C_5(q;a,b)a^q + C_6(q;a,b)b^q\right)^{1/q}\right. \\ &\left.+\left(C_6(q;b,a)a^q + C_5(q;b,a)b^q\right)^{1/q}\right\},\end{aligned} \quad (53)$$

where $q > 1$, $1/p + 1/q = 1$, and C_4, C_5, and C_6 are defined as in Theorem 11.

Proof. The assertion follows from inequality (25) in Theorem 11, for $f:(0,\infty) \to \mathbb{R}$, $f(x) = x^2$. □

Proposition 24. *Let $0 < a < b$ and $\lambda \in [0,1]$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^2 + \lambda A\left(a^2,b^2\right) - G^2\right| \\ &\leq \frac{ab(b-a)^{1-1/p}}{(2p-1)^{1/p}} \\ &\times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q} \\ &\times \left\{C_7^{1/p}(a,b;p)\left[C_9(\lambda,q)a^q + C_{10}(\lambda,q)b^q\right]^{1/q}\right. \\ &\left.+ C_8^{1/p}(a,b;p)\left[C_{10}(\lambda,q)a^q + C_9(\lambda,q)b^q\right]^{1/q}\right\},\end{aligned} \quad (54)$$

where $q > 1$, $1/p+1/q = 1$, and C_7, C_8, C_9, and C_{10} are defined as in Theorem 15.

Proof. The assertion follows from inequality (33) in Theorem 15, for $f:(0,\infty) \to \mathbb{R}$, $f(x) = x^2$. □

Proposition 25. *Let $0 < a < b$, $n \in (-1,\infty) \setminus \{0\}$, $\lambda \in [0,1]$, and $q \geq 1$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^{n+2} + \lambda A\left(a^{n+2},b^{n+2}\right) - G^2 \cdot L_n^n\right| \\ &\leq \frac{ab(b-a)(n+2)}{2} \\ &\times \left\{C_1^{1-1/q}(\lambda;a,b)\right. \\ &\times \left[C_2(\lambda;a,b)a^{(n+1)q} + C_3(\lambda;a,b)b^{(n+1)q}\right]^{1/q} \\ &+ C_1^{1-1/q}(\lambda;b,a) \\ &\left.\times\left[C_3(\lambda;b,a)a^{(n+1)q} + C_2(\lambda;b,a)b^{(n+1)q}\right]^{1/q}\right\},\end{aligned} \quad (55)$$

where C_1, C_2, and C_3 are defined as in Theorem 7.

Proof. The assertion follows from inequality (14) in Theorem 7, for $f:(0,\infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1,\infty) \setminus \{0\}$. □

Proposition 26. *Let $0 < a < b$, $\lambda \in [0,1]$, and $n \in (-1,\infty) \setminus \{0\}$. Then one has the following inequality:*

$$\begin{aligned}&\left|(1-\lambda)H^{n+2} + \lambda A\left(a^{n+2},b^{n+2}\right) - G^2 \cdot L_n^n\right| \\ &\leq \frac{ab(b-a)(n+2)C_4^{1/p}(\lambda,p)}{4\left[(1-q)(1-2q)(b-a)^2\right]^{1/q}} \\ &\times \left\{\left(C_5(q;a,b)a^{(n+1)q} + C_6(q;a,b)b^{(n+1)q}\right)^{1/q}\right. \\ &\left.+\left(C_6(q;b,a)a^{(n+1)q} + C_5(q;b,a)b^{(n+1)q}\right)^{1/q}\right\},\end{aligned} \quad (56)$$

where $q > 1$, $1/p + 1/q = 1$, *and* C_5 *and* C_6 *are defined as in Theorem 11.*

Proof. The assertion follows from inequality (25) in Theorem 11, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1, \infty) \setminus \{0\}$. □

Proposition 27. *Let* $0 < a < b$, $\lambda \in [0, 1]$, *and* $n \in (-1, \infty) \setminus \{0\}$. *Then one has the following inequality:*

$$\begin{aligned} &\left|(1-\lambda) H^{n+2} + \lambda A\left(a^{n+2}, b^{n+2}\right) - G^2 \cdot L_n^n\right| \\ &\quad \le \frac{ab(n+2)(b-a)^{1-1/p}}{2(2p-1)^{1/p}} \\ &\quad\quad \times \left[\frac{1}{4(q+1)(q+2)}\right]^{1/q} \\ &\quad\quad \times \left\{C_7^{1/p}(a, b; p)\right. \\ &\quad\quad\quad \times \left[C_9(\lambda, q) a^{(n+1)q} + C_{10}(\lambda, q) b^{(n+1)q}\right]^{1/q} \\ &\quad\quad\quad + C_8^{1/p}(a, b; p) \\ &\quad\quad\quad \left.\times \left[C_{10}(\lambda, q) a^{(n+1)q} + C_9(\lambda, q) b^{(n+1)q}\right]^{1/q}\right\}, \end{aligned} \tag{57}$$

where $q > 1$, $1/p + 1/q = 1$, *and* C_7, C_8, C_9, *and* C_{10} *are defined as in Theorem 15.*

Proof. The assertion follows from inequality (33) in Theorem 15, for $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^{n+2}$, $n \in (-1, \infty) \setminus \{0\}$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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