Large mass self-similar solutions of the parabolic-parabolic Keller–Segel model of chemotaxis

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Abstract In two space dimensions, the parabolic-parabolic Keller–Segel system shares many properties with the parabolic-elliptic Keller–Segel system. In particular, solutions globally exist in both cases as long as their mass is less than a critical threshold M_c. However, this threshold is not as clear in the parabolic-parabolic case as it is in the parabolic-elliptic case, in which solutions with mass above M_c always blow up. Here we study forward self-similar solutions of the parabolic-parabolic Keller–Segel system and prove that, in some cases, such solutions globally exist even if their total mass is above M_c, which is forbidden in the parabolic-elliptic case.

Keywords Keller–Segel model · chemotaxis · self-similar solution · nonlocal parabolic equations · critical mass · existence · blowup

Mathematics Subject Classification (2000) 35B30 · 35K40 · 35K57 · 35J60

1 Introduction

The Keller–Segel model and its generalizations has been widely studied for almost forty years. It models the behavior of a slime mold of myxamoebae, *Dictyostelium Discoideum*, which have the peculiarity of organizing themselves to form aggregates by moving towards regions of a higher concentration of a chemoattractant. This chemoattractant, the *cyclic adenosine monophosphate*, is secreted by the amoebae themselves when they are lacking of nutrients. The Keller–Segel model is considered as a prototypical (and very simplified) model for pattern formation in chemotaxis, and has attracted

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a lot of attention as a test case for more complex taxis phenomena driven by chemical substances. See [10–12, 18, 19] for further references.

The simplest version of the model is made of two parabolic equations, one for the density n of the amoebae and another one for the density c of the chemoattractant, that is

$$n_t = \Delta n - \nabla \cdot (n \nabla c)\,, \tag{1}$$

$$\tau\, c_t = \Delta c + n\,, \tag{2}$$

where the parameter τ takes into account the difference of the time scales of the diffusive processes undergone by n and c. Here, all the other biologically relevant coefficients (amoebae diffusivity, sensitivity function, etc.) are put equal to 1, which can be obtained after a suitable scaling and adimensionalization of the original system of Keller and Segel. An even simpler reduced version has been widely considered by neglecting the time-derivative of the chemoattractant density in (2). Therefore, in the sequel we shall refer to the complete version of (1)–(2) with $\tau > 0$ as the parabolic-parabolic model, and to the latter ($\tau = 0$) as the parabolic-elliptic Keller–Segel model.

All variants of the above model involve diffusions in the equations for the density of the amoebae and for the density of the chemoattractant. The coupling is due to the fact that amoebae move according to the gradient of the chemoattractant, and that the emission of the chemoattractant is proportional to the density of amoebae. A crude insight into the main features of the model can be gained from the simplest case, that is when the nonlinear term in equation (1) is quadratic, but more realistic models should probably involve more complex nonlinearities.

Interesting mathematical questions are related to qualitative properties of problem (1)–(2) such as global in time existence *versus* finite time blowup of solutions describing chemotactic concentration phenomena. After the pioneering works of Keller and Segel, a huge literature has dealt with the mathematical modelling of chemotaxis and its analysis. We recommend the reading of [10] for a recent review from both biological and mathematical points of view.

Since the slime mold moves over a planar substrate, it makes sense to consider two-dimensional geometries. In some cases, boundary effects are important, but they are out of the purpose of this paper and we shall therefore assume that the model is set on the two-dimensional Euclidean plane. In that case, for the parabolic-elliptic model, there is a critical mass M_c, ($M_c = 8\,\pi$ for the adimensionalized system (1)–(2)), whose role is now rather well understood; see [9, 5]. Below such a mass, the diffusion predominates, in the sense that amoebae are unable to emit enough chemoattractant to aggregate. Therefore, on large times, the population diffuses and locally vanishes, although the behavior significantly differs from a pure diffusion. Above M_c, at least one singularity appears in finite time, which is interpreted as the occurrence of an aggregate. In the critical case $M = M_c$, the solutions are known to be global in time but for initially not dispersed data the density n grows and mass concentration occurs in infinite time; see [3, 4].

Since singularities are local, it is widely believed that the critical mass M_c should be a threshold between the diffusion dominated regime and the regime of aggregation also in the parabolic-parabolic model. This is certainly the case in some sense, for appropriate initial data, but the situation is not as simple as in the parabolic-elliptic case. It turns out that if, initially, the population of amoebae is scattered enough, and for a well chosen initial distribution of the chemoattractant, there are solutions for which the diffusion predominates for large times, even for masses larger than M_c. It is the purpose of this paper to establish such a fact, for a special class of solutions and in a certain range of the parameters of the model.

In this paper, we shall consider the parabolic-parabolic Keller–Segel system (1)–(2) for any $t > 0$, $x \in \mathbb{R}^2$, supplemented with initial conditions n_0 and c_0. From now on we shall assume that n_0 and c_0 are nonnegative and that n_0 is integrable on $\mathbb{R}^2$. As a consequence, for solutions with sufficiently fast decay at infinity, the total mass is conserved, *i.e.*,

$$M := \int_{\mathbb{R}^2} n(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^2} n_0(x)\,\mathrm{d}x$$

does not depend on t.

Throughout the paper, τ is a positive parameter. The qualitative properties of n and c (such as the asymptotic behavior for large values of t) strongly depend on τ and the stability of system (1)–(2) with respect to τ is expected, *i.e.* solutions of the parabolic-parabolic Keller–Segel system are expected to converge to those of the parabolic-elliptic system when $\tau \searrow 0$. This has been recently proved, at least for solutions with a suitably small mass M, in [17]. However, here we are interested in the differences between the parabolic-elliptic Keller–Segel system ($\tau = 0$) and the parabolic-parabolic Keller–Segel system ($\tau > 0$). According to [7], when $\tau > 0$, solutions of (1)–(2) globally exist for any $M < 8\,\pi$, cf. also [1] for the case of a bounded domain. On the other hand, it has not yet been proved that explosion occurs in finite time as soon as $M > 8\,\pi$, eventually under some additional assumptions like a smallness condition on $\int_{\mathbb{R}^2} |x|^2\,n_0(x)\,\mathrm{d}x$. If $M = 8\,\pi$, it is known that there is an infinite number of steady states (see [3]), but no other result is available, apart from self-similar solutions.

Motivated by this lack of results for (1)–(2), this paper deals with the existence of *positive forward self-similar solutions* of (1)–(2), *i.e.*, solutions which can be written as

$$n(t,x) = \frac{1}{t}\,u\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad c(t,x) = v\left(\frac{x}{\sqrt{t}}\right), \tag{3}$$

with *a large total mass* (that is, larger than $8\,\pi$). Indeed, since we are dealing with the two-dimensional case, any self-similar solution n in $L^1(\mathbb{R}^2)$ preserves mass, *i.e.*, for each $t \geq 0$

$$\int_{\mathbb{R}^2} n(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^2} u(\xi)\,\mathrm{d}\xi = M\,.$$

Therefore, for any given $\tau > 0$, we are interested in the optimal range of M for the existence of such solutions, and in uniqueness or multiplicity issues for a given M in the optimal range. Actually, our goal is double. The main one is to prove the above mentioned existence result. Second, we will give an as complete as possible review of the numerous existing results on the topic and also simplified, new proofs of them. The remainder of the introduction will be primarily devoted to the state of the art on self-similar solutions.

Self-similar solutions can be obtained through various approaches. The first method for the study of self-similar solutions (see for example [1] and the references therein) amounts to look for mild solutions of (1)–(2), that is, solutions of

$$n(t,\cdot) = \mathrm{e}^{(t-t_0)\varDelta}\,n(t_0,\cdot) - \int_{t_0}^{t}\left(\nabla \mathrm{e}^{(t-s)\varDelta}\right)\cdot\big(n(s,\cdot)\,\nabla c(s,\cdot)\big)\,\mathrm{d}s\,,$$

$$c(t,\cdot) = \mathrm{e}^{\frac{t-t_0}{\tau}\varDelta}\,c(t_0,\cdot) + \frac{1}{\tau}\int_{t_0}^{t}\mathrm{e}^{\frac{t-s}{\tau}\varDelta}\,n(s,\cdot)\,\mathrm{d}s\,,$$

for any $t > t_0 \geq 0$. Roughly speaking, such self-similar solutions are obtained by a fixed point theorem. However, smallness conditions on the initial data are required in order to apply a contraction mapping principle; see [15], where this method has been applied to (1)–(2) with $\tau = 1$. Therefore, covering the whole range of masses for which solutions exists seems out of reach in this setting.

Alternatively, one can prove the existence of self-similar solutions through the direct analysis of the elliptic system satisfied by (u, v), *i.e.*,

$$\Delta u - \nabla \cdot \left(u \, \nabla v - \frac{1}{2} \, \xi \, u \right) = 0 \,, \tag{4}$$

$$\Delta v + \frac{\tau}{2} \, \xi \cdot \nabla v + u = 0 \,, \tag{5}$$

where $\xi = x/\sqrt{t}$ and the differential operators in (4)–(5) are taken with respect to ξ. In this case, a natural functional space to be considered for both u and v is the subspace $C^2_0(\mathbb{R}^2)$ of functions in the space $C^2(\mathbb{R}^2)$ such that

$$\lim_{|\xi| \to \infty} u(\xi) = 0 \quad \text{and} \quad \lim_{|\xi| \to \infty} v(\xi) = 0 \,.$$

For such classical solutions, equation (4) can be written equivalently as either

$$\nabla \cdot \left[u \, \nabla \left(\log u - v + \frac{|\xi|^2}{4} \right) \right] = 0 \,,$$

or

$$\nabla \cdot \left[\mathrm{e}^{v} \, \mathrm{e}^{-|\xi|^2/4} \, \nabla \left(u \, \mathrm{e}^{-v} \, \mathrm{e}^{|\xi|^2/4} \right) \right] = 0 \,.$$

Then, using the fact that u, v, and consequently $|\nabla v|$ are bounded, it has been proved in [16] that there exists a constant σ such that

$$u(\xi) = \sigma \, \mathrm{e}^{v(\xi)} \mathrm{e}^{-\frac{|\xi|^2}{4}} \tag{6}$$

for any $\xi \in \mathbb{R}^2$. Since u is positive by the maximum principle, it follows that σ is positive. As a consequence, $u \in L^1(\mathbb{R}^2)$, and the stationary system (4)–(5) reduces to a family of nonlinear elliptic equations for v, namely

$$\Delta v + \frac{\tau}{2} \, \xi \cdot \nabla v + \sigma \, \mathrm{e}^{v} \, \mathrm{e}^{-\frac{|\xi|^2}{4}} = 0 \,, \tag{7}$$

parametrized by $\sigma > 0$. Again by the maximum principle applied to (7), the following upper bound for v can be proved

$$v(\xi) \leq C \, \mathrm{e}^{-\min\{1,\tau\} \frac{|\xi|^2}{4}} \,, \tag{8}$$

where C is any positive constant such that $C \min\{1, \tau\} \geq \sigma \, \mathrm{e}^{\|v\|_\infty}$; see for instance [16]. Therefore, $v \in L^1(\mathbb{R}^2)$ holds true for any solution of (7) in $C^2_0(\mathbb{R}^2)$.

The range of M for which self-similar solutions exist in $C^2_0(\mathbb{R}^2)$ gives an indication on the range of M for which some solutions of (1)–(2) may globally exist. Self-similar solutions indeed provide explicit examples of global solutions, even with smooth initial data, up to a time-shift: take for instance u and v as the initial data for (1)–(2). Moreover, if self-similar solutions describe the asymptotic behavior of any solution of (1)–(2) under appropriate conditions on initial data, then the ranges of global existence of solutions should be exactly the same. This property has been established in [5] for $\tau = 0$. In the case $\tau > 0$, this might not be as simple as in the case $\tau = 0$ if one can prove that blowup may occur for any $M > 8\,\pi$. However, at least for initial data close enough to u and v, one can expect that the ranges of global existence are the same.

In view of our main goal, we are actually more interested in parametrizing the set of $C_0^2(\mathbb{R}^2)$ self-similar solutions in terms of mass rather than in terms of σ. This is possible using in (7) the relation

$$M = \sigma \int_{\mathbb{R}^2} \mathrm{e}^{v(\xi)}\,\mathrm{e}^{-\frac{|\xi|^2}{4}}\,\mathrm{d}\xi\,. \tag{9}$$

However, by doing that, equation (7) is not anymore local, as was the original system (1)–(2), and the problem is definitely more difficult to handle. Another not less important reason to consider a different but equivalent formulation of problem (4)–(5) is that the correspondence between σ and M is not clear (see Remark 1 for further comments on σ).

For the sake of completeness, we have to say that equation (7), written as

$$\nabla\cdot\left(\mathrm{e}^{\frac{\tau}{4}|\xi|^2}\,\nabla v\right) + \sigma\,\mathrm{e}^{v}\,\mathrm{e}^{\frac{\tau-1}{4}|\xi|^2} = 0\,,$$

has been studied using variational methods in [14,20]. The weighted functional space $H^1(\mathbb{R}^2; \exp(\frac{\tau}{4}|\xi|^2)\,\mathrm{d}\xi)$ is then natural, but working in this space introduces a condition on the values of τ, which have to be in the interval $(0,2)$. Under such a restriction, it has been established that solutions exist if $0 < \sigma < \sigma^*$, for some $\sigma^* > 0$. These solutions are positive and belong to $C_0^2(\mathbb{R}^2)$, but due to the restriction on τ, one has to look for alternative approaches.

Another important and useful result has been obtained in [16] using the moving planes technique: any positive solution $v \in C_0^2(\mathbb{R}^2)$ of (7) must be radially symmetric. As a consequence, system (4)–(5) reduces to the ODE system

$$u' - u\,v' + \frac{1}{2}\,r\,u = 0\,, \tag{10}$$

$$v'' + \left(\frac{1}{r} + \frac{\tau}{2}\,r\right)\,v' + u = 0\,, \tag{11}$$

where u and v are considered as functions of the radial variable $r = |\xi|$ only. Equations (6)–(7) then become

$$\begin{aligned} &u(r) = \sigma\,\mathrm{e}^{v(r)}\,\mathrm{e}^{-r^2/4},\\ &v'' + \left(\frac{1}{r} + \frac{\tau}{2}\,r\right)v' + \sigma\,\mathrm{e}^{v}\,\mathrm{e}^{-r^2/4} = 0\,. \end{aligned} \tag{12}$$

Equation (12) has been studied in [13,16]. More specifically, the authors proved in [13] the existence of a positive decreasing solution of (12) endowed with the initial and integrability conditions

$$v'(0) = 0 \quad \text{and} \quad \int_0^\infty r\,v(r)\,\mathrm{d}r < \infty\,, \tag{13}$$

for any $\tau > 0$ and $\sigma > 0$ such that $\sigma\,\frac{\log\tau}{\tau-1} < \frac{1}{\mathrm{e}}$ (see Remark 2 for more details).

It is worth noticing that the boundary conditions (13) and the following ones,

$$v'(0) = 0 \quad \text{and} \quad \lim_{r\to\infty} v(r) = 0\,, \tag{14}$$

are equivalent for classical decreasing solutions. Indeed, (13) implies (14) and the converse holds true by (8). Using (14), equation (12) turns out to be equivalent to

$$w'' + \left(\frac{1}{r} + \frac{\tau}{2}\,r\right) w' + \mathrm{e}^{w}\,\mathrm{e}^{-r^2/4} = 0\,, \tag{15}$$

$$w'(0) = 0 \quad \text{and} \quad w(0) = s\,, \tag{16}$$

with $w = \log\sigma + v$, for some shooting parameter $s \in \mathbb{R}$. Indeed, if $w(r;s)$ is a classical solution of (15)–(16) for a given $s \in \mathbb{R}$, then $w(\infty;s) = \lim_{r\to\infty} w(r;s)$ exists and is finite and $v(r) = w(r;s) - w(\infty;s)$ is a classical solution of (12)–(14) with $\sigma = \mathrm{e}^{w(\infty;s)}$. Conversely, if v is a classical solution of (12)–(14), then $w(r;s) = v(r) + \log\sigma$ is a classical solution of (15)–(16) with $s = v(0) + \log\sigma$ and again $\sigma = \mathrm{e}^{w(\infty;s)}$ holds true. It follows that all solutions of (12)–(14) can be parametrized in terms of s. See [16] for more details. Using this equivalence, the authors of [16] analyze the structure of the set of solutions of (12)–(14) seen as a one-parameter family; see Remark 1 for more details. Computations presented in Figs. 1 and 2 have been based on this parametrization of the solution set.

Last but not least, the parametrization of the solutions of (15)–(16) in terms of s allows us to parametrize the total mass M in term of s by

$$M(s) = 2\,\pi \int_0^\infty \mathrm{e}^{w(r;s)}\,\mathrm{e}^{-r^2/4}\,r\,\mathrm{d}r\,. \tag{17}$$

Computations presented in Fig. 2 (left) are based on this parametrization. But again, this does not provide an explicit computation for the optimal range of M.

Being this the state of the art, we will establish that the formulation of system (10)–(11) in terms of *cumulated densities* is better adapted to the qualitative description of u and v. This is a classical technique used previously, for example, in the context of the parabolic-elliptic Keller–Segel system and astrophysical models; see [1,3], further references therein and for instance [8] for a recent application. For $\tau > 0$, many qualitative properties of the solutions can still be proved in this framework. These will allow us to build *positive forward self-similar solutions* of (1)–(2) satisfying (3), which have an arbitrarily large mass when τ is large enough. Our results are summarized in Theorem 1 below. One may interpret it by saying that the diffusion of c described by (2) for large positive τ and some $M > 8\,\pi$ may prevent the blowup of the solutions of the parabolic-parabolic Keller–Segel system. This is a major difference with the parabolic-elliptic case $\tau = 0$, for which the response of c to the variations of n being instantaneous, any smooth solution with mass $M > 8\,\pi$ must concentrate and blow up in finite time.

Theorem 1 *All nontrivial solutions $(u, v) \in (C_0^2(\mathbb{R}^2))^2$ of (4)–(5) are radial, decreasing in $|x|$ for $x \neq 0$, such that $u > 0$ and $v > 0$, with exponential decay at infinity, and hence attain their maximum at $x = 0$. They are uniquely determined by $a := u(0)/2$, which in turn uniquely determines their mass $M = M(a, \tau)$. Moreover, $\lim_{a\to\infty} M(a, \tau) = 8\,\pi$, while, as $a \to \infty$, $u/(8\,\pi)$ weakly converges to the Dirac delta distribution located at $x = 0$.*

For any $M \in (0, 8\pi)$, there is at least one solution of (4)–(5) with mass M for any $\tau > 0$, i.e. *there exists an $a > 0$ such that $M = M(a, \tau)$. For any $M > 8\pi$, there exists some $\tilde{\tau}(M) > 1/2$ such that for any $\tau \geq \tilde{\tau}(M)$ there is also at least one solution $(u, v) \in (C_0^2(\mathbb{R}^2))^2$ of (4)–(5) with $u > 0$ and mass M.*

In other words, for any $\tau > 0$, let $M^* = M^*(\tau)$ be the supremum of $M > 0$ such that (4)–(5) has at least one solution with $u > 0$ for any mass in $(0, M)$. Then, $M^* \geq 8\pi$ for any $\tau > 0$ and $\lim_{\tau\to\infty} M^*(\tau) = \infty$.

Moreover, M^* is finite, achieved if $M^* > 8\pi$, there is no solution if $M > M^*$ and there are at least two solutions if $M \in (8\pi, M^*)$.

More detailed statements will be given in Theorems 3 and 4, in the framework of cumulated densities. For estimates on $M(a, \tau)$ and $M^*(\tau)$, see respectively Theorems 2 and 4.

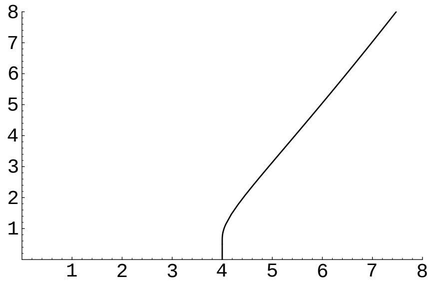

Fig. 0 A numerical computation of $\tilde{\tau}(M)$ as a function of $\frac{M}{2\pi}$. For more details on numerics see Section 6. If $M > 8\pi$, $\tau \geq \tilde{\tau}(M)$ means $M \leq M^*$. The function $M \mapsto \tilde{\tau}(M)$ is a generalized inverse of $\tau \mapsto M^*(\tau)$; see Fig. 6 for an enlarged plot of M^* for $\tau < 1$.

This paper is organized as follows. We shall first establish the main *a priori* estimates for Theorem 1 in the next section. The framework of cumulated densities is developed in Section 3 and detailed statements are given in Theorems 2–4. The remaining *a priori* estimates and proofs are given in Sections 4 and 5, respectively. Section 6 is devoted to some numerical results and Section 7 to concluding remarks.

2 Large mass positive forward self-similar solutions

Before restating the question of self-similar solutions in terms of cumulated densities, let us establish the key *a priori* estimate for Theorem 1, which proves that solutions of (4)–(5) may have an arbitrary large mass when τ is large enough. This result is entirely new. Such an estimate can be obtained either using equation (12) or in the cumulated densities formulation. In this section, we shall establish this *a priori* estimate in the first setting. It will be translated in the cumulated densities framework in Section 4.

From now on, we shall parametrize M in term of a and τ, *i.e.* $M = M(a, \tau)$, where $a = u(0)/2$ will also be the shooting parameter in the cumulated densities shooting problem, see (29)–(30) and (33)–(34) below.

A positive classical solution v of (12), (14) solves

$$\left(r\,\mathrm{e}^{\tau\,r^2/4}\,v'\right)' + \sigma\,r\,\mathrm{e}^{(\tau-1)\,r^2/4}\,\mathrm{e}^{v} = 0\,,$$

which, after an integration on $(0,r)$, gives

$$v'(r) = -\frac{\sigma}{r}\, e^{-\tau\, r^2/4} \int_0^r e^{(\tau-1)\, z^2/4}\, e^{v(z)}\, z\, dz\,. \tag{18}$$

As a consequence, v' is nonpositive, so that $v(z) \le v(0)$ for any $z \ge 0$ and, for $\tau \ne 1$,

$$v'(r) \ge -\frac{\sigma}{r}\, e^{-\tau\, r^2/4}\, e^{v(0)} \int_0^r e^{(\tau-1)\, z^2/4}\, z\, dz = -\frac{2}{\tau-1}\,\frac{\sigma}{r}\, e^{v(0)} \left(e^{-r^2/4} - e^{-\tau\, r^2/4}\right). \tag{19}$$

We observe that

$$\frac{d}{d\tau} \int_0^\infty \left(e^{-r^2/4} - e^{-\tau\, r^2/4}\right) \frac{2\, dr}{r} = \int_0^\infty e^{-\tau\, r^2/4}\, \frac{r}{2}\, dr = \frac{1}{\tau}\,.$$

Hence, after one more integration of (19) on $(0,\infty)$, we get, for any $\tau \ne 1$,

$$v(0) \le \sigma\, e^{v(0)}\, I(\tau) \quad \text{with} \quad I(\tau) := \frac{\log \tau}{\tau - 1}\,. \tag{20}$$

Actually, it is easy to check that estimate (20) holds true also for $\tau = 1$ with $I(1) = 1$. Since from (6) we have

$$\sigma\, e^{v(0)} = u(0) = 2\, a\,, \tag{21}$$

we have proved that

$$0 = \lim_{z\to\infty} v(z) \le v(r) \le v(0) \le 2\, a\, I(\tau) \tag{22}$$

for any $r \in \mathbb{R}_+$. On the other hand, by (9), (21) and (22), the mass M can be estimated for any positive a and τ by

$$M = 2\,\pi\,\sigma \int_0^\infty e^{v(r)}\, e^{-r^2/4}\, r\, dr \ge 2\,\pi\,\sigma \int_0^\infty e^{-r^2/4}\, r\, dr = 4\,\pi\,\sigma \ge 8\,\pi\, a\, e^{-2\, a\, I(\tau)}\,. \tag{23}$$

As a function of a, $\widetilde{M}(a,\tau) := 8\,\pi\, a\, e^{-2\, a\, I(\tau)}$ achieves its maximum at $a_*(\tau) := \frac{1}{2\, I(\tau)}$, which proves that $M = M(a,\tau)$ verifies for each $\tau > 0$

$$\max_{a>0} M(a,\tau) \ge \widetilde{M}(a_*(\tau),\tau) = \frac{4\,\pi}{e\, I(\tau)}\,,$$

and it is clear that the right hand side can be made arbitrarily large for τ large enough. Hence, the corresponding density $u(r) = \sigma\, e^{v(r)} e^{-r^2/4}$ has mass $M > 8\,\pi$ if $\frac{4\,\pi}{e\, I(\tau)} > 8\,\pi$, that is for any $\tau > \bar\tau$ with $\bar\tau$ such that $I(\bar\tau) = \frac{1}{2\, e}$, *i.e.* $\bar\tau \approx 16.1109$. Also observe that for any $\tau > \bar\tau$ the density u corresponding to $a = a_*(\tau)$ satisfies $u(0) = 2\, a_*(\tau) > 2\, e$. Finally, using $v(z) \ge v(r)$ in (18) and integrating the inequality on $(0,\infty)$, one obtains $e^{-v(0)} - \lim_{r\to\infty} e^{-v(r)} \le -\,\sigma\, I(\tau)$, for any $\tau > 0$. As a consequence, using (21) and $\lim_{r\to\infty} e^{-v(r)} = 1$, we obtain that

$$1 - e^{v(0)} \le -\,\sigma\, I(\tau)\, e^{v(0)} = -\,2\, a\, I(\tau)\,.$$

This gives the estimate

$$v(0) \ge \log(2\, a\, I(\tau) + 1)\,, \tag{24}$$

which implies that $v(0)$ becomes arbitrarily large as $a \to \infty$, for any $\tau > 0$.

Estimates (22) and (24) can be read also as lower and upper bounds for $\sigma = 2\,a\,\mathrm{e}^{-v(0)}$, namely

$$2\,a\,\mathrm{e}^{-2\,a\,I(\tau)} \leq \sigma \leq \min\left\{\frac{M}{4\,\pi}\,,\,\frac{2\,a}{2\,a\,I(\tau)+1}\right\}\,, \tag{25}$$

hence showing that σ takes arbitrarily large values for τ large enough (since the maximum of $2\,a\,\mathrm{e}^{-2\,a\,I(\tau)}$ becomes arbitrarily large as $\tau \to \infty$).

Remark 1 Estimates (25) on σ are new. The authors of [16] analyzed the map $s \mapsto \sigma(s)$, where s is the shooting parameter defined in (16), and they proved that it is a continuous map from $\mathbb{R}$ into $\mathbb{R}_+$ with $\lim_{s\to\pm\infty}\sigma(s) = 0$. Therefore, σ must be bounded for any fixed τ by $\sigma^* = \sigma(s^*)$, for some $s^* \in \mathbb{R}$, and problem (12)–(14) admits no solution for $\sigma > \sigma^*$, at least one solution for $\sigma = \sigma^*$ and finally (at least) two distinct solutions for $0 < \sigma < \sigma^*$. However, estimates on σ (or σ^*) were missing.

Remark 2 Estimate (20) says that, for any fixed $\sigma > 0$ and $\tau > 0$, $v(0)$ satisfies

$$v(0) - \,\sigma\,I(\tau)\,\mathrm{e}^{v(0)} \leq 0\,.$$

Since the function $x \mapsto x - \sigma\,I(\tau)\,\mathrm{e}^{x}$ is strictly concave and attains its maximum at $x = -\log(\sigma\,I(\tau))$, we deduce that whenever $\sigma\,I(\tau) < 1/\mathrm{e}$, there exists an open bounded interval $J \subset \mathbb{R}_+$ of nonexistence of solutions of (12) satisfying (14), with $v(0) \in J$. On the other hand if $\sigma\,I(\tau) \geq 1/\mathrm{e}$, the above inequality induces no restriction on $v(0)$.

3 Cumulated densities and main results

Let us introduce the *cumulated densities* formulation of the parabolic-parabolic Keller–Segel model as in [1], in terms of the functions u and v which solve problem (10)–(11), by defining

$$\phi(y) := \frac{1}{2\,\pi}\int_{B(0,\sqrt{y})} u(\xi)\,\mathrm{d}\xi = \int_0^{\sqrt{y}} r\,u(r)\,\mathrm{d}r\,,$$

$$\psi(y) := \frac{1}{2\,\pi}\int_{B(0,\sqrt{y})} v(\xi)\,\mathrm{d}\xi = \int_0^{\sqrt{y}} r\,v(r)\,\mathrm{d}r\,.$$

Using the relations

$$\phi'(y) = \frac{1}{2}\,u\left(\sqrt{y}\right) \quad \text{and} \quad \phi''(y) = \frac{1}{4\,\sqrt{y}}\,u'\left(\sqrt{y}\right)\,, \tag{26}$$

$$\psi'(y) = \frac{1}{2}\,v\left(\sqrt{y}\right) \quad \text{and} \quad \psi''(y) = \frac{1}{4\,\sqrt{y}}\,v'\left(\sqrt{y}\right)\,,$$

it follows from (10)–(11) that the cumulated densities ϕ and ψ solve the second order ODE system

$$\phi'' + \frac{1}{4}\,\phi' - 2\,\phi'\psi'' = 0\,, \tag{27}$$

$$4\,y\,\psi'' + \tau\,y\,\psi' - \tau\,\psi + \phi = 0\,, \tag{28}$$

where (11) has been multiplied by r and integrated on $(0,\sqrt{y})$. Observing that equation (28) can be written as

$$4\,(y\,\psi'-\psi)'+\tau\,(y\,\psi'-\psi)+\phi=0\,,$$

and defining $S(y):=4\,(\psi(y)-y\,\psi'(y))'=-4\,y\,\psi''(y)=-\sqrt{y}\,v'(\sqrt{y})$ as in [2,16], system (27)–(28) becomes, after a differentiation of (28) with respect to y, a first order system in the (ϕ',S) variables

$$\phi''+\frac{1}{4}\,\phi'+\frac{1}{2\,y}\,\phi' S=0\,, \tag{29}$$

$$S'+\frac{\tau}{4}\,S=\phi'\,. \tag{30}$$

The last formulation of the ODE system can be equivalently written as a single integro-differential equation, hence nonlocal, for ϕ',

$$\phi''+\frac{1}{4}\,\phi'+\frac{1}{2\,y}\,\phi'\,\mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{\tau\,z/4}\,\phi'(z)\,\mathrm{d}z=0\,, \tag{31}$$

since, by (30),

$$S(y)=\mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{\tau\,z/4}\,\phi'(z)\,\mathrm{d}z\,, \tag{32}$$

and as a single, local but nonlinear second order ODE for S,

$$S''+\frac{1}{4}\,(\tau+1)\,S'+\frac{\tau}{16}\,S+\frac{1}{2\,y}\left(S\,S'+\frac{\tau}{4}\,S^2\right)=0\,,$$

which is obtained by differentiating (30). We will use in the sequel all these formulations in order to get *a priori* estimates.

For any positive self-similar solution $(u,v)\in(C_0^2(\mathbb{R}^2))^2$, the natural initial conditions for (29)–(30) are

$$\phi(0)=0\,,\quad \phi'(0)=a>0\quad\text{and}\quad S(0)=0\,, \tag{33}$$

in view of the definition of ϕ and of (32). Moreover, for any self-similar solution $u\in L^1(\mathbb{R}^2)$, the corresponding cumulated density ϕ satisfies the boundary condition

$$\phi(\infty):=\lim_{y\to\infty}\phi(y)=\frac{M(a,\tau)}{2\,\pi}\,. \tag{34}$$

The problem is now formulated in terms of a shooting parameter problem (29)–(30), (33), with a new shooting parameter a which is directly related to the concentration of the self-similar density u around the origin, since $a=u(0)/2$ by definition. This has been obtained in Section 2 and will be made more precise below. Let us observe that the relation between a and the shooting parameter s defined in (16) is $2\,a=\mathrm{e}^s$ by (21), since $s=v(0)+\log\sigma$. Thus, a one-to-one relation is established between the initial valued problems (29)–(30), (33) and (15)–(16) as soon as an existence and uniqueness result is established for one of them. Moreover, we have

$$v(0)=v(\sqrt{y})+\frac{1}{2}\int_0^y \frac{S(z)}{z}\,\mathrm{d}z\,,$$

and the boundary condition $\lim_{r\to\infty} v(r) = 0$ is equivalent to

$$v(0) = \frac{1}{2}\int_0^\infty \frac{S(z)}{z}\,\mathrm{d}z\,. \tag{35}$$

We also have by (6)

$$\sigma = \lim_{r\to\infty} u(r)\,\mathrm{e}^{r^2/4} = 2\lim_{y\to\infty}\phi'(y)\,\mathrm{e}^{y/4}\,. \tag{36}$$

Hence we can reparametrize $v(0)$ and σ in terms of a and τ using (21).

The main statements we are going to prove are summarized in the following theorems. The *a priori* estimates will be established in Section 4. The proofs will be given in Section 5. We shall say that (ϕ, S) is a *positive* solution if both ϕ and S are positive functions.

Theorem 2 *For any $(a,\tau)\in\mathbb{R}_+^2$ there exists a unique positive solution (ϕ, S) of (29)–(30), (33) such that $\phi\in C^2(0,\infty)\cap C^1[0,\infty)$ and $S\in C^1[0,\infty)$. Moreover, for any fixed $\tau>0$, $\phi\in C^2[0,\infty)$, the maps $a\mapsto(\phi,S)$ and $a\mapsto M(a,\tau)$ on $\mathbb{R}_+$ are continuous and*

$$g(a,\tau)\le\frac{M(a,\tau)}{2\,\pi}\le f(a,\tau)\,,$$

where

$$f(a,\tau)=\begin{cases}\min\{4,4\,a\} & \text{if}\quad \tau\in\left(0,\frac{1}{2}\right]\,,\\[2mm] \min\left\{4\,a,\frac{2}{3}\,\pi^2\right\} & \text{if}\quad \tau\in\left(\frac{1}{2},1\right]\,,\\[2mm] \min\left\{4\,a,\frac{2}{3}\,\pi^2\,\tau,4\,(\tau+1)\right\} & \text{if}\quad \tau>1\,,\end{cases} \tag{37}$$

and

$$g(a,\tau)=\begin{cases}\max\left\{4\,a\,\mathrm{e}^{-2\,a\,\frac{\log\tau}{\tau-1}},\frac{4\,a\,\tau}{a+\tau}\right\} & \text{if}\quad \tau\in(0,1]\,,\\[2mm] \max\left\{4\,a\,\mathrm{e}^{-2\,a\,\frac{\log\tau}{\tau-1}},\frac{4\,a}{a+1}\right\} & \text{if}\quad \tau>1\,.\end{cases} \tag{38}$$

For consistency, it is worth noticing that the inequality $g(a,\tau)\le f(a,\tau)$ holds for all $\tau>0$ and $a>0$.

Theorem 3 *Given any fixed $\tau>0$, for any positive sequence $\{a_k\}$ such that $a_k\to\infty$ as $k\to\infty$, there exists a sequence of positive self-similar solutions $(u_k,v_k)\in(C_0^2(\mathbb{R}^2))^2$ satisfying (4)–(5) and $u_k(0)=2\,a_k$, $v_k'(0)=0$ such that*

$$u_k\rightharpoonup 8\,\pi\,\delta_0\quad\text{as}\quad k\to\infty$$

in the sense of weak convergence of measures. Moreover, $\lim_{k\to\infty}\int_{\mathbb{R}^2}u_k\,\mathrm{d}x=8\,\pi$ and $\lim_{k\to\infty}\|v_k\|_{L^\infty(\mathbb{R}^2)}=\infty$.

Theorem 3 has already been proved in [16, Th. 2, (iii)] using a classical result by Brezis and Merle in [6]. However, here we shall give a simplified and quite direct proof using the cumulated densities formulation. This result and the next one are precise versions of Theorem 1.

Theorem 4 *For any fixed $\tau > 0$ there exists $M^* = M^*(\tau) \geq \max\{8\,\pi\,;\,\frac{4\pi}{\mathrm{e}}\,\frac{\tau-1}{\log\tau}\}$ such that problem* (29)–(30) *with the boundary conditions*

$$\phi(0) = 0\,, \quad \lim_{y\to\infty}\phi(y) = \frac{M}{2\,\pi}\,, \quad S(0) = 0\,,$$

has no positive solution $(\phi, S) \in C^2[0,\infty) \times C^1[0,\infty)$ if $M > M^*$ and has at least one positive solution $(\phi, S) \in C^2[0,\infty) \times C^1[0,\infty)$ in the following cases:

(i) $M \in (0, M^*]$ if $M^* > 8\,\pi$,
(ii) $M \in (0, M^*)$ if $M^* = 8\,\pi$.

Moreover, there exist τ^* and τ^{**} with $1/2 < \tau^* \leq \tau^{**}$ such that M^* satisfies: $M^* = 8\,\pi$ if $0 < \tau \leq \tau^*$ and $M^* > 8\,\pi$ if $\tau > \tau^{**}$. Finally, when $M^* > 8\,\pi$, there are at least two positive solutions for any $M \in (8\,\pi, M^*)$.

Remark 3 When $M^* = 8\,\pi$, it is still an open question to decide if there is a positive solution $(\phi, S) \in C^2[0,\infty) \times C^1[0,\infty)$ such that $M = M^*$ or to prove a uniqueness result for any $M \in (0, 8\,\pi)$.

Remark 4 Asymptotically, we have $M^* = \mathcal{O}(\tau)$ by (37). The reader interested in the qualitative behavior of the curve $\tau \mapsto M^*(\tau)$ is invited to look at the plot of Fig. 6.

Remark 5 The estimate $\tau^* > 1/2$ will be given in Proposition 1, as well as refined estimates on $M(a,\tau)$. Theoretical results show that $\tau^* \in (0.5, 16.1109\ldots)$, see Th. 2, (37)–(38) and Sec. 2, while numerical computations suggest that $\tau^* \in (0.62, 0.64)$, see Fig. 2 (right). Moreover, it is an interesting open question to decide whether $\tau^* = \tau^{**}$, as again the numerical results suggest (cf. Fig. 2, Fig. 3, right, Fig. 4, right and Fig. 5), or not. Exact multiplicities of solutions for $M > 8\,\pi$ are not known although computations suggest that it is two for $M < M^*$ and one for $M = M^*$. Let us observe that for $\tau > \tau^*$ the function $M(a,\tau)$ depends on a in a nonmonotone manner. This is a significant difference with the monotone dependence of self-similar solutions of the parabolic-elliptic Keller–Segel system (see [3, Sec. 4]).

4 Qualitative properties of ϕ and S

In the present section we will derive all *a priori* estimates on ϕ and S which are necessary to prove Theorems 2, 3 and 4. Some of them are new while other were already known. In any case, we shall give a unified and simplified proof of all of them in terms of cumulated densities.

4.1 Preliminary estimates

Let $(u,v) \in (C_0^2(\mathbb{R}^2))^2$ be a positive solution of (4)–(5) with $u \in L^1(\mathbb{R}^2)$. The corresponding (ϕ, S) satisfies (29)–(30), (33) with $a = u(0)/2$, see (10)–(11) and (26). Moreover, for any $y > 0$, it immediately holds true that: ϕ is a positive, strictly increasing and concave function on $(0,\infty)$, see (10), (18) and (26), while $0 < S(y) < \phi(y)$ for any $y > 0$ since $S' < \phi'$ on $(0,\infty)$ by (30). More precisely, an integration by parts in (32) gives

$$S(y) = \phi(y) - \frac{\tau}{4}\,\mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{\tau\,z/4}\,\phi(z)\,\mathrm{d}z\,. \qquad (39)$$

On the other hand, in (39), the increasing monotonicity property of ϕ gives us

$$S(y) \geq \phi(y) - \frac{\tau}{4}\, \mathrm{e}^{-\tau\, y/4}\, \phi(y) \int_0^y \mathrm{e}^{\tau\, z/4}\, \mathrm{d}z = \mathrm{e}^{-\tau\, y/4}\, \phi(y)\,, \tag{40}$$

while the decreasing monotonicity property of ϕ' in (32) leads to

$$S(y) \geq \mathrm{e}^{-\tau\, y/4}\, \phi'(y) \int_0^y \mathrm{e}^{\tau\, z/4}\, \mathrm{d}z = \frac{4}{\tau}\, \phi'(y) \left(1 - \mathrm{e}^{-\tau\, y/4}\right) \tag{41}$$

for each $y \geq 0$. From (40) and (41), we get

$$\frac{\tau}{2}\, S(y) \geq \left(\phi(y) - \phi(y)\, \mathrm{e}^{-\tau\, y/4}\right)'.$$

Since $\frac{\tau}{2}\, S = 2\,\phi' - 2\, S'$, the last inequality gives

$$S(y) \leq \frac{1}{2}\, \phi(y) \left(1 + \mathrm{e}^{-\tau\, y/4}\right)$$

for each $y \geq 0$, which is a better estimate than $S < \phi$ but still not yet satisfactory for large y.

Let us now estimate ϕ. Looking closer at system (29)–(30), one observes that the quantity $\mathrm{e}^{\,y/4}\, \phi'(y)$ is positive and decreasing. Hence

$$l(a,\tau) := \lim_{z\to\infty} \mathrm{e}^{z/4}\, \phi'(z) \leq \mathrm{e}^{\,y/4}\, \phi'(y) \leq \phi'(0) = a\,, \tag{42}$$

for any $y \geq 0$. Notice that $l(a,\tau) = \sigma/2$, which proves that $\lim_{s\to-\infty} \sigma(s) = 0$, being $s = \log(2a)$ (see (36) and also Remark 1). Integrating once more the above inequalities on $[0,y]$ we have

$$4\, l(a,\tau) \left(1 - \mathrm{e}^{-y/4}\right) \leq \phi(y) \leq 4\, a \left(1 - \mathrm{e}^{-y/4}\right)\,. \tag{43}$$

In particular, for each $\tau > 0$, $M(a,\tau)$ is finite,

$$l(a,\tau) \leq \frac{M(a,\tau)}{8\,\pi} \leq a\,, \tag{44}$$

and we see that, whatever τ is, the shooting parameter a has to be large enough ($a > 1$) in order to obtain a self-similar solution u with mass $M > 8\,\pi$.

We can improve estimate (43) as follows. Since $\lim_{y\to\infty} \phi'(y) = 0$, integrating the inequality $\phi'' + \frac{1}{4}\, \phi' < 0$ on $[y,\infty)$, we get

$$\phi'(y) + \frac{1}{4}\, \phi(y) \geq \frac{M(a,\tau)}{8\,\pi}\,,$$

and therefore, by integrating once more on $[0,y]$,

$$\phi(y) \geq \frac{M(a,\tau)}{2\,\pi} \left(1 - \mathrm{e}^{-y/4}\right)\,.$$

In conclusion, using the previous estimate for ϕ, we obtain for each $y \geq 0$

$$\frac{M(a,\tau)}{2\,\pi} \left(1 - \mathrm{e}^{-y/4}\right) \leq \phi(y) \leq \min\left\{4\, a \left(1 - \mathrm{e}^{-y/4}\right), \frac{M(a,\tau)}{2\,\pi}\right\}\,, \tag{45}$$

where equality in the minimum is achieved for $\overline{y} = -4\log\left(1-\frac{M}{8\pi a}\right) \in (0,\infty]$. In particular, equalities hold in (45), *i.e.* $\phi(y) = \frac{M}{2\pi}\left(1-\mathrm{e}^{-y/4}\right)$, if and only if $M = 8\pi a$, in which case $\overline{y} = \infty$. But since $\phi(y) = \frac{M}{2\pi}\left(1-\mathrm{e}^{-y/4}\right)$ is not a solution of (29)–(30), estimate (45) holds true with strict inequalities as well as $M < 8\pi a$.

Coming back to the function S, using estimate (42) and identity (32), we have

$$S(y) \le a\,\mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{(\tau-1)\,z/4}\,\mathrm{d}z\,,$$

i.e., for each $y \ge 0$ and $\tau > 0$,

$$S(y) \le a\;y\;h(y;\tau) \tag{46}$$

where

$$h(y;\tau) = \begin{cases} \mathrm{e}^{-y/4} & \text{if } \tau = 1\,, \\ \frac{4}{y\,(\tau-1)}\left(\mathrm{e}^{-y/4}-\mathrm{e}^{-\tau\,y/4}\right) & \text{if } \tau \ne 1\,. \end{cases} \tag{47}$$

As a consequence, it holds true that

$$\lim_{y\to\infty} S(y) = \lim_{y\to\infty}\frac{S(y)}{y} = 0\,,$$

$S(y)/y$ is integrable near $y = 0$ and, using (41), $S'(0) = a$.

The above asymptotic behavior of S at infinity, together with the initial condition $S(0) = 0$, allow us to integrate equation (30) on $[0,\infty)$ to obtain

$$\frac{M(a,\tau)}{2\pi} = \phi(\infty) = \frac{\tau}{4}\int_0^\infty S(y)\,\mathrm{d}y\,. \tag{48}$$

Therefore, any appropriate bound for S would give a bound for the total mass M. However, let us observe that if we plug estimates (46) into (48), we found again the upper bound in (44). Finally, thanks to the integrability of $S(y)/y$ near $y = 0$, equation (29) written as

$$\phi'' + \phi'\left(y/4 + \frac{1}{2}\int_0^y \frac{S(z)}{z}\,\mathrm{d}z\right)' = 0$$

and integrated on $[0,y]$ gives the relation

$$\phi'(y) = a\,\mathrm{e}^{-y/4}\;\exp\left(-\frac{1}{2}\int_0^y \frac{S(z)}{z}\,\mathrm{d}z\right)\,. \tag{49}$$

4.2 Further estimates

First, let us improve on the lower bound in (45) for ϕ. As far as we know, all estimates of this section are new. Using the fact that $S < \phi$ in (29), for $y > 0$ we have

$$\phi'' + \frac{1}{4}\,\phi' + \frac{1}{2\,y}\,\phi'\phi > 0\,.$$

After a multiplication by y, an integration on $[0,y]$ leads to

$$y\,\phi' - \phi + \frac{y}{4}\,\phi + \frac{1}{4}\,\phi^2 > \frac{1}{4}\int_0^y \phi(z)\,\mathrm{d}z\,.$$

Dividing by $\phi^2\,\mathrm{e}^{\,y/4}$ we obtain the differential inequality

$$\left(-\frac{y}{\phi}\,\mathrm{e}^{-y/4}\right)' + \frac{1}{4}\,\mathrm{e}^{-y/4} > \frac{1}{4}\,\frac{1}{\phi^2}\,\mathrm{e}^{-y/4}\int_0^y \phi(z)\,\mathrm{d}z\,.$$

Finally, dropping the positive term on the right hand side, and integrating once again on $[0,y]$ gives us a lower bound for any $\tau>0$ and $a>0$, namely,

$$\phi(y) \geq \frac{y}{\left(1+\frac{1}{a}\right)\,\mathrm{e}^{\,y/4}-1} \tag{50}$$

for each $y\geq 0$. This is, of course, a better estimate than (45) but only for y near the origin since the inequality $S(y)<\phi(y)$ is a good approximation for y near the origin but not for large y. However, we can now replace (45) with

$$\max\left\{\tfrac{M(a,\tau)}{2\,\pi}\left(1-\mathrm{e}^{-y/4}\right),\,\tfrac{y}{\left(1+\frac{1}{a}\right)\mathrm{e}^{\,y/4}-1}\right\} \leq \phi(y) \leq \min\left\{4\,a\left(1-\mathrm{e}^{-y/4}\right),\,\tfrac{M(a,\tau)}{2\,\pi}\right\}. \tag{51}$$

The maximum on the left hand side of (51) is achieved at some $\tilde{y}>0$ and

$$\max\left\{\tfrac{M(a,\tau)}{2\,\pi}\left(1-\mathrm{e}^{-y/4}\right),\,\tfrac{y}{\left(1+\frac{1}{a}\right)\mathrm{e}^{\,y/4}-1}\right\} = \frac{y}{\left(1+\frac{1}{a}\right)\,\mathrm{e}^{\,y/4}-1}$$

for each $y\in[0,\tilde{y}]$. Moreover, for any $y\geq y^*$, we have

$$\frac{M(a,\tau)}{2\,\pi} > \phi(y) \geq \phi(y^*) \geq \frac{y^*}{\left(1+\frac{1}{a}\right)\,\mathrm{e}^{\,y^*/4}-1} = \frac{4}{1+\frac{1}{a}} \;\to 4^- \quad \text{as } a\to\infty\,,$$

if y^* is the point where the maximum of $y\mapsto\frac{y}{\left(1+\frac{1}{a}\right)\mathrm{e}^{\,y/4}-1}$ is achieved.

Next, let us apply estimates (46)–(47) to (49). For $\tau\neq 1$, we have

$$\begin{aligned}\int_0^y h(z;\tau)\,\mathrm{d}z \; &= \frac{4}{\tau-1}\int_0^y \frac{1}{z}\int_\tau^1 \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{e}^{-\frac{t}{4}z})\,\mathrm{d}t\,\mathrm{d}z = \frac{1}{\tau-1}\int_0^y\int_1^\tau \mathrm{e}^{-\frac{t}{4}z}\,\mathrm{d}t\,\mathrm{d}z\\ &= \frac{4}{\tau-1}\int_1^\tau \frac{1}{t}\left(1-\mathrm{e}^{-t\,y/4}\right)\,\mathrm{d}t = \frac{4}{\tau-1}\log\tau - \frac{4}{\tau-1}\int_1^\tau \frac{1}{t}\,\mathrm{e}^{-t\,y/4}\,\mathrm{d}t\,,\end{aligned}$$

$$\int_0^y \frac{S(z)}{z}\,\mathrm{d}z \leq 4\,a\,I(\tau)$$

with $I(\tau)=\frac{\log\tau}{\tau-1}$ and

$$\phi'(y) \geq a\,\mathrm{e}^{-y/4}\mathrm{e}^{-2\,a\,I(\tau)}\,. \tag{52}$$

Integrating (52) on $[0,\infty)$, we get the same estimate as in (23) giving arbitrarily large mass M for τ large enough, *i.e.*

$$\frac{M(a,\tau)}{2\,\pi} \geq 4\,a\,\mathrm{e}^{-2\,a\,I(\tau)}\,. \tag{53}$$

For $\tau=1$, since $h(y;1)=\mathrm{e}^{-y/4}$, one obtains, for all $a>0$,

$$\frac{M(a,1)}{2\,\pi} \geq 2\left(1-\mathrm{e}^{-2\,a}\right)\,.$$

Remark 6 The lower bound (53) is compatible with the upper bounds for M known from [2], *i.e.*

$$\frac{M}{2\pi} \leq 4 \text{ if } \tau \in (0,1/2]\,, \quad \frac{M}{2\pi} \leq \frac{2}{3}\pi^2 \text{ if } \tau \in (1/2,1]$$
$$\text{and} \quad \frac{M}{2\pi} \leq \min\left\{\tfrac{2}{3}\pi^2\tau, 4(\tau+1)\right\} \text{ if } \tau > 1\,.$$

Finally, following [16], define the new function

$$W(y) := \int_0^y \phi'(z)\,\mathrm{e}^{\tau z/4}\,\mathrm{d}z = \mathrm{e}^{\tau y/4}\,S(y)\,,$$

where the second equality follows from (32). After a multiplication of (31) by $\mathrm{e}^{\tau y/4}$, it is easy to see that W satisfies the initial value problem

$$W'' + \frac{1-\tau}{4}\,W' + \frac{1}{4\,y}\left(W^2\right)'\mathrm{e}^{-\tau y/4} = 0\,,$$
$$W(0) = 0\,, \quad W'(0) = a\,.$$

Next, a multiplication by y and an integration on $[0,y]$ gives us

$$y\,W' - W + \frac{1-\tau}{4}\int_0^y z\,W'(z)\,\mathrm{d}z + \frac{1}{4}\,\mathrm{e}^{-\tau y/4}\,W^2 + \frac{\tau}{16}\int_0^y \mathrm{e}^{-\tau z/4}\,W^2(z)\,\mathrm{d}z = 0\,.$$

Dividing by W^2 the equation becomes

$$\left(-\frac{y}{W}\right)' + \frac{1-\tau}{4}\,\frac{1}{W^2}\int_0^y z\,W'(z)\,\mathrm{d}z + \frac{1}{4}\,\mathrm{e}^{-\tau y/4} + \frac{\tau}{16}\,\frac{1}{W^2}\int_0^y \mathrm{e}^{-\tau z/4}\,W^2(z)\,\mathrm{d}z = 0\,. \tag{54}$$

This last identity is a useful reformulation of the problem for $0 < \tau \leq 1$, since in this case the two integral terms in the equation are positive. Then, eliminating both of them and integrating on $[0,y]$, we get for each $y \geq 0$

$$\frac{y}{W(y)} \geq \frac{1}{a} + \frac{1}{\tau}\left(1 - \mathrm{e}^{-\tau y/4}\right),$$

i.e.

$$S(y) \leq \frac{\tau\,a\,y}{\tau\,\mathrm{e}^{\tau y/4} + a\left(\mathrm{e}^{\tau y/4} - 1\right)}\,. \tag{55}$$

For $\tau > 1$ it is more convenient to integrate by parts the first integral term in (54) to obtain

$$\left(-\frac{y}{W}\right)' + \frac{1-\tau}{4}\,\frac{y}{W} - \frac{1-\tau}{4}\,\frac{1}{W^2}\int_0^y W(z)\,\mathrm{d}z + \frac{1}{4}\,\mathrm{e}^{-\tau y/4}$$
$$+ \frac{\tau}{16}\,\frac{1}{W^2}\int_0^y \mathrm{e}^{-\tau z/4}\,W^2(z)\,\mathrm{d}z = 0\,.$$

Again, eliminating the two positive integral terms and multiplying by $\mathrm{e}^{(\tau-1)\,y/4}$, we obtain

$$\left(\mathrm{e}^{(\tau-1)\,y/4}\,\frac{y}{W}\right)' \geq \frac{1}{4}\,\mathrm{e}^{-y/4}.$$

After an integration on $[0,y]$, this gives

$$\mathrm{e}^{(\tau-1)\,y/4}\,\frac{y}{W} \geq \frac{1}{a} + 1 - \mathrm{e}^{-y/4}\,,$$

i.e.

$$S(y) \leq \frac{a\,y}{\mathrm{e}^{\,y/4} + a\left(\mathrm{e}^{\,y/4} - 1\right)}\,. \tag{56}$$

Summarizing, estimates (55) and (56) read

$$S(y) \leq a\;y\;g(y;a,\tau)\,, \tag{57}$$

for all $y \geq 0$, $a > 0$, $\tau > 0$, where

$$g(y;a,\tau) := \begin{cases} \dfrac{\tau}{(\tau+a)\,\mathrm{e}^{\tau\,y/4} - a} & \text{if } 0 < \tau \leq 1\,, \\[2ex] \dfrac{1}{(1+a)\,\mathrm{e}^{\,y/4} - a} & \text{if } \tau > 1\,. \end{cases} \tag{58}$$

As an important consequence of (57)–(58), for any $\tau > 0$, S is bounded uniformly with respect to $a > 0$:

$$S(y) \leq \frac{\min\{\tau,1\}\,y}{\mathrm{e}^{\min\{\tau,1\}\,y/4} - 1} \tag{59}$$

for each $y > 0$. Such an estimate does not follow from (46)–(47).

Estimate (57) is better than estimate (46) for $\tau = 1$. For $\tau \neq 1$, this depends on the values of τ and a. Therefore, it is interesting to reproduce the computations giving (53) by using the function g instead of h. For $\tau \geq 1$ and each $y \geq 0$, we obtain

$$\int_0^y g(z;a,\tau)\,\mathrm{d}z = \frac{4}{a}\,\log\left[(1+a)\,\mathrm{e}^{\,y/4} - a\right] - \frac{y}{a}\,,$$

and from equation (49)

$$\phi'(y) \geq a\,\frac{\mathrm{e}^{\,y/4}}{\left[(1+a)\,\mathrm{e}^{\,y/4} - a\right]^2}\,. \tag{60}$$

This gives, for $a > 0$ and $\tau \geq 1$,

$$\frac{M(a,\tau)}{2\,\pi} \geq \frac{4\,a}{a+1}\,. \tag{61}$$

Such a lower bound is definitely worse than (53) for large values of τ or, to be precise, as soon as $I(\tau) \leq \log(a+1)/(2\,a)$. On the other hand, for $\tau < 1$, we have

$$\int_0^y g(z;a,\tau)\,\mathrm{d}z = \frac{4}{a}\log\left[\left(\frac{a}{\tau}+1\right)\mathrm{e}^{\tau\,y/4} - \frac{a}{\tau}\right] - \frac{\tau}{a}\,y\,,$$

and again from equation (49),

$$\phi'(y) \geq a\,\mathrm{e}^{-y/4}\,\mathrm{e}^{\tau\frac{y}{2}}\,\frac{1}{\left[\left(\frac{a}{\tau}+1\right)\mathrm{e}^{\tau\,y/4} - \frac{a}{\tau}\right]^2}\,.$$

Finally, it holds true that, for $a > 0$ and $\tau < 1$,

$$\frac{M(a,\tau)}{2\,\pi} \geq \frac{4\,a\,\tau}{a+\tau}\,. \tag{62}$$

To conclude, integrating (60) on $[0,y]$ and using estimate (57) gives us, for any $\tau \geq 1$,

$$S(y) \leq \frac{y}{(1+\frac{1}{a})\,\mathrm{e}^{y/4}-1} \leq \frac{4\,(\mathrm{e}^{y/4}-1)}{(1+\frac{1}{a})\,\mathrm{e}^{y/4}-1} \leq \phi(y)$$

for each $y \geq 0$, which is a good approximation of S and ϕ near the origin since it takes into account the condition $S'(0) = \phi'(0) = a$. Moreover, (51) is improved and replaced with

$$\max\left\{\tfrac{M(a,\tau)}{2\,\pi}\left(1-\mathrm{e}^{-y/4}\right), \tfrac{4\,(\mathrm{e}^{y/4}-1)}{(1+\frac{1}{a})\mathrm{e}^{y/4}-1}\right\} \leq \phi(y) \leq \min\left\{4\,a\left(1-\mathrm{e}^{-y/4}\right), \tfrac{M(a,\tau)}{2\,\pi}\right\}$$

for any $\tau \geq 1$ and $y \geq 0$.

Remark 7 As an additional consequence of the above estimates, we observe that

$$a\int_0^y g(z;a,\tau)\,\mathrm{d}z = 4\,\log\left(1+\tfrac{a}{\min\{1,\tau\}} - \tfrac{a}{\min\{1,\tau\}}e^{-\min\{1,\tau\}\,y/4}\right)$$

converges as $y \to \infty$, so that

$$\exp\left[-\frac{a}{2}\int_0^\infty g(z;a,\tau)\,\mathrm{d}z\right] = \left(1+\tfrac{a}{\min\{1,\tau\}}\right)^{-2}.$$

According to (42), (49) and (57), we find the estimate

$$\tfrac{\sigma}{2} = l(a,\tau) = \lim_{y\to+\infty} a\,\exp\left(-\frac{1}{2}\int_0^y \frac{S(z)}{z}\,\mathrm{d}z\right) \geq a\left(1+\tfrac{a}{\min\{1,\tau\}}\right)^{-2},$$

which, taking into account the change of parametrization $s = \log(2\,a)$, refines the estimate $\lim_{s\to+\infty}\sigma(s) = 0$ found in [16] and our estimate (25) (also see Remark 1).

4.3 New upper bounds

Using the previous estimates on S and an argument in [2], we can improve on the upper bound in (44). Let

$$j(\tau) := \begin{cases} \infty & \text{if } 0 < \tau \leq \frac{1}{2}\,, \\ \tau\frac{\mathrm{e}^{1-\frac{1}{2\tau}}}{2\,\tau-\mathrm{e}^{1-\frac{1}{2\tau}}} & \text{if } \frac{1}{2} < \tau \leq 1\,, \\ \frac{\mathrm{e}^{1-\frac{1}{2\tau}}}{2\,\tau-\mathrm{e}^{1-\frac{1}{2\tau}}} & \text{if } \tau > 1\,. \end{cases}$$

Proposition 1 *For any $\tau > 0$, if $a \leq \max\{j(\tau),1\}$, then $M(a,\tau) \leq 8\,\pi\min\{1,a\}$.*

The above estimate gives us a nonoptimal set of parameters (a,τ) that guarantees $M(a,\tau) \leq 8\,\pi$. It is interesting to notice that $\lim_{\tau\to(1/2)^+} j(\tau) = \infty$.

Proof Let $M = M(a,\tau)$. From the identity

$$\left(\frac{M}{2\,\pi}\right)^2 - 4\left(\frac{M}{2\,\pi}\right) = \int_0^\infty \left(2\,\phi(y)\,\phi'(y) + 4\,y\,\phi''(y)\right)\,\mathrm{d}y\,,$$

and $4\,y\,\phi'' = -y\,\phi' - 2\,\phi' S$ which follows from (29), we have, after an integration by parts and using (30),

$$\left(\frac{M}{2\,\pi}\right)^2 - 4\left(\frac{M}{2\,\pi}\right) = \int_0^\infty \phi'(2\,\phi - 2\,S - y)\,\mathrm{d}y = \int_0^\infty \left(\phi - \frac{M}{2\,\pi}\right)'(2\,\phi - 2\,S - y)\,\mathrm{d}y$$
$$= -\int_0^\infty \left(\phi - \frac{M}{2\,\pi}\right)(2\,\phi' - 2\,S' - 1)\,\mathrm{d}y = -\int_0^\infty \left(\phi - \frac{M}{2\,\pi}\right)\left(\frac{\tau}{2}\,S - 1\right)\,\mathrm{d}y\,.$$

Hence we have $\frac{M}{2\,\pi} \le 4$ if

$$\frac{\tau}{2}\,S(y) \le 1 \tag{63}$$

for each $y > 0$. From (59) it follows that $S(y) < 4$ for all $y \ge 0$, for any $\tau > 0$ and $a > 0$. Therefore, the above sufficient condition (63) is satisfied whenever $\tau \le 1/2$. For $\tau > 1/2$ we have to use one of the previous upper bounds for S.

(a) Using (46), we have

$$1 - \frac{\tau}{2}\,S(y) \ge 1 - 2\,a\,\frac{\tau}{\tau - 1}\left(\mathrm{e}^{-y/4} - \mathrm{e}^{-\tau\,y/4}\right)$$

for any $\tau \ne 1$ and each $y \ge 0$, and condition (63) is satisfied if

$$a \le \min_{y>0} \frac{1}{2}\,\frac{\tau - 1}{\tau}\,\frac{1}{\mathrm{e}^{-y/4} - \mathrm{e}^{-\tau\,y/4}} = \frac{1}{2}\,\tau^{\frac{1}{\tau-1}}\,.$$

For $\tau = 1$, using (46) as before (or by continuity of the previous argument as $\tau \to 1$), we similarly obtain

$$a \le \min_{y>0} \frac{2\,\mathrm{e}^{\,y/4}}{y} = \frac{\mathrm{e}}{2}\,.$$

(b) Using (57), we have for $\tau > 1$ and each $y \ge 0$

$$\frac{\tau}{2}\,S(y) - 1 \le \frac{1}{2}\,\frac{\tau\,a\,y}{(1+a)\,\mathrm{e}^{\,y/4} - a} - 1\,,$$

Then condition (63) is satisfied if

$$a \le \min_{0<y<\bar{y}} \frac{2\,\mathrm{e}^{\,y/4}}{\tau\,y - 2\,(\mathrm{e}^{\,y/4} - 1)} = \frac{\mathrm{e}^{1-\frac{1}{2\,\tau}}}{2\,\tau - \mathrm{e}^{1-\frac{1}{2\,\tau}}}\,,$$

where we take into account that $\tau\,y - 2\,(\mathrm{e}^{\,y/4} - 1) < 0$ for $y > \bar{y}$, $\bar{y}$ being the unique solution of the equation $\frac{\tau}{2}\,y + 1 = \mathrm{e}^{\,y/4}$. Similarly, for $\frac{1}{2} < \tau \le 1$ and each $y \ge 0$, we get

$$\frac{\tau}{2}\,S(y) - 1 \le \frac{\tau}{2}\,\frac{\tau\,a\,y}{(\tau + a)\,\mathrm{e}^{\frac{\tau}{4}\,y} - a} - 1\,.$$

Then condition (63) is satisfied if

$$a \le \min_{0<\tau\,y<\bar{y}} \frac{2\,\tau\,\mathrm{e}^{\frac{\tau}{4}\,y}}{\tau^2\,y - 2\left(\mathrm{e}^{\frac{\tau}{4}\,y} - 1\right)} = \tau\,\frac{\mathrm{e}^{1-\frac{1}{2\,\tau}}}{2\,\tau - \mathrm{e}^{1-\frac{1}{2\,\tau}}}\,.$$

Comparing the results obtained in (a) and (b), the proof of Proposition 1 is completed. □

5 Proofs

This section is devoted to the proof of Theorems 2, 3 and 4. As a byproduct of these results, we obtain Theorem 1.

5.1 Proof of Theorem 2

Given any fixed $(a,\tau)\in\mathbb{R}_+^2$, the local existence issue of the (singular) system (29)–(30) with initial conditions (33) can be solved using a fixed point argument applied to the operator

$$\mathcal{T}[\Phi](y) = a\,\mathrm{e}^{-y/4} - \frac{1}{2}\,\mathrm{e}^{-y/4}\int_0^y \frac{1}{z}\,\mathrm{e}^{(1-\tau)\,z/4}\,\Phi(z)\int_0^z \mathrm{e}^{\tau\,\xi/4}\,\Phi(\xi)\;\mathrm{d}\xi\;\;\mathrm{d}z,$$

defined on the complete metric space $X_a := \{\Phi\in C[0,y_a]\;:\;\Phi(0)=a, 0\le\Phi(y)\le a, 0\le y\le y_a\}$ endowed with the usual supremum norm. Indeed, an appropriate choice of y_a gives that $\mathcal{T}$ maps X_a into X_a and that $\mathcal{T}$ is a contraction. If $\mathcal{T}[\Phi]=\Phi$, it is then enough to define $\phi(y) := \int_0^y \Phi(z)\;\mathrm{d}z$ and $S(y) := \mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{\tau\,z/4}\,\Phi(z)\,\mathrm{d}z$ in order that (ϕ,S) is a solution of (29)–(30), (33) with $\phi\in C^1[0,y_a]\cap C^2(0,y_a]$ and $S\in C^1[0,y_a]$. The continuation of the local solution to a global one is standard since system (29)–(30) is no more singular away from the origin, and both ϕ, S are locally bounded on $\mathbb{R}_+$ by (51) and (59).

The fact that $\phi\in C^2[0,\infty)$ follows from (49) and $\lim_{y\to 0_+} S(y)/y = a = S'(0)$. Estimates (37) and (38) have been proved in Section 4, (see (53), (61) and (62) for the lower bound, and (44) and Remark 6 for the upper bound).

Finally, uniqueness of global solutions of (29)–(30), (33) is a consequence of the contraction property of $\mathcal{T}$ and the Cauchy–Lipschitz theorem.

Concerning the continuity of the map $a\in\mathbb{R}_+\mapsto(\phi,S)$, let us denote by (ϕ_i,S_i) the solution associated with the shooting parameter a_i, $i=1,2$. Following [13] we have

$$|\log\phi_1'(y)-\log\phi_2'(y)| \le |\log a_1-\log a_2| + \frac{1}{2}\int_0^y \frac{1}{z}\,|S_1(z)-S_2(z)|\,\mathrm{d}z \tag{64}$$

and

$$\begin{aligned}|S_1(y)-S_2(y)| &\le \mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{(\tau-1)\,z/4}\left|\mathrm{e}^{z/4}\phi_1'(z)-\mathrm{e}^{z/4}\phi_2'(z)\right|\,\mathrm{d}z\\ &\le \mathrm{e}^{\max\{\log a_1,\log a_2\}}\,\mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{(\tau-1)\,z/4}\,|\log\phi_1'(z)-\log\phi_2'(z)|\,\mathrm{d}z\,,\end{aligned} \tag{65}$$

where the decreasing monotonicity property of the function $\mathrm{e}^{\,y/4}\,\phi'(y)$ has been used in the last inequality. Plugging (65) into (64) and denoting $C=\mathrm{e}^{\max\{\log a_1,\log a_2\}}$, we obtain

$$\begin{aligned}&|\log\phi_1'(y)-\log\phi_2'(y)|\\ &\le |\log a_1-\log a_2| + \frac{C}{2}\int_0^y \frac{1}{z}\,\mathrm{e}^{-\tau\,z/4}\int_0^z \mathrm{e}^{(\tau-1)\,\zeta/4}|\log\phi_1'(\zeta)-\log\phi_2'(\zeta)|\,\mathrm{d}\zeta\,\mathrm{d}z\\ &\le |\log a_1-\log a_2| + \frac{C}{2}\int_0^y |\log\phi_1'(\zeta)-\log\phi_2'(\zeta)|\,f(\zeta)\,\mathrm{d}\zeta\,,\end{aligned} \tag{66}$$

where $f(\zeta) = \mathrm{e}^{(\tau-1)\,\zeta/4}\int_\zeta^\infty \frac{1}{z}\,\mathrm{e}^{-\tau\,z/4}\,\mathrm{d}z$. Next, $f \in L^1(0,\infty)$ with $\int_0^\infty f(\zeta)\,\mathrm{d}\zeta = 4\,\frac{\log\tau}{\tau-1} = 4\,I(\tau)$. Therefore, the Gronwall lemma applied to (66) gives us

$$|\log\phi_1'(y)-\log\phi_2'(y)| \le |\log a_1-\log a_2|\,\mathrm{e}^{\frac{C}{2}\int_0^y f(\zeta)\,\mathrm{d}\zeta} \le |\log a_1-\log a_2|\,\mathrm{e}^{2\,C\,I(\tau)}. \quad (67)$$

Estimate (67) implies the continuity of the map $a \mapsto \phi'$. The continuity of the maps $a \mapsto S$ and $a \mapsto \phi$ follows by (65)–(67) and by the identity $\phi(y) = S(y) + \frac{\tau}{4}\int_0^y S(z)\,\mathrm{d}z$ respectively. Finally, the continuity of $a \mapsto M$ follows by (48). □

5.2 Proof of Theorem 3

The existence of a sequence of positive self-similar solutions (u_k, v_k) corresponding to a positive sequence $\{a_k\}$ is an immediate consequence of the existence of a positive solution (ϕ_k, S_k) of (29)–(30), (33) by Theorem 2. Indeed, it is sufficient to define

$$u_k(r) = 2\,\phi_k'(r^2) \quad \text{and} \quad v_k(r) = \frac{1}{2}\int_{r^2}^\infty \frac{S_k(z)}{z}\,\mathrm{d}z\,,$$

as follows from (26) and (35). Moreover, $u_k \in C^1[0,\infty)$ and $v_k \in C^2[0,\infty)$. Whenever $a_k \to \infty$, the limit $\|v_k\|_{L^\infty(\mathbb{R}^2)} \to \infty$ follows from $\|v_k\|_{L^\infty(\mathbb{R}^2)} = v_k(0)$ and (24).

Let us prove that $\lim_{k\to\infty} u_k(r) = 0$ for any $r > 0$. By (40) and (50), we know that

$$\frac{S_k(z)}{z} \ge \frac{\mathrm{e}^{-\tau\,z/4}}{\left(1+\frac{1}{a_k}\right)\,\mathrm{e}^{\,z/4}-1}\,.$$

According to (49), we have

$$\begin{aligned} 0 \le \phi_k'(y) &\le a_k\,\mathrm{e}^{-y/4}\,\exp\left[-\frac{1}{2}\int_0^y \frac{\mathrm{e}^{-\tau\,z/4}}{\left(1+\frac{1}{a_k}\right)\,\mathrm{e}^{\,z/4}-1}\,\mathrm{d}z\right] \\ &\le a_k\,\mathrm{e}^{-y/4}\,\exp\left[-\frac{1}{2}\,\mathrm{e}^{-\tau\,y/4}\underbrace{\int_0^y \frac{1}{\left(1+\frac{1}{a_k}\right)\,\mathrm{e}^{\,z/4}-1}\,\mathrm{d}z}_{=4\,\log\left(a_k\,(\mathrm{e}^{y/4}-1)+\mathrm{e}^{y/4}\right)-y}\right]. \end{aligned}$$

For any $y > 0$, this proves that

$$0 \le \phi_k'(y) \le a_k^{1-2\,\mathrm{e}^{-\tau\,y/4}}\,\mathrm{e}^{-y/4}\,\left(\mathrm{e}^{y/4}-1-y/4\right)^{-2\,\mathrm{e}^{-\tau\,y/4}}\,\left(1+\mathcal{O}\left(\tfrac{1}{a_k}\right)\right)$$

as $k \to \infty$, thus showing that $\lim_{k\to\infty}\phi_k'(y) = 0$ for any $y \in (0, 4\,\tau^{-1}\,\log 2)$. Recalling that $u_k(r) = 2\,\phi_k'(r^2)$ and using the fact that u_k is positive and monotone decreasing, this proves that $\lim_{k\to\infty} u_k(r) = 0$ for any $r > 0$.

Next, let us define $M_k := \|u_k\|_{L^1(\mathbb{R}^2)}$. By Theorem 2 and (37), the sequence $\{M_k\}$ is bounded from above by a constant depending only on τ. Hence, there exist two subsequences, still denoted M_k and u_k, such that $M_k \to \alpha$ and $u_k \rightharpoonup \alpha\,\delta_0$ as $k \to \infty$, the delta measure being centered at the origin since u_k is radially symmetric decreasing (and $u_k(0) = 2\,a_k \to \infty$). Actually $\alpha = 8\,\pi$ for any $\tau > 0$, as an immediate consequence of the identity obtained in the proof of Proposition 1, i.e.

$$\left(\frac{M_k}{2\,\pi}\right)^2 - 4\left(\frac{M_k}{2\,\pi}\right) = \int_0^\infty \phi_k'(2\,\phi_k - 2\,S_k - y)\,\mathrm{d}y\,.$$

Hence, we also have

$$\left(\frac{M_k}{2\,\pi}\right)^2 - 4\left(\frac{M_k}{2\,\pi}\right) = \frac{1}{\pi}\int_{\mathbb{R}^2} u_k(\xi)\left(\phi_k(|\xi|^2) - S_k(|\xi|^2) - \tfrac{1}{2}|\xi|^2\right)\mathrm{d}\xi\,. \tag{68}$$

Letting $k \to \infty$ and observing that:

(i) $\phi_k(y) - S_k(y) \le M_k \to \alpha$ is bounded uniformly with respect to $a_k \to \infty$,
(ii) $u_k(r) = 2\,\phi_k'(r^2)$ is exponentially decaying, uniformly with respect to $a_k \to \infty$, for large values of r, as a consequence of (41) and (59),
(iii) $\lim_{r\to\infty}\sup_k \int_{|\xi|>r} u_k(\xi)\,|\xi|^2\,\mathrm{d}\xi = 0$, again as a consequence of (41) and (59),

we obtain that the right hand side in (68) converges to 0. On the other hand, α is necessarily positive by (61) and (62), which proves that $\alpha = 8\,\pi$. □

Remark 8 Let us observe that the identity

$$4\,M + 2\int_{\mathbb{R}^2} u(\xi)\,\nabla v(\xi)\cdot\xi\,\mathrm{d}\xi - \int_{\mathbb{R}^2}|\xi|^2\,u(\xi)\,\mathrm{d}\xi = 0$$

follows from equation (4) multiplied by $|\xi|^2$ and from the integrability of u given by (6) and (8). Mimicking a standard computation for the parabolic-elliptic Keller–Segel system by writing $v = -\frac{1}{2\,\pi}\log(\cdot) * u + \tilde{v}$, the above identity reads

$$4\,M - \frac{M^2}{2\,\pi} + 2\int_{\mathbb{R}^2} u(\xi)\,\nabla\tilde{v}(\xi)\cdot\xi\,\mathrm{d}\xi - \int_{\mathbb{R}^2}|\xi|^2 u(\xi)\,\mathrm{d}\xi = 0\,.$$

See for instance [5,9] for more details. Therefore, we have found that $\nabla\tilde{v}(\xi)\cdot\xi = \phi(|\xi|^2) - S(|\xi|^2) \ge 0$. This is consistent with the fact that, from equation (11), one easily finds that $\phi(r^2) - S(r^2) = -\frac{\tau}{2}\int_0^r s^2\,v'(s)\,\mathrm{d}s$.

5.3 Proof of Theorem 4

For any fixed τ, let us define $M^*(\tau) = \sup_{a>0} M(a,\tau)$. Since M is bounded from above by a constant depending on τ, uniformly in a, continuous with respect to a, such that $M(0,\tau) = 0$ and $\lim_{a\to\infty} M(a,\tau) \to 8\,\pi$, $M^*(\tau)$ is well defined and finite. The theorem is then a straightforward consequence of Theorem 2, Theorem 3 and Proposition 1. □

6 Numerical results

In this section, we numerically illustrate the above results. In particular, we show the existence of positive forward self-similar solutions with mass above $8\,\pi$ and their multiplicity when τ is large enough. We follow two different approaches: first the formulation (15)–(16), and then the cumulated densities formulation based on (29)–(30).

6.1 Bifurcation diagrams

The computations giving rise to Figs. 1 and 2 are based on the parametrization provided by (15)–(16). Numerically, one has to be careful with the origin and solve (15) on the interval (ε,∞) with the initial conditions

$$w(\varepsilon\,;s)=s-\frac{1}{4}\,\varepsilon^2\,\mathrm{e}^s \quad \text{and} \quad w'(\varepsilon\,;s)=-\frac{1}{2}\,\varepsilon\,\mathrm{e}^s\,,$$

obtained by the Taylor expansion at $\varepsilon>0$, small enough, thus dropping higher order terms in ε. Observe that by (15) $w''(0\,;s)=-\mathrm{e}^s/2$. In case of Fig. 2, one has to compute $M(s)$, which is given by (17), by solving $M'(r)=2\,\pi\,\mathrm{e}^{w(r;s)}\,\mathrm{e}^{-r^2/4}\,r$ with the approximate initial condition $M(\varepsilon)=\pi\,\varepsilon^2\,\mathrm{e}^s$.

In Fig. 2, we recover that $M(s)\to 8\,\pi$ as $s\to+\infty$. Moreover, for τ large enough, there are two solutions corresponding to a given M larger than $8\,\pi$, with $M-8\,\pi$ not too large. Since it is of interest to decide for which values of τ solutions may have mass larger than $8\,\pi$, the small rectangle in Fig. 2 (left) is enlarged in Fig. 2 (right).

It can be numerically checked that solving the equations on $(\varepsilon, r_{\max})$ with $r_{\max}=10$ gives a good approximation of the solution. Furthermore, here we took $\varepsilon=10^{-8}$ and $s\in[-10,20]$.

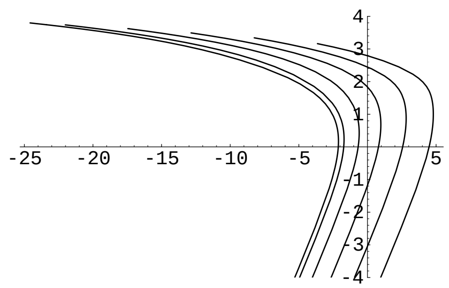

Fig. 1 The set of all positive solutions of $\Delta v_\sigma+\frac{\tau}{2}\,\xi\cdot\nabla v_\sigma+\sigma\,\mathrm{e}^{v_\sigma}\,\mathrm{e}^{-|\xi|^2/4}=0$ in $C^2_0(\mathbb{R}^2)$, where $\sigma=\sigma(s)=\mathrm{e}^{w(\infty;s)}$, is parametrized by $s\mapsto(\log\sigma,\log v_\sigma(0))$ for $\tau=10^\alpha$, $\alpha=-2$, -1, ..., 3. Recall that the solutions v_σ are radial and decreasing so that $v_\sigma(0)=\|v_\sigma\|_{L^\infty(\mathbb{R}^2)}$. We observe that $\max_{s\in\mathbb{R}}\log\sigma(s)$ appears as an increasing function of τ.

6.2 Cumulated densities

Plots and bifurcation diagrams of forward self-similar solutions can be computed in the framework of cumulated densities (29)–(30), (33). However, again one has to be careful with the singularity at the origin. As above, since for $\varepsilon>0$ small enough, $S'\sim\phi'\sim a$ on $(0,\varepsilon)$ and so

$$S(y)=a\,y+\mathcal{O}(\varepsilon^2) \quad \text{and} \quad \phi''(y)\sim-\frac{a}{4}\,(1+2\,a)+\mathcal{O}(\varepsilon)\,,$$

we practically solve (29)–(30) on $(\varepsilon, y_{\max})$ with the initial data

$$\phi'(\varepsilon)=a-\frac{a}{4}\,(1+2\,a)\,\varepsilon\,,\quad \phi(\varepsilon)=a\,\varepsilon-\frac{a}{8}\,(1+2\,a)\,\varepsilon^2 \quad \text{and} \quad S(\varepsilon)=a\,\varepsilon$$

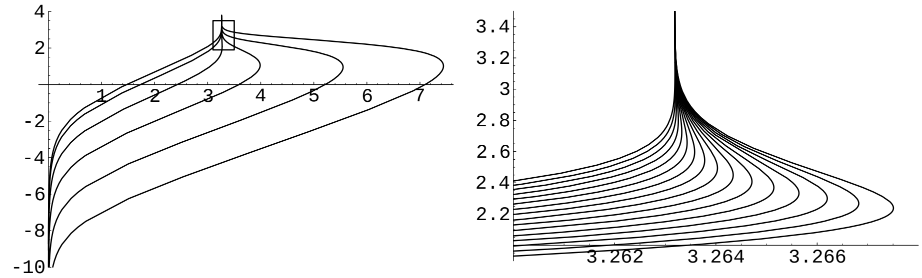

Fig. 2 *Left:* The set of all positive solutions of $\Delta v_\sigma + \frac{\tau}{2}\,\xi \cdot \nabla v_\sigma + \sigma\, \mathrm{e}^{v_\sigma}\, \mathrm{e}^{-|\xi|^2/4} = 0$ in $C^2_0(\mathbb{R}^2)$ is now parametrized by $s \mapsto (\log(1 + M(s)), \log v_\sigma(0))$ for $\tau = 10^\alpha$, $\alpha = -2, -1, \ldots, 3$. We observe that $\max_{s\in\mathbb{R}} M(s)$ appears as an increasing function of τ.
Right: The plot is an enlargement of the rectangle of Fig. 2 (left), with $\tau = 0.60$, 0.62, 0.64, $\ldots$, 0.90. Numerically, the first solution with mass larger than $8\,\pi$ appears for $\tau \in (0.62, 0.64)$, which is far below the bound found in Section 2. This is not easy to read on the above figure, but it can be shown graphically by enlarging it enough.

for any $y \in (0, \varepsilon)$. Obviously, having fixed $\varepsilon > 0$, one has to take a in such a way that $\phi'(\varepsilon) - a = o(a)$. Here, we choose $\varepsilon = 10^{-6}$. Finally, we shall approximate M from below by $\phi(y_{\max})$ with $y_{\max}$ large enough. Figs. 3 and 4 correspond to the cases $\tau = 0.1$ and $\tau = 10$ respectively. For $\tau = 0.1$, the value $8\,\pi$ for the total mass is achieved only asymptotically in the limit $a \to \infty$. For $\tau = 10$, self-similar solutions with mass M larger than $8\,\pi$ exist for a large enough. Finally, Figs. 5 and 6 show the total mass as a function of a and τ.

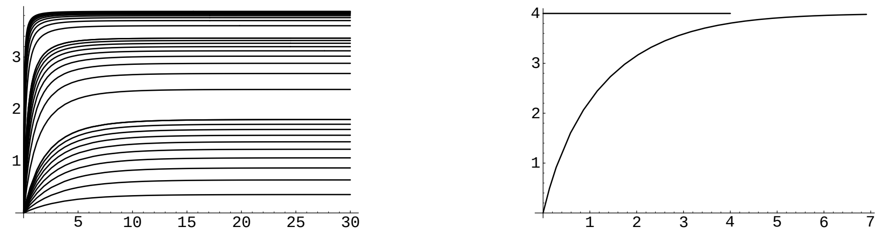

Fig. 3 *Left*: Plots of ϕ for $\phi'(0) = a$, with $a = 10^b\, c$, $b = -1$, 0, 1, $c \in \{1, \ldots, 10\}$ for $\tau = 0.1$. *Right*: Plot of $b \mapsto \phi(y_{\max})$ in the logarithmic scale, with $\phi'(0) = a$, $a = \mathrm{e}^b - 1$, $y_{\max} = 30$.

7 Conclusions

Self-similar solutions are much more than an example of a family of solutions. The experience of various nonlinear diffusion equations shows that they are likely to be attracting a whole class of solutions, although this is still an open question for the parabolic-parabolic Keller–Segel model with large mass (see [15] for a result for small mass solutions). It is quite reasonable to expect that well chosen perturbations of these solutions asymptotically converge in self-similar variables to the stationary solutions

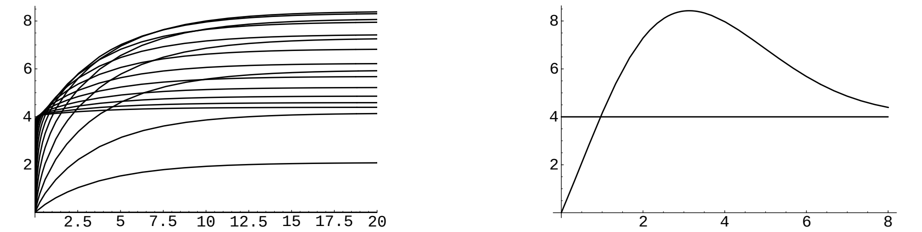

Fig. 4 *Left*: Plots of ϕ for $\phi'(0) = \mathrm{e}^{\alpha}$, with $\alpha = 1, 2, \ldots, 20$ for $\tau = 10$. *Right*: Plot of $\phi(y_{\max})$ as a function of b (in the logarithmic scale), with $\phi'(0) = a$, $a = \mathrm{e}^{b} - 1$. Here $\tau = 10$, $y_{\max} = 30$.

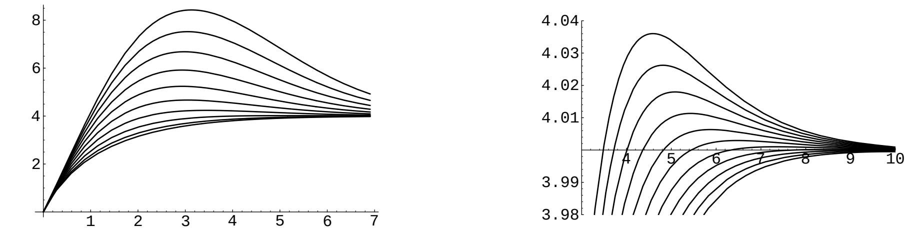

Fig. 5 *Left*: The value of mass $\phi(\infty) = M(a,\tau)/(2\,\pi)$ in the logarithmic scale as a function of a, for $\tau = 0.1\,k^2$ with $k = 1, 2, \ldots, 10$. *Right*: An enlargement around the value $M(a,\tau)/(2\,\pi) = 4$ in the logarithmic scale as a function of a, for $\tau = 0.50, 0.55, 0.60, \ldots, 1.00$.

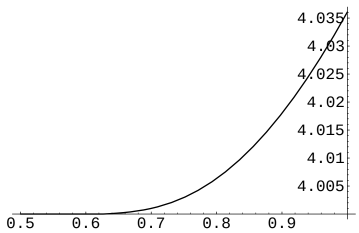

Fig. 6 The value of the maximal (in terms of a) mass $\phi(\infty) = M_*(\tau)/(2\,\pi)$ as a function of τ. Numerically, the first solution with mass larger than $8\,\pi$ appears for $\tau \in (0.62, 0.64)$, as already noticed at the level of Fig. 2 (right). This is again not easy to read on the above figure, but it can be shown graphically by enlarging it enough.

we have found. This actually raises a much more interesting question, which is how to determine the basin of attraction of these self-similar solutions and to understand where is the threshold between solutions for which diffusion predominates and solutions which aggregate. Clearly, it is not going to be as simple as in the parabolic-elliptic case, where a single parameter, the total mass, determines the asymptotic regime. We can conjecture that blowup occurs for mass large enough and even, maybe, as soon as the total mass of the system is above $8\,\pi$ if initial data are sufficiently concentrated.

One may expect that for $M < 8\,\pi$ self-similar solutions are dynamically stable as solutions of (1)–(2) while the upper branch of self-similar solutions (for $M > 8\,\pi$) is unstable. An interesting stability question also arises for the critical value $M = 8\,\pi$, since for each $\tau \geq 0$ there exists an infinite number of stationary solutions n of (1)–(2) decaying algebraically as $|x| \to \infty$, (cf. [3]), while for $\tau > \tau^{**}$ there is a self-similar solution n of mass 8π decaying exponentially at infinity. As far as we know all these stability issues are open.

The model considered in this paper is by many aspects ridiculously simple. See, for instance, [10] to get a taste of the variety of the nonlinearities that make sense even for a rather crude modelling purpose. Still, these models, in limiting regimes, asymptotically exhibit scaling properties similar to the ones of the parabolic-parabolic Keller–Segel model considered here. Therefore, we believe that the information gathered above, together with the methods that have been introduced, for instance, the cumulated densities reformulation of the model, should definitely be some valuable piece of information in the study of the asymptotic behaviors of the equations used in chemotaxis.

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