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LIMIT THEOREMS FOR FUNCTIONALS OF MOVING AVERAGES

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Let $X_n = \sum_{i=1}^{\infty} a_i \varepsilon_{n-i}$, where the ε_i are i.i.d. with mean 0 and finite second moment and the a_i are either summable or regularly varying with index $\in (-1, -1/2)$. The sequence $\{X_n\}$ has short memory in the former case and long memory in the latter. For a large class of functions K, a new approach is proposed to develop both central ($\sqrt{N}$ rate) and noncentral (non-$\sqrt{N}$ rate) limit theorems for $S_N \equiv \sum_{n=1}^{N}[K(X_n) - EK(X_n)]$. Specifically, we show that in the short-memory case the central limit theorem holds for S_N and in the long-memory case, S_N can be decomposed into two asymptotically uncorrelated parts that follow a central limit and a noncentral limit theorem, respectively. Further we write the noncentral part as an expansion of uncorrelated components that follow noncentral limit theorems. Connections with the usual Hermite expansion in the Gaussian setting are also explored.

1. Introduction. This paper focuses on infinite moving averages defined by $X_n = \sum_{i=1}^{\infty} a_i \varepsilon_{n-i}$, $n \in Z$, where the innovations ε_i are mean-zero i.i.d. random variables having at least finite second moments, and the moving average coefficients a_i satisfy $\sum_{i=1}^{\infty} a_i^2 < \infty$ and certain conditions to be described later. The goal is to investigate the asymptotic behavior of $S_N \equiv \sum_{n=1}^{N}[K(X_n) - EK(X_n)]$, as $N \to \infty$, for a general class of measurable functions K under those conditions. Recall that for $K(x) = x$, a complete characterization of the weak limit of S_N in terms of the a_i was given by Davydov (1970).

The first and primary case that we will consider is where the a_i's are regularly varying with exponent $-\beta$, denoted by $a_i \in RV_{-\beta}$, for some $\beta \in (1/2, 1)$, that is, $a_i = i^{-\beta}L(i)$ and $L(i)$ is slowly varying at ∞. Notice that the covariance function $\rho(m) \equiv EX_0X_m$ of $\{X_n\}$ in this case is regularly varying with exponent $1-2\beta \in (-1, 0)$ and hence it is not summable. The study of stationary sequences with correlations decaying at hyperbolic rates presents interesting and challenging probabilistic as well as statistical problems. Progress has been steadily achieved for the last two decades or so. The expressions "long-memory sequence" and "sequence with long-range dependence" are nowadays commonly accepted as the descriptive terms for stationary time series whose correlation function is not summable. Detailed accounts on the development of long-memory sequences, both in theory and application, can be found in a large volume of review papers including Cox (1984), Rosenblatt (1984), Taqqu (1985), Sun and Ho (1985), Künsch (1986), Robinson (1994), Beran (1992) and

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Heyde (1995). A recent book, Beran (1994), also provides an up-to-date description of the statistical development in the area. The long-memory moving average processes considered in this paper cover the model known as the fractional ARIMA process [cf. Granger and Joyeux (1980), Hosking (1981)], which has motivated considerable interest in applied areas such as econometrics and hydrology [see, e.g., Lo (1991), Hipel and McLeod (1994)]. Suppose the X_n are normal and $EK^2(X_1) < \infty$. Then the asymptotics for S_N can be studied by exploiting the Hermite expansion of K along with the tool of multiple Wiener–Itô integrals, and are already well-known [see Taqqu (1979), Dobrushin and Major (1979), Breuer and Major (1983), Ho and Sun (1987), Giraitis and Surgailis (1985)]. Not much is known, however, about the case of nonnormal X_n. The difficulty is in part due to the lack of orthogonal polynomial expansion of $K(X_n)$ in terms of X_n in a general setting. With the notion of so-called Appell polynomials, Surgailis (1982) and Giraitis (1985) proved non-$\sqrt{N}$ (noncentral) and $\sqrt{N}$ (central) limit theorems, respectively, under very restrictive conditions that require K be analytic and ε_i have finite moments of all orders. Using the result in Surgailis (1982), Avram and Taqqu (1987) characterized the noncentral limits when K is any Appell polynomial of X_n. In this paper, we establish the asymptotic behavior of S_N using a new approach that bypasses the traditional polynomial expansion and hence requires considerably weaker regularity conditions for K than those of Surgailis (1982) and Giraitis (1985). It is the first time that a central limit theorem for S_N is proved in the long-memory setting without using the method of moments (cumulants).

The second case we shall consider is where the moving average coefficients a_i are summable and so $\{X_n\}$ has short memory. A very straightforward question is whether one can prove a central limit theorem for the empirical process of $X_1, \dots, X_n$ under no additional assumption on the a_i. This seems to be a natural problem and should have been thoroughly considered. Surprisingly, not much has been done towards this. Although there is a sizable literature on the theory of empirical processes for dependent random variables, with the exception of Chanda and Ruymgaart (1990) and Hesse (1990), the focus has been primarily on results based on various abstract conditions of short-range dependence, for example, strong-mixing conditions [see, e.g., Babu and Singh (1978), Basawa and Prakasa Rao (1980), Billingsley (1968), Deo (1973), Gastwirth and Rubin (1975), Mehra and Rao (1975), Sen (1971), Silverman (1983), Withers (1975)]. In the case of linear processes, the mixing conditions typically translate to very stringent conditions on the a_i and the ε_i. See Andrews (1984), Athreya and Pantula (1986), Bradley (1986), Chanda (1974), Gorodetskii (1977), Pham and Tran (1985). In our approach, we study S_N using a modification of the argument used for proving the central limit theorem in the long-memory setting and show that the assumption $\sum_{i=1}^{\infty} |a_i| < \infty$ alone guarantees the central limit theorem for S_N for a very general class of K.

This paper is structured as follows. Section 2 introduces the notation and technical conditions. Sections 3 and 4, respectively, contain the results for the long and short-memory cases. The results are then applied to the Gaussian moving average process in Section 5 where certain known results are rediscovered and connections with the theory based on Hermite expansion are given. For ease of reading, most of the proofs are collected in Section 6.

2. Notation and preliminaries. We first introduce some notation that will be useful for the whole paper. Throughout, the notation $\sum_{1\le j_1<\cdots<j_r<\infty}$ denotes a sum over the set of all positive integers $j_1, \dots, j_r$ such that $j_1 < \cdots < j_r$. Define

$$Y_{N,0} = N,$$

and for $r \ge 1$,

$$Y_{N,r} = \sum_{n=1}^{N} \sum_{1\le j_1<\cdots<j_r<\infty} \prod_{s=1}^{r} a_{j_s}\varepsilon_{n-j_s}.$$

If $a_i \in RV_{-\beta}$ for some $\beta \in (1/2, 1)$ and r is any positive integer such that $r(2\beta-1) < 1$, then the process $\{\sum_{1\le j_1<\cdots<j_r<\infty}\prod_{s=1}^{r} a_{j_s}\varepsilon_{n-j_s}\colon n \ge 1\}$ also has long memory and it is easily seen that

$$\sigma^2_{N,r} \equiv \mathrm{var}(Y_{N,r}) \sim N^{2-r(2\beta-1)}L^{2r}(N), \tag{2.1}$$

where L is the slowly varying component of a_i. Here and elsewhere in this paper, the notation $b_N \sim c_N$ means $b_N/c_N \to 1$ as $N \to \infty$.

Define $X_{n,0} = 0$ and

$$X_{n,j} = \sum_{1\le i\le j} a_i\varepsilon_{n-i}, \qquad \tilde{X}_{n,j} = X_n - X_{n,j}, \qquad j \ge 1$$

and

$$\tilde{X}_{n,j,\ell} = \tilde{X}_{n,j} - \tilde{X}_{n,\ell}, \qquad 1 \le j \le \ell.$$

Let $F_j, \tilde{F}_j, G_j$ and G be the distribution functions of $X_{n,j}, \tilde{X}_{n,j}, a_j\varepsilon_1$, and ε_1, respectively. For $j \ge 0$, define

$$K_j(x) = \int K(x+y)\,dF_j(y), \qquad K_\infty(x) = \int K(x+y)\,dF(y) \tag{2.2}$$

and

$$K_\infty^{(r)}(x) = \frac{d^r}{dx^r}\int K(x+y)\,dF(y)$$

whenever they are well defined. For $p \ge 0$, define

$$S_{N,p} = \sum_{n=1}^{N} K(X_n) - \sum_{r=0}^{p} K_\infty^{(r)}(0)Y_{N,r},$$

and denote

$$S_N \equiv S_{N,0} = \sum_{n=1}^{N}[K(X_n) - EK(X_n)].$$

Now we define a technical condition that will be used throughout the paper. If the tth derivative $K_j^{(t)}$ of K_j exists, define

$$K_{j,\lambda}^{(t)}(x) = \sup_{|y|\le\lambda} \left|K_j^{(t)}(x+y)\right|, \qquad \lambda \ge 0.$$

Let τ and t be nonnegative integers, and let λ be a nonnegative real number. We say that the condition $C(t,\tau,\lambda)$ holds, if:

1. $K_\tau^{(t)}(x)$ exists for all x and $K_\tau^{(t)}$ is continuous;
2. For all $x \in \Re$,

$$\sup_{I\subset\{1,2,\dots\}} E\left[K_{\tau,\lambda}^{(t)}\left(x + \sum_{i\in I} a_i\varepsilon_i\right)\right]^4 < \infty,$$

where the sup is taken over all subsets I of $\{1,2,\dots\}$.

REMARK 1. We will not require the full power of $C(t,\tau,\lambda)$. However, $C(t,\tau,\lambda)$ conveniently implies a number of conditions required for the proof of Lemma 6.2. On the other hand, condition $C(t,\tau,\lambda)$ is clearly satisfied if the tth derivative of $K(\cdot)$ is bounded and continuous, in which case one can simply take $\tau = 0$. Similarly, any polynomial K also ensures $C(t,\tau,\lambda)$, provided that ε_i has finite moments of sufficiently high orders. The novelty here is that $C(t,\tau,\lambda)$ can hold without K being smooth at all. One important example is $K(x) = I(x \le u)$ which is encountered in the case of the empirical process, where it is not difficult to see that if G has a continuous and integrable second derivative then, for $1 \le t \le \tau$,

$$K_\tau^{(t)}(x) = \int\cdots\int K(x + a_1y_1 + \cdots + a_\tau y_\tau)G^{(2)}(y_1)\cdots G^{(2)}(y_t) \times G^{(1)}(y_{t+1})\cdots G^{(1)}(y_\tau)\,dy_1\cdots dy_\tau$$

is bounded and continuous.

The following basic result has a number of applications in this paper.

LEMMA 2.1. *Let q be a nonnegative integer. Suppose that there exist τ and λ for which the condition $C(t,\tau,\lambda)$ holds for all $t = 0,1,\dots,q$. Then for each $j \ge \tau+1$, K_j is q times continuously differentiable and satisfies*

$$K_j^{(t)}(x) = EK_\tau^{(t)}(x + \tilde{X}_{1,\tau,j}) = \int K_{j-1}^{(t)}(x+u)\,dG_j(u) \quad \text{for all } t = 0,1,\dots,q. \tag{2.3}$$

PROOF. If $t = 0$, then (2.3) follows from the Fubini theorem and $C(0,\tau,\lambda)$. Next, suppose that we have shown (2.3) for $t = 0,\dots,s$ for some $s < q$. Then,

$$\frac{K_j^{(s)}(x+\delta) - K_j^{(s)}(x)}{\delta} = E\left[\frac{K_\tau^{(s)}(x+\delta+\tilde{X}_{1,\tau,j}) - K_\tau^{(s)}(x+\tilde{X}_{1,\tau,j})}{\delta}\right].$$

Hence, under $C(s+1,\tau,\lambda)$, the first and the second equalities of (2.3) for $t=s+1$ follow from the Lebesgue dominated convergence theorem and the Fubini theorem, respectively. □

We also need the following definition.

DEFINITION. We say that K has power rank k for some positive integer k, if $K_\infty^{(k)}(0)$ exists and is nonzero and $K_\infty^{(r)}(0)=0$ for $1\le r<k$.

REMARK 2. Suppose X_n is standard normal. Then for every function K with $EK^2(X_n)<\infty$, the following L_2-expansion holds:

$$K(X_n)=\sum_{j=0}^{\infty}\frac{h_j}{j!}H_j(X_n),$$

where $H_j(x)=(-1)^j e^{x^2/2}(d^j e^{-x^2/2}/dx^j)$ denotes the jth Hermite polynomial and $h_j=\int K(x)H_j(x)\,d\Phi(x)$. In Taqqu (1979), the smallest j for which h_j is nonzero is called the Hermite rank of K. It is easy to check that $h_j=K_\infty^{(j)}(0)$ for each j and hence the power rank is identical to the Hermite rank.

3. Limit theorems for long-memory moving averages.

In this section we consider limit theorems for S_N in the long-memory case; $a_i\in RV_{-\beta}$ and $\beta\in(1/2,1)$. Theorem 3.1 below contains a stochastic Taylor expansion for S_N using the uncorrelated terms $Y_{N,j}$. Loosely speaking, the number of terms that can be included in the expansion is p, the integer part of $(2\beta-1)^{-1}$. Theorem 3.2 shows that $S_{N,p}$, the remainder of the aforementioned expansion, follows a central limit theorem. The proof is based on a surprising argument that $S_{N,p}$ can be approximated in distribution by the remainder of the same expansion for a finite moving average process. Theorems 3.1 and 3.2 are then utilized to determine the asymptotic distribution of S_N in Corollary 3.3.

THEOREM 3.1. *Let $\beta\in(1/2,1)$ and p be any positive integer satisfying $p<(2\beta-1)^{-1}$. Assume that $E\varepsilon_1^8<\infty$, $EK^2(X_1)<\infty$ and for some $\tau\ge 0$ and $\lambda>0$, condition $C(t,\tau,\lambda)$ holds for $t=0,\dots,p+2$. Then for any $\zeta>0$, there exists a constant $C<\infty$ such that for all $N\ge 1$,*

$$\operatorname{var}\left(S_{N,p}\right)\le C\left(N\vee N^{2-(p+1)(2\beta-1)+\zeta}\right). \tag{3.1}$$

Moreover, for any $\lambda<((2\beta-1)\wedge(1-p(2\beta-1)))/2$,

$$\frac{N^\lambda}{\sigma_{N,p}}S_{N,p}\to 0 \quad \text{a.s. as } N\to\infty, \tag{3.2}$$

where $\sigma_{N,p}$ is defined by (2.1).

It follows from (3.1) that if $p+1 > (2\beta-1)^{-1}$ then $\mathrm{var}(S_{N,p}) \le CN$, from which one might conjecture that $S_{N,p}$ follows a central limit theorem under proper conditions. That turns out to be true for a reason which is natural but far from obvious at first sight, as explained later. Fix $\ell \ge 1$ and consider the moving average process $\{X_{n,\ell},\ n \ge 1\}$ with coefficients $\{a_i,\ i \le \ell\}$ and innovations $\{\varepsilon_j\}$. Clearly, $\{X_{n,\ell}\}$ has short-range dependence (in fact, ℓ-dependence). Now define

$$Y_{N,r,\ell} = \sum_{n=1}^{N} \sum_{1 \le j_1 < \cdots < j_r \le \ell} \prod_{s=1}^{r} a_{j_s} \varepsilon_{n-j_s}, \qquad 1 \le r \le \ell$$

and

$$S_{N,p,\ell} = \sum_{n=1}^{N} K(X_{n,\ell}) - \sum_{r=0}^{p} K_\ell^{(r)}(0) Y_{N,r,\ell}, \qquad 0 \le p \le \ell.$$

The essence of the following result is that $S_{N,p}$ behaves asymptotically like $S_{N,p,\ell}$, the partial sum of a short-memory process, and therefore follows a central limit theorem. It is important to remark that this approach is completely different from the traditional approach of the method of moments [cf. Arcones (1994), Breuer and Major (1983), Ho and Sun (1987)].

THEOREM 3.2. *Let $\beta \in (1/2, 1)$ and p be any positive integer satisfying $p+1 > (2\beta-1)^{-1}$. Assume that $E\varepsilon_1^8 < \infty$, $EK^2(X_1) < \infty$ and*

$$E[K(X_1) - K(X_{1,\ell})]^2 \to 0 \quad \text{as } \ell \to \infty, \tag{3.3}$$

In addition, assume that there exist some $\tau \ge 0$ and $\lambda > 0$ such that condition $C(t, \tau, \lambda)$ holds for $t = 0, \ldots, p+2$. Then

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} N^{-1} \mathrm{var}\big(S_{N,p} - S_{N,p,\ell}\big) = 0 \tag{3.4}$$

and

$$N^{-1/2} S_{N,p} \to_d N(0, \sigma^2), \tag{3.5}$$

where

$$\sigma^2 = \lim_{N\to\infty} N^{-1} \mathrm{var}(S_{N,p}) = \lim_{\ell\to\infty} \lim_{N\to\infty} N^{-1} \mathrm{var}(S_{N,p,\ell}) \in [0, \infty). \tag{3.6}$$

The proofs of Theorems 3.1 and 3.2 are given in Section 6. Although we do not formally prove it here, we remark that under the conditions of Theorem 3.2, for each $r < (2\beta-1)^{-1}$, $S_{N,p}$ and $Y_{N,r}$ are asymptotically uncorrelated in the sense that

$$\lim_{N\to\infty} \mathrm{cov}\big(N^{-1/2} S_{N,p}, \sigma_{N,r}^{-1} Y_{N,r}\big) = 0.$$

In the Gaussian case, $S_{N,p}$ and $Y_{N,r}$ are in fact uncorrelated for every N. See Section 5.

Combining the messages of Theorems 3.1 and 3.2 and assuming that $(2\beta-1)^{-1}$ is not an integer, we can write down the decomposition

$$
\begin{aligned}
S_N &= \text{noncentral part} + \text{central part} \\
&= \sum_{r=1}^{[(2\beta-1)^{-1}]} K_\infty^{(r)}(0) Y_{N,r} + \left(\sum_{n=1}^{N} K(X_n) - \sum_{r=0}^{[(2\beta-1)^{-1}]} K_\infty^{(r)}(0) Y_{N,r} \right),
\end{aligned}
\tag{3.7}
$$

where, as explained earlier, the noncentral and central parts are asymptotically uncorrelated. If the noncentral part is present then the asymptotic distribution of S_N is determined by the leading term there, and if the noncentral part is absent then the asymptotic distribution is determined by the central part. This is stated more formally in the following corollary.

Corollary 3.3. *Let $\beta \in (1/2, 1)$ and k be the power rank of $K(\cdot)$. Assume, that for some τ and λ, condition $C(t, \tau, \lambda)$ holds for $t = 0, \dots, k+2$, and $EK^2(X_1) < \infty$. If $k < (2\beta-1)^{-1}$ and $E\varepsilon_1^{2k\vee 8} < \infty$ then*

$$
\sigma_{N,k}^{-1} S_N \to_d K_\infty^{(k)}(0) Z_k \quad \text{as } N \to \infty, \tag{3.8}
$$

where the random variable Z_k can be represented by the multiple Wiener–Itô integral

$$
Z_k = \kappa(\beta, k) \int_{-\infty < u_1 < \cdots < u_k < 1} \int \left\{ \int_0^1 \prod_{j=1}^{k} [(v - u_j)^+]^{-\beta} dv \right\} dB(u_1) \cdots dB(u_k),
$$

with B denoting standard Brownian motion and

$$
\kappa(\beta, k) = \left\{ k!(1 - k(\beta - 1/2))(1 - k(2\beta - 1)) \left[\int_0^\infty (x + x^2)^{-\beta} \, dx \right]^{-k} \right\}^{1/2},
$$

ensuring $EZ_k^2 = 1$. If $k > (2\beta-1)^{-1}$, $E\varepsilon_1^8 < \infty$ and (3.3) holds, then

$$
N^{-1/2} S_N \to_d N(0, \sigma^2) \quad \text{as } N \to \infty, \tag{3.9}
$$

where $\sigma^2 < \infty$ is as in (3.6).

Proof. Since the power rank of K is k, from the definition of $S_{N,p}$ we readily deduce

$$
S_N = S_{N,k-1} = S_{N,k} + K_\infty^{(k)}(0) Y_{N,k}. \tag{3.10}
$$

First, consider the case $k < (2\beta-1)^{-1}$. Avram and Taqqu (1987) imposed $E\varepsilon_1^{2k} < \infty$ and showed that $\sigma_{N,k}^{-1} Y_{N,k}$ converges in distribution to Z_k [defined in (3.8)] as $N \to \infty$. Thus, (3.8) follows from Theorem 3.1 and the second equality of (3.10). Next, if $k > (2\beta-1)^{-1}$, then (3.9) follows from Theorem 3.2 and the first equality of (3.10). □

In the preceding results if $k = (2\beta - 1)^{-1}$ is an integer, then the asymptotic behavior of the various quantities in Corollary 3.3 can also be derived by slightly modifying the proofs, provided that the precise form of the slowly varying component of a_i is available. In general, it is not easy to write limiting variance σ^2 of (3.6) or (3.9) in a closed form, unless some very specific conditions are assumed about the functional K and the distribution of X_n. In Theorem 5.1 of Section 5, we derive a formula for σ^2 when X_n is Gaussian.

4. A central limit theorem for short-memory moving averages. In this section we focus on short-memory linear processes $\{X_n\}$, where the coefficients a_i satisfy $\sum_{i=1}^{\infty}|a_i| < \infty$. In Theorem 4.1 we show that S_N is asymptotically normal. The proof, similar in spirit to that of Theorem 3.2, takes full advantage of the linear structure and, in particular, does not rely on whether certain mixing conditions hold. To the best of our knowledge, this is the first attempt to prove a central limit theorem for S_N in this general setting.

THEOREM 4.1. *Assume that $E\varepsilon_1^4 < \infty$, (3.3) holds and for some τ and λ, condition $C(t, \tau, \lambda)$ holds for $t = 0, 1$. Then*

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} N^{-1}\,\mathrm{var}(S_{N,0} - S_{N,0,\ell}) = 0 \tag{4.1}$$

and

$$N^{-1/2} S_N \to_d N(0, \sigma^2),$$

where

$$\sigma^2 = \lim_{N\to\infty} N^{-1}\,\mathrm{var}(S_N) = \lim_{\ell\to\infty}\lim_{N\to\infty} \mathrm{var}(S_{N,0,\ell}) \in [0, \infty).$$

5. Gaussian processes. We revisit some results in the literature obtained by different arguments in the context of the Gaussian process.

Assume in this section that the ε_i are i.i.d. standard normal random variables and the a_i satisfy $\sum_{i=1}^{\infty} a_i^2 = 1$ so that $\mathrm{var}\,(X_n) = 1$. We apply Corollary 3.3 to obtain the following result in which the noncentral limit theorem was originally obtained by Dobrushin and Major (1979) and Taqqu (1979), and the central limit theorem by Breuer and Major (1983).

THEOREM 5.1. *Suppose that $a_j \in RV_{-\beta}$ for some $\beta \in (1/2, 1)$ and K has Hermite rank k (cf. Remark 2) and satisfies $EK^2(X_n) < \infty$. If $k < (2\beta - 1)^{-1}$, then*

$$\sigma_{N,k}^{-1} S_N \to_d h_k Z_k \quad \text{as } N \to \infty, \tag{5.1}$$

where Z_k is as described in Theorem 3.2. If $k > (2\beta - 1)^{-1}$, then

$$N^{-1/2} S_N \to_d N(0, \sigma^2), \tag{5.2}$$

where $0 < \sigma^2 < \infty$ *and is equal to*

$$\sigma^2 = \sum_{j=k}^{\infty} \frac{h_j^2}{j!} \sum_{m=-\infty}^{\infty} \rho^j(m). \tag{5.3}$$

Note that Theorem 5.1 has no assumptions on K other than $EK^2(X_n) < \infty$, since normal distribution has bounded derivatives and moments of all order (cf. Remark 1). Our proof actually uses truncated Hermite expansions of K to approximate K. We note also that Theorem 4.1 together with the argument used for proving the central limit theorem above can be utilized to prove a central limit theorem for short-memory linear Gaussian processes. Taking a broader perspective, the following result explains the fundamental role of the $Y_{N,i}$ for the Gaussian moving average process. Note that we do not distinguish between the long- and short-memory cases here.

THEOREM 5.2. *Assume the setting explained at the start of this section. Then for each* $N \geq 1$, *the infinite series* $\sum_{i=1}^{\infty} K_{\infty}^{(i)}(0) Y_{N,i}$ *converges in* L_2, *and* $Y_{N,i}$, $1 \leq i \leq \infty$, *are mutually uncorrelated, where* $Y_{N,\infty} \equiv S_N - \sum_{i=1}^{\infty} K_{\infty}^{(i)}(0) Y_{N,i}$.

The preceding result shows that we can write S_N as an infinite L_2-expansion using the mutually uncorrelated terms $Y_{N,i}$, $1 \leq i \leq \infty$. This expansion, which collects all the ‘diagonal” terms $a_j^k \varepsilon_{n-j}^k (k \geq 2)$ in $Y_{N,\infty}$, is more natural than the Hermite expansion $\sum_{i=1}^{\infty} (K_{\infty}^{(i)}(0)/i!) \sum_{n=1}^{N} H_i(X_n)$, since it gives a clear picture of the fundamental role of each term in the expansion. For example, in the setting of Theorem 3.1 and under the terminology of (3.7) of Section 3, if the noncentral part is present in (3.7) then the expansion based on the $Y_{N,i}$ reflects the fact that the diagonal terms make no contribution to the limiting distribution.

6. Proofs.

We first give a representation of $S_{N,p} - S_{N,p,\ell}$ which will be central in the proofs on a number of occasions. We will assume the existence of $K_j^{(t)}$, $j \geq \tau$, $t \leq p$ and the L_2 convergence of a number of telescoping series, where the justifications will be given when these representations are applied to specific cases. With that in mind, for a given nonnegative integer p, define for $\ell = 0$ and $\ell = \tau+1, \tau+2, \dots,$

$$\begin{aligned} T_{N,1,\ell}^{(t)} = \Bigg[& \sum_{n=1}^{N} \sum_{j=\tau+1}^{\infty} \big(K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})\big) \\ & - \sum_{r=1}^{t-1} K_{\infty}^{(r)}(0) \sum_{n=1}^{N} \sum_{\tau+1 \leq j_1 < \cdots < j_r < \infty} \prod_{s=1}^{r} a_{j_s} \varepsilon_{n-j_s} \\ & - \sum_{n=1}^{N} \sum_{\tau+1 \leq j_1 < \cdots < j_t < \infty} \left(\prod_{s=1}^{t} a_{j_s} \varepsilon_{n-j_s} \right) K_{j_t}^{(t)}(\tilde{X}_{n,j_t}) \Bigg] \end{aligned}$$

$$-\Big[\sum_{n=1}^{N}\sum_{j=\tau+1}^{\infty}\big(K_{j-1}(\tilde{X}_{n,j-1,\ell})-K_j(\tilde{X}_{n,j,\ell})\big)$$

$$-\sum_{r=1}^{t-1}K_\ell^{(r)}(0)\sum_{n=1}^{N}\sum_{\tau+1\le j_1<\cdots<j_r\le\ell}\prod_{s=1}^{r}a_{j_s}\varepsilon_{n-j_s}$$

$$-\sum_{n=1}^{N}\sum_{\tau+1\le j_1<\cdots<j_t\le\ell}\bigg(\prod_{s=1}^{t}a_{j_s}\varepsilon_{n-j_s}\bigg)K_{j_t}^{(t)}(\tilde{X}_{n,j_t,\ell})\Big],\qquad 0\le t\le p,$$

$$T_{N,2,\ell}=\Big[\sum_{n=1}^{N}\sum_{\tau+1\le j_1<\cdots<j_p<\infty}\bigg(\prod_{s=1}^{p}a_{j_s}\varepsilon_{n-j_s}\bigg)\big(K_{j_p}^{(p)}(\tilde{X}_{n,j_p})-K_\infty^{(p)}(0)\big)\Big]$$

$$-\Big[\sum_{n=1}^{N}\sum_{\tau+1\le j_1<\cdots<j_p\le\ell}\bigg(\prod_{s=1}^{p}a_{j_s}\varepsilon_{n-j_s}\bigg)\big(K_{j_p}^{(p)}(\tilde{X}_{n,j_p,\ell})-K_\ell^{(p)}(0)\big)\Big],$$

$$T_{N,3,\ell}=\Big[\sum_{n=1}^{N}\sum_{j=1}^{\tau}\big(K_{j-1}(\tilde{X}_{n,j-1})-K_j(\tilde{X}_{n,j})\big)\Big]$$

$$-\Big[\sum_{n=1}^{N}\sum_{j=1}^{\tau}\big(K_{j-1}(\tilde{X}_{n,j-1,\ell})-K_j(\tilde{X}_{n,j,\ell})\big)\Big],$$

$$T_{N,4,\ell}=-\sum_{r=1}^{p}K_\infty^{(r)}(0)\sum_{n=1}^{N}\sum_{\substack{j_1<\cdots<j_r<\infty,\\ 1\le j_1\le\tau}}\bigg(\prod_{s=1}^{r}a_{j_s}\varepsilon_{n-j_s}\bigg)$$

$$+\sum_{r=1}^{p}K_\ell^{(r)}(0)\sum_{n=1}^{N}\sum_{\substack{j_1<\cdots<j_r\le\ell,\\ 1\le j_1\le\tau}}\bigg(\prod_{s=1}^{r}a_{j_s}\varepsilon_{n-j_s}\bigg)$$

$$=-\sum_{r=1}^{p}K_\infty^{(r)}(0)\sum_{n=1}^{N}\sum_{\substack{j_1<\cdots<j_r,\\ 1\le j_1\le\tau,\,j_r\ge\ell+1}}\bigg(\prod_{s=1}^{r}a_{j_s}\varepsilon_{n-j_s}\bigg)$$

$$+\sum_{r=1}^{p}\big(K_\ell^{(r)}(0)-K_\infty^{(r)}(0)\big)\sum_{n=1}^{N}\sum_{\substack{j_1<\cdots<j_r\le\ell,\\ 1\le j_1\le\tau}}\bigg(\prod_{s=1}^{r}a_{j_s}\varepsilon_{n-j_s}\bigg).$$

Note. We interpret $\sum_{j=m}^{n}=0$ when $m>n$; for example, if $\ell=0$ then the second terms of the various quantities of T defined above are taken to be 0. Further, we write

$$T_{N,1,\ell}^{(1)}=\sum_{n=1}^{N}\sum_{j=\tau+1}^{\infty}U_{n,j,\ell},$$

$$T_{N,1,\ell}^{(t+1)}-T_{N,1,\ell}^{(t)}=\sum_{n=1}^{N}\sum_{\tau+1\le j_1<\cdots<j_{t+1}<\infty}\bigg(\prod_{s=1}^{t}a_{j_s}\varepsilon_{n-j_s}\bigg)V_{n,j_{t+1},\ell}^{(t)},$$

$$T_{N,2,\ell} = \sum_{n=1}^{N} \sum_{\tau+1 \le j_1 < \cdots < j_p < \infty} \left(\prod_{s=1}^{p} a_{j_s} \varepsilon_{n-j_s} \right) W_{n,j_p,\ell},$$

$$T_{N,3,\ell} = \sum_{n=1}^{N} \sum_{j=1}^{\tau} P_{n,j,\ell},$$

where

$$\begin{aligned} U_{n,j,\ell} &= [K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j}) - a_j \varepsilon_{n-j} K_{j-1}^{(1)}(\tilde{X}_{n,j})] \\ &\quad - [K_{j-1}(\tilde{X}_{n,j-1,\ell}) - K_j(\tilde{X}_{n,j,\ell}) - a_j \varepsilon_{n-j} K_{j-1}^{(1)}(\tilde{X}_{n,j,\ell})] I(j \le \ell), \\ V_{n,j,\ell}^{(t)} &= [K_{j-1}^{(t)}(\tilde{X}_{n,j-1}) - K_j^{(t)}(\tilde{X}_{n,j}) - a_j \varepsilon_{n-j} K_j^{(t+1)}(\tilde{X}_{n,j})] \\ &\quad - [K_{j-1}^{(t)}(\tilde{X}_{n,j-1,\ell}) - K_j^{(t)}(\tilde{X}_{n,j,\ell}) - a_j \varepsilon_{n-j} K_j^{(t+1)}(\tilde{X}_{n,j,\ell})] I(j \le \ell), \\ W_{n,j,\ell} &= [K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0)] - [K_j^{(p)}(\tilde{X}_{n,j,\ell}) - K_\ell^{(p)}(0)] I(j \le \ell) \end{aligned}$$

and

$$\begin{aligned} P_{n,j,\ell} &= [K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j}))] \\ &\quad - [K_{j-1}(\tilde{X}_{n,j-1,\ell}) - K_j(\tilde{X}_{n,j,\ell})] I(j \le \ell). \end{aligned} \tag{6.1}$$

Now comes the most crucial observation in this paper. Fix n, n', j, j' and write

$$K_i(\tilde{X}_{m,i}) = E(K(X_m) | \mathscr{F}_{m-i-1})$$

where $\mathscr{F}_s$ is the σ-field generated by $\varepsilon_k, k \le s$. Suppose $n - j \ne n' - j'$ and without loss of generality assume that $n - j < n' - j'$. Then

$$\begin{aligned} &E[K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})][K_{j'-1}(\tilde{X}_{n',j'-1}) - K_{j'}(\tilde{X}_{n',j'})] \\ &\quad = E\{[E(K(X_n)|\mathscr{F}_{n-j}) - E(K(X_n)|\mathscr{F}_{n-j-1})] \\ &\qquad \times [E(K(X_{n'})|\mathscr{F}_{n'-j'}) - E(K(X_{n'})|\mathscr{F}_{n'-j'-1})]\} \\ &\quad = E\{[E(K(X_n)|\mathscr{F}_{n-j}) - E(K(X_n)|\mathscr{F}_{n-j-1})] \\ &\qquad \times E[E(K(X_{n'})|\mathscr{F}_{n'-j'}) - E(K(X_{n'})|\mathscr{F}_{n'-j'-1})|\mathscr{F}_{n-j}]\} \\ &\quad = E\{[E(K(X_n)|\mathscr{F}_{n-j}) - E(K(X_n)|\mathscr{F}_{n-j-1})] \\ &\qquad \times [E(K(X_{n'})|\mathscr{F}_{n-j}) - E(K(X_{n'})|\mathscr{F}_{n-j})]\} \\ &\quad = 0. \end{aligned}$$

By repeated applications of arguments of this nature, we obtain

$$\operatorname{cov}(U_{n,j,\ell}, U_{n',j',\ell}) = 0 \quad \text{if } n - j \ne n' - j', \tag{6.2}$$

$$\operatorname{cov}\left(\left(\prod_{s=1}^{t} a_{j_s} \varepsilon_{n-j_s} \right) V_{n,j_{t+1},\ell}^{(t)}, \left(\prod_{s=1}^{t} a_{j'_s} \varepsilon_{n'-j'_s} \right) V_{n',j'_{t+1},\ell}^{(t)} \right) = 0 \tag{6.3}$$
$$\text{if } n - j_s \ne n' - j'_s \text{ for some } 1 \le s \le t,$$

$$\operatorname{cov}(P_{n,j,\ell}, P_{n',j',\ell}) = 0 \quad \text{if } n - j \neq n' - j', \tag{6.4}$$

$$\operatorname{cov}\left(\left(\prod_{s=1}^{p} a_{j_s}\varepsilon_{n-j_s}\right)W_{n,j_p,\ell},\ \left(\prod_{s=1}^{p} a_{j'_s}\varepsilon_{n'-j'_s}\right)W_{n',j'_p,\ell}\right) = 0 \quad \text{if } n - j_s \neq n' - j'_s \text{ for some } 1 \leq s \leq p, \tag{6.5}$$

$$\operatorname{cov}\left(\prod_{s=1}^{r_1} a_{j_s}\varepsilon_{n-j_s},\ \prod_{s=1}^{r_2} a_{j'_s}\varepsilon_{n'-j'_s}\right) = 0 \quad \text{if } r_1 \neq r_2 \text{ or if } r_1 = r_2 = r \text{ but } n - j_s \neq n' - j'_s \text{ for some } 1 \leq s \leq r. \tag{6.6}$$

In the following, let $j' = (n' - n) + j$. By (6.2)–(6.6), the Cauchy–Schwarz inequality, and the symmetry of the covariance function, we obtain

$$\begin{aligned} \operatorname{var}\big(T^{(1)}_{N,1,\ell}\big) &\leq 4\sum_{n=1}^{N}\sum_{j=\tau+1}^{\ell}\sum_{n'=n}^{n+\ell-j}\operatorname{cov}(U_{n,j,\ell}, U_{n',j',\ell}) \\ &\quad + 4\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{j=\ell+1}^{\infty}\operatorname{cov}(U_{n,j,0}, U_{n',j',0}), \end{aligned} \tag{6.7}$$

$$\begin{aligned} &\operatorname{var}\big(T^{(t+1)}_{N,1,\ell} - T^{(t)}_{N,1,\ell}\big) \\ &\quad \leq 4\sum_{n=1}^{N}\sum_{\tau+1\leq j_1<\cdots<j_{t+1}\leq\ell}\ \sum_{n'=n}^{n+\ell-j_{t+1}}\prod_{s=1}^{t} a_{j_s}a_{j'_s}\operatorname{cov}\big(V^{(t)}_{n,j_{t+1},\ell}, V^{(t)}_{n',j'_{t+1},\ell}\big) \\ &\qquad + 4\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{\substack{\tau+1\leq j_1<\cdots<j_{t+1},\\ j_{t+1}\geq\ell+1}}\prod_{s=1}^{t} a_{j_s}a_{j'_s}\operatorname{cov}\big(V^{(t)}_{n,j_{t+1},0}, V^{(t)}_{n',j'_{t+1},0}\big), \end{aligned} \tag{6.8}$$

$$\begin{aligned} &\operatorname{var}(T_{N,2,\ell}) \\ &\quad \leq 4\sum_{n=1}^{N}\sum_{\tau+1\leq j_1<\cdots<j_p\leq\ell}\ \sum_{n'=n}^{n+\ell-j_p}\left(\prod_{s=1}^{p} a_{j_s}a_{j'_s}\right)\operatorname{cov}\big(W_{n,j_p,\ell},\ W_{n',j'_p,\ell}\big) \\ &\qquad + 4\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{\substack{\tau+1\leq j_1<\cdots<j_p,\\ j_p\geq\ell+1}}\left(\prod_{s=1}^{p} a_{j_s}a_{j'_s}\right)\operatorname{cov}\big(W_{n,j_p,0},\ W_{n',j'_p,0}\big), \end{aligned} \tag{6.9}$$

$$\operatorname{var}(T_{N,3,\ell}) \leq 2\sum_{n=1}^{N}\sum_{j=1}^{\tau}\sum_{n'=n}^{n+\tau-j}\operatorname{cov}\big(P_{n,j,\ell}, P_{n',j',\ell}\big), \tag{6.10}$$

$$
\begin{aligned}
&\operatorname{var}(T_{N,4,\ell}) \\
&\quad \le 4E(\varepsilon_1^2)\sum_{r=1}^{p}\sum_{n=1}^{N}\big(K_\infty^{(r)}(0)-K_\ell^{(r)}(0)\big)^2 \sum_{\substack{j_1<\cdots<j_r\le\ell,\\ 1\le j_1\le\tau}}\sum_{n'=n}^{n+\tau-j_1}\bigg(\prod_{s=1}^{r} a_{j_s}a_{j'_s}\bigg) \\
&\qquad + 4E(\varepsilon_1^2)\sum_{r=1}^{p}(K_\infty^{(r)}(0))^2\sum_{n=1}^{N}\sum_{\substack{j_1<\cdots<j_r,\\ 1\le j_1\le\tau,\ j_r\ge\ell+1}}\sum_{n'=n}^{n+\tau-j_1}\bigg(\prod_{s=1}^{r} a_{j_s}a_{j'_s}\bigg).
\end{aligned}
\tag{6.11}
$$

PROOF OF THEOREM 3.1. We prove (3.1) only, as (3.2) easily follows from (3.1) [cf. Lai and Stout (1980), Ho and Hsing (1996), Csörgő and Mielniczuk (1995)]. First, the derivatives $K_j^{(t)}$, $0 \le t \le p$, $j \ge \tau$, exist by conditions $C(t, \tau, \lambda)$, $0 \le t \le p+2$, and Lemma 2.1. Next, relations (6.12)–(6.15) below justify the existence of various infinite series in the following expression (in the L_2 sense):

$$T_{N,1,0}^{(1)} + \sum_{t=1}^{p-1}\Big(T_{N,1,0}^{(t+1)} - T_{N,1,0}^{(t)}\Big) + T_{N,2,0} + T_{N,3,0} + T_{N,4,0}.$$

By the proof of (iii) of Lemma 6.2, it is easy to see that

$$K_j(\tilde{X}_{n,j}) \to_{L_2} K_\infty(0) \quad \text{as } j\to\infty,$$

which implies that

$$
\begin{aligned}
\sum_{n=1}^{N}[K(X_n) - EK(X_n)] &= \sum_{n=1}^{N}[K(X_n) - K_\infty(0)] \\
&= \sum_{n=1}^{N}\sum_{j=1}^{\infty}\big[K_{j-1}(\tilde{X}_{n,j-1}) - K_j(\tilde{X}_{n,j})\big],
\end{aligned}
$$

where the infinite sum converges in L_2. Therefore we can write

$$
\begin{aligned}
S_{N,p} &= T_{N,1,0}^{(p)} + T_{N,2,0} + T_{N,3,0} + T_{N,4,0} \\
&= T_{N,1,0}^{(1)} + \sum_{t=1}^{p-1}\Big(T_{N,1,0}^{(t+1)} - T_{N,1,0}^{(t)}\Big) + T_{N,2,0} + T_{N,3,0} + T_{N,4,0}.
\end{aligned}
$$

Our plan is to show that for some universal constant $B < \infty$,

$$\operatorname{var}\Big(T_{N,1,0}^{(1)}\Big) \le BN, \tag{6.12}$$

$$\sum_{t=1}^{p-1}\operatorname{var}\Big(T_{N,1,0}^{(t+1)} - T_{N,1,0}^{(t)}\Big) \le BN, \tag{6.13}$$

$$\operatorname{var}(T_{N,3,0}) + \operatorname{var}(T_{N,4,0}) \le BN, \tag{6.14}$$

and for any given small $\zeta > 0$, there exists $C < \infty$, independent of k, such that

$$\text{var}(T_{N,2,0}) \le C(N \vee N^{2-(p+1)(2\beta-1)+\zeta}). \tag{6.15}$$

Clearly, by (i) of Lemma 6.2 and the Cauchy–Schwarz inequality, for each $n, n', j, \ j' \ge 1$, there exists a universal constant B such that

$$\left|\text{cov}(U_{n,j,0}, U_{n',j',0})\right| \le Ba_j^2 a_{j'}^2, \tag{6.16}$$

$$\left|\text{cov}\left(V^{(t)}_{n,j_{t+1},0}, V^{(t)}_{n',j'_{t+1},0}\right)\right| \le Ba^2_{j_{t+1}-1}a^2_{j'_{t+1}-1}. \tag{6.17}$$

Since the a_j^2 are summable, (6.12) clearly follows from (6.7) and (6.16). Now, by (6.8) and (6.17),

$$\begin{aligned}&\text{var}\left(T^{(t+1)}_{N,1,0} - T^{(t)}_{N,1,0}\right)\\ &\quad \le B\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{\tau+1\le j_1<\cdots<j_t<\infty}\left(\prod_{s=1}^{t}|a_{j_s}a_{j'_s}|\right)\sum_{j_{t+1}>j_t}a^2_{j_{t+1}}a^2_{j'_{t+1}}.\end{aligned} \tag{6.18}$$

For any $\zeta > 0$ there exists $C < \infty$ such that

$$|a_j| \le C j^{-\beta+\zeta}.$$

Choose $0 < \zeta < \beta - 1/2$. Simple algebra shows that the right-hand side of (6.18) is bounded by

$$CN\sum_{n=0}^{N-1}\sum_{\tau+1\le j_1<\cdots<j_t<\infty}\left(\prod_{s=1}^{t}[j_s(n+j_s)]^{-\beta+\zeta}\right)\sum_{j_{t+1}>j_t}[j_{t+1}(n+j_{t+1})]^{-2\beta+\zeta}$$

for some $C < \infty$. By Lemma 6.3, for $n \ge 1$,

$$\begin{aligned}\sum_{\tau+1\le j_1<j_2<\cdots<j_t<\infty}&\left(\prod_{s=1}^{t}[j_s(n+j_s)]^{-\beta+\zeta}\right)\sum_{j_{t+1}>j_t}[j_{t+1}(n+j_{t+1})]^{-2\beta+\zeta}\\ &\le C\sum_{j=\tau+1}^{\infty}[j(n+j)]^{-[(t+1)(\beta-\zeta-1/2)+\beta+1/2]}\\ &\le C\sum_{j=\tau+1}^{\infty}[j(n+j)]^{-(\beta+1/2)} \le Cn^{-(\beta+1/2)},\end{aligned}$$

which is summable and ensures (6.13). Next we show (6.14). By the Jensen inequality,

$$E\left(K_j^2(\tilde{X}_{n,j})\right) = E\left(E^2\left(K(X_n)|\tilde{X}_{n,j}\right)\right) \le EK^2(X_n) < \infty \quad \text{for any } n, j \ge 1,$$

and hence by the Cauchy–Schwarz inequality,

$$\left|\text{cov}(P_{n,j,0}, P_{n',j',0})\right| \le 4EK^2(X_n).$$

This, together with (6.10), takes care of the first term of (6.14). The second term there can be handled easily using (6.11). Finally, we show (6.15). It follows from

(iii) of Lemma 6.2 and the fact that $a_i \in RV_{-\beta}$ that for any $0 < \zeta < \beta - 1/2$ there exists C such that

$$EW^2_{n,j,0} \le C \sum_{i=j}^{\infty} a_i^2 < C j^{-2[(\beta-1/2)-\zeta]}.$$

This and (6.9) show that

$$\begin{aligned}
&\mathrm{var}(T_{N,2,0}) \\
&\quad \le C \sum_{n=1}^{N} \sum_{n'=n}^{N} \sum_{\tau+1 \le j_1 < \cdots < j_p < \infty} \left(\prod_{s=1}^{p} |a_{j_s} a_{j'_s}| \right) (j_p j'_p)^{-(\beta-1/2)+\zeta} \\
&\quad \le CN \sum_{n=0}^{N-1} \sum_{\tau+1 \le j_1 < \cdots < j_p < \infty} \left(\prod_{s=1}^{p} [j_s(n+j_s)]^{-\beta+\zeta} \right) [j_p(n+j_p)]^{-(\beta-1/2)+\zeta}.
\end{aligned}$$

By Lemma 6.3, for $n \ge 1$,

$$\begin{aligned}
&\sum_{\tau+1 \le j_1 < \cdots < j_p < \infty} \left(\prod_{s=1}^{p} [j_s(n+j_s)]^{-\beta+\zeta} \right) [j_p(n+j_p)]^{-(\beta-1/2)+\zeta} \\
&\qquad \le C \sum_{j=\tau+1}^{\infty} [j(n+j)]^{-[(p+1)(\beta-\zeta-1/2)+1/2]}.
\end{aligned}$$

Note that there are two possibilities: for small $\zeta > 0$, $(p+1)(\beta-\zeta-1/2)+1/2$ is greater or less than one. By Lemma 6.3, in the former case, the right-hand side of the preceding inequality is summable in n, and in the latter case, it is bounded by $Cn^{-2(p+1)(\beta-\zeta-1/2)}$. Thus, for some $C < \infty$,

$$\mathrm{var}(T_{N,2,0}) \le C(N \vee N^{2-2(p+1)(\beta-\zeta-1/2)}).$$

This shows (6.15) and concludes the proof of (3.1). □

Proof of Theorem 3.2. Here is the strategy of our proof. Since $S_{N,p,\ell}$ is the partial sum of an ℓ-dependent sequence for any fixed ℓ, it satisfies

$$N^{-1/2} S_{N,p,\ell} \xrightarrow{d} N(0, \sigma_\ell^2) \tag{6.19}$$

for some finite σ_ℓ^2, where

$$\sigma_\ell^2 = \lim_{N\to\infty} N^{-1} \mathrm{var}(S_{N,p,\ell}).$$

By the triangle inequality and the Cauchy–Schwarz inequalities,

$$\mathrm{var}(S_{N,p,\ell_1} - S_{N,p,\ell_2}) \le 2[\mathrm{var}(S_{N,p} - S_{N,p,\ell_1}) + \mathrm{var}(S_{N,p} - S_{N,p,\ell_2})].$$

Thus, if (3.4) holds, then

$$\lim_{\ell\to\infty} \sup_{\ell_1,\ell_2 \ge \ell} \limsup_{N\to\infty} N^{-1} \mathrm{var}(S_{N,p,\ell_1} - S_{N,p,\ell_2}) = 0,$$

which implies that $\{\sigma_\ell^2,\ \ell = 1, 2, \ldots\}$ is a Cauchy sequence and therefore σ_ℓ^2 tends to some finite value σ^2 as $\ell \to \infty$. Hence, assuming (3.4), both (3.5) and

(3.6) follow readily as a result of this fact and (6.19). Thus, the remaining part of the proof is dedicated to showing (3.4).

Using arguments similar to those used in the proof of Theorem 3.1, we obtain

$$S_{N,\,p} - S_{N,\,p,\,\ell} = T^{(1)}_{N,\,1,\,\ell} + \sum_{t=1}^{p-1} \bigl(T^{(t+1)}_{N,\,1,\,\ell} - T^{(t)}_{N,\,1,\,\ell}\bigr) + T_{N,\,2,\,\ell} + T_{N,\,3,\,\ell} + T_{N,\,4,\,\ell}.$$

Our goal is to prove

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} N^{-1}\,\mathrm{var}\bigl(T^{(1)}_{N,\,1,\,\ell}\bigr) = 0, \tag{6.20}$$

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} N^{-1} \sum_{t=1}^{p-1} \mathrm{var}\bigl(T^{(t+1)}_{N,\,1,\,\ell} - T^{(t)}_{N,\,1,\,\ell}\bigr) = 0 \tag{6.21}$$

and

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} N^{-1}\,\mathrm{var}(T_{N,\,i,\,\ell}) = 0, \qquad i = 2, 3, 4. \tag{6.22}$$

We first show (6.20) and (6.21) through bounding the terms on the right-hand side of (6.7) and (6.8). We first handle the first of the two terms in each of the bounds in (6.7) and (6.8). By (ii) of Lemma 6.2 and the Cauchy–Schwarz inequality,

$$\bigl|\mathrm{cov}(U_{n,\,j,\,\ell}, U_{n',\,j',\,\ell})\bigr| \le C\Biggl(\sum_{i\ge \ell+1} a_i^2\Biggr) a_j^2 a_{j'}^2$$

and

$$\bigl|\mathrm{cov}\bigl(V_{n,\,j_{t+1},\,\ell}, V_{n',\,j'_{t+1},\,\ell}\bigr)\bigr| \le C\Biggl(\sum_{i\ge \ell+1} a_i^2\Biggr) a_{j_{t+1}}^2 a_{j'_{t+1}}^2.$$

Since the a_i^2 are summable, it is clear that

$$\lim_{\ell\to\infty} \limsup_{N\to\infty} N^{-1} \Biggl| \sum_{n=1}^{N} \sum_{j=\tau+1}^{\ell} \sum_{n'=n}^{n+\ell-j} \mathrm{cov}(U_{n,\,j,\,\ell}, U_{n',\,j',\,\ell}) \Biggr| = 0 \tag{6.23}$$

and

$$\begin{aligned} \lim_{\ell\to\infty} \limsup_{N\to\infty} \frac{1}{N} \Biggl| \sum_{n=1}^{N} \sum_{\tau+1\le j_1<\cdots<j_{t+1}\le \ell} \sum_{n'=n}^{n+\ell-j_{t+1}} \Biggl(\prod_{s=1}^{t} a_{j_s} a_{j'_s} \Biggr) \\ \times \mathrm{cov}\Bigl(V^{(t)}_{n,\,j_{t+1},\,\ell}, V^{(t)}_{n',\,j'_{t+1},\,\ell}\Bigr) \Biggr| = 0. \end{aligned} \tag{6.24}$$

Next, we consider the second of the two terms of each of the right-hand sides of (6.7) and (6.8). By (6.16), for any $0 < \zeta < 2\beta - 1$, there exists a finite constant

C such that

$$\left|\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{j=\ell+1}^{\infty}\operatorname{cov}(U_{n,j,0},U_{n',j',0})\right| \le C\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{j=\ell+1}^{\infty}(jj')^{-2\beta+\zeta}$$
$$\le CN\sum_{n=0}^{N-1}\sum_{j=\ell+1}^{\infty}[j(n+j)]^{-2\beta+\zeta},$$

and it follows from Lemma 6.3 that the right hand side is bounded by

$$CN\sum_{n=0}^{N-1}(n\vee\ell)^{-2\beta+\zeta}\ell^{1-2\beta+\zeta} \le CN\ell^{2(1-2\beta+\zeta)}.$$

Since $1-2\beta+\zeta<0$, we have

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}N^{-1}\left|\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{j=\ell+1}^{\infty}\operatorname{cov}\big(U_{n,j,0},U_{n',j',0}\big)\right|=0. \tag{6.25}$$

Similarly, by (6.17) and Lemma 6.3,

$$\left|\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{\substack{\tau+1\le j_1<\cdots<j_{t+1},\\ j_{t+1}\ge\ell+1}}\prod_{s=1}^{t}a_{j_s}a_{j'_s}\operatorname{cov}\big(V^{(t)}_{n,j_{t+1},0},V^{(t)}_{n',j'_{t+1},0}\big)\right|$$
$$\le CN\sum_{n=0}^{N-1}\Bigg(\sum_{1\le j_1<\cdots<j_t<\infty}\prod_{s=1}^{t}[j_s(n+j_s)]^{-\beta+\zeta}\Bigg)\Bigg(\sum_{j=\ell+1}^{\infty}[j(n+j)]^{-2\beta+\zeta}\Bigg)$$
$$\le CN\ell^{1-2\beta+\zeta}\sum_{n=1}^{N-1}n^{-2\beta+\zeta}$$
$$\le CN\ell^{1-2\beta+\zeta}.$$

Therefore, we have

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}N^{-1}\left|\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{\substack{1\le j_1<\cdots<j_{t+1},\\ j_{t+1}\ge\ell+1}}\Bigg(\prod_{s=1}^{t}a_{j_s}a_{j'_s}\Bigg)\times\operatorname{cov}\Big(V^{(t)}_{n,j,0},V^{(t)}_{n',j',0}\Big)\right|=0. \tag{6.26}$$

Thus, (6.20) follows from (6.7), (6.23) and (6.25) and (6.21) follows from (6.8), (6.24) and (6.26). Next we show (6.22) for $i=2$ through (6.9). The first term on the right of (6.9) is handled as follows. By (iv) of Lemma 6.2,

$$EW^2_{n,j,\ell}\le C\sum_{i\ge\ell}a_i^2 \quad \text{for all } n,j,$$

for some finite constant C. Then, there is a constant C independent of n, j, j' and ℓ such that

$$\big|\operatorname{cov}\big(W_{n,j,\ell},W_{n',j',\ell}\big)\big|\le C\sum_{i\ge\ell}a_i^2\le C\ell^{-2(\beta-1/2-\zeta)},$$

where we choose $\zeta > 0$ small enough so that $2(p+1)(\beta - 1/2 - \zeta) > 1$. Consequently,

$$\begin{aligned}
&\left|\sum_{n=1}^{N} \sum_{\tau+1\le j_1<\cdots<j_p\le \ell} \sum_{n'=n}^{n+\ell-j_p} \left(\prod_{s=1}^{p} a_{j_s} a_{j'_s}\right) \operatorname{cov}\left(W_{n,j_p,\ell},\ W_{n',j'_p,\ell}\right)\right| \\
&\quad \le C\ell^{-2(\beta-1/2-\zeta)} \sum_{n=1}^{N} \sum_{n'=n}^{n+\ell-1} \sum_{\tau+1\le j_1<\cdots<j_p\le \ell} \left(\prod_{s=1}^{p} |a_{j_s} a_{j'_s}|\right) \\
&\quad \le C\ell^{-2(\beta-1/2-\zeta)} \sum_{n=1}^{N} \sum_{n'=n}^{n+\ell-1} \left(\sum_{j=1}^{\ell} |a_j a_{j'}|\right)^p \\
&\quad \le C\ell^{-2(\beta-1/2-\zeta)} N \sum_{n=0}^{\ell-1} \left(\sum_{j=1}^{\ell} [j(n+j)]^{-\beta+\zeta}\right)^p.
\end{aligned}$$

By Lemma 6.4 below, the last expression is bounded by

$$\begin{aligned}
CN\ell^{-2(\beta-1/2-\zeta)} \sum_{n=1}^{\ell-1} n^{-2p(\beta-1/2-\zeta)} &\le CN\ell^{-2(\beta-1/2-\zeta)} \ell^{-2p(\beta-1/2-\zeta)+1} \\
&= CN\ell^{-2(p+1)(\beta-1/2-\zeta)+1}.
\end{aligned}$$

Hence, in view of the the restriction on ζ, we conclude that

$$\begin{aligned}
\lim_{\ell\to\infty} \limsup_{N\to\infty} N^{-1} \Bigg| \sum_{n=1}^{N} \sum_{n'=n}^{N} \sum_{\tau+1\le j_1<\cdots<j_p\le \ell} \left(\prod_{s=1}^{p} a_{j_s} a_{j'_s}\right) \\
\times \operatorname{cov}\left(W_{n,j_p,\ell},\ W_{n',j'_p,\ell}\right)\Bigg| = 0.
\end{aligned} \tag{6.27}$$

Now we consider the second term on the right of (6.9). By (iii) of Lemma 6.2, for any $\zeta > 0$ there exists $C < \infty$ such that

$$\begin{aligned}
&\left|\sum_{n=1}^{N} \sum_{n'=n}^{N} \sum_{\substack{\tau+1\le j_1<\cdots<j_p,\\ j_p\ge \ell+1}} \left(\prod_{s=1}^{p} a_{j_s} a_{j'_s}\right) \operatorname{cov}\left(W_{n,j_p,0},\ W_{n',j'_p,0}\right)\right| \\
&\quad \le C \sum_{n=1}^{N} \sum_{n'=n}^{N} \sum_{\substack{1\le j_1<\cdots<j_p,\\ j_p\ge \ell+1}} \left(\prod_{s=1}^{p} |a_{j_s} a_{j'_s}|\right) (j_p j'_p)^{-(\beta-1/2)+\zeta} \\
&\quad \le CN \sum_{n=0}^{N-1} \sum_{\substack{1\le j_1<\cdots<j_p,\\ j_p\ge \ell+1}} \left(\prod_{s=1}^{p} [j_s(n+j_s)]^{-\beta+\zeta}\right) [j_p(n+j_p)]^{-\beta+1/2+\zeta} \\
&\quad = CN \sum_{n=0}^{N-1} \sum_{\substack{1\le j_1<\cdots<j_p,\\ j_p\ge \ell+1}} \left(\prod_{s=1}^{p-1} [j_s(n+j_s)]^{-\beta+\zeta}\right) [j_p(n+j_p)]^{-2\beta+1/2+2\zeta}.
\end{aligned}$$

By Lemma 6.6, ζ here could be chosen small enough so that the preceding inequality gives

$$\lim_{\ell\to\infty}\limsup_{N\to\infty} N^{-1}\left|\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{\substack{\tau+1\le j_1<\cdots<j_p,\\ j_p\ge\ell+1}}\left(\prod_{s=1}^{p} a_{j_s}a_{j'_s}\right)\operatorname{cov}\Big(W_{n,\,j_p,0},\; W_{n',\,j'_p,0}\Big)\right| = 0.$$

This and (6.27) imply (6.22) for $i = 2$.

The proof of (6.22) for $i = 3$ follows from (6.10), using the fact that (3.3) implies that

$$\sup_{j\le\ell} E\big[K_j(\tilde{X}_{1,\,j}) - K_j(\tilde{X}_{1,\,j,\,\ell})\big]^2 \le E[K(X_1) - K(X_{1,\,\ell})]^2 \to 0 \quad \text{as } \ell\to\infty,$$

and hence

$$P_{n,\,j,\,\ell} \to_{L_2} 0 \quad \text{as } \ell\to\infty \text{ uniformly in } n \text{ and } j\le\tau. \tag{6.28}$$

The proof of (6.22) for $i = 4$ follows from (6.11) together with

$$K_\infty^{(r)}(0) - K_\ell^{(r)}(0) \to 0 \quad \text{as } \ell\to\infty \text{ for all } r = 1,\ldots,p,$$

which is straightforward (see Lemmas 6.2 and 6.3). □

PROOF OF THEOREM 4.1. The proof is similar to that of Theorem 3.2 with $p = 0$; it suffices to show (4.1). As before,

$$\operatorname{var}\big(S_{N,0} - S_{N,0,\ell}\big) \le R_{N,1,\ell} + R_{N,2,\ell} + R_{N,3,\ell}$$

where, with the $P_{n,\,j,\,\ell}$ defines by (6.1),

$$R_{N,1,\ell} = 8\sum_{n=1}^{N}\sum_{j=\tau+1}^{\ell}\sum_{n'=n}^{n+\ell-j}\operatorname{cov}(P_{n,\,j,\,\ell}, P_{n',\,j',\,\ell}),$$

$$R_{N,2,\ell} = 8\sum_{n=1}^{N}\sum_{n'=n}^{N}\sum_{j=\ell+1}^{\infty}\operatorname{cov}(P_{n,\,j,\,\ell}, P_{n',\,j',\,\ell}),$$

$$R_{N,3,\ell} = 4\sum_{n=1}^{N}\sum_{j=1}^{\tau}\sum_{n'=n}^{n+\tau-j}\operatorname{cov}\big(P_{n,\,j,\,\ell}, P_{n',\,j',\,\ell}\big).$$

By (v) and (vi) of Lemma 6.2 and the fact that the $|a_j|$ are summable,

$$\lim_{\ell\to\infty}\lim_{N\to\infty} N^{-1}\operatorname{var}(R_{N,1,\ell}) \le C\lim_{\ell\to\infty}\left(\sum_{i=\ell+1}^{\infty} a_i^2\right)\left(\sum_{j=1}^{\infty}|a_j|\right)^2 = 0$$

and

$$\lim_{\ell\to\infty}\lim_{N\to\infty} N^{-1}\operatorname{var}(R_{N,2,\ell}) \le C\lim_{\ell\to\infty}\left(\sum_{j=\ell+1}^{\infty}|a_j|\right)^2 = 0.$$

Finally, it follows readily from (6.28) that

$$\lim_{\ell\to\infty}\lim_{N\to\infty} N^{-1}\operatorname{var}(R_{N,3,\ell}) = 0.$$

This shows (4.1) and concludes the proof. □

To prove Theorems 5.1 and 5.2, we need a result due to Avram and Taqqu (1987). We first review some notation. For any fixed, preassigned positive integer r, let $\pi = (p_1, \dots, p_l)$ be a vector of positive integers such that $1 \le p_1 < \cdots < p_l$ and $p_1 + \cdots + p_l = r$. We denote by $(i)_t = (i_1, \dots, i_t)$ the t-tuples whose components i_s's are all distinct positive integers. Any t-tuple $(i)_t = (i_1, \dots, i_t)$ satisfying $1 \le i_s \le \ell$, $1 \le s \le t$, is denoted by $(i)_{t,\ell}$.

PROOF OF THEOREM 5.1. Let $\rho(m) = EX_n X_{n+m}$. Define

$$X_{n,\ell} = \sum_{i=1}^{\ell} a_i \varepsilon_{n-i} \quad \text{and} \quad X'_{n,\ell} = b_\ell^{-1} \sum_{i=1}^{\ell} a_i \varepsilon_{n-i},$$

where $b_\ell = \sum_{i=1}^{\ell} a_i^2$ ensuring $E(X'_{n,\ell})^2 = 1$. We first note that

$$\rho(m) = \sum_{i=1}^{\infty} a_i a_{i+m}, \qquad EX'_{n,\ell} X'_{n+m,\ell} = b_\ell^{-2} \sum_{i=1}^{\ell} a_i a_{i+m-n} I(m-n \le \ell - i). \tag{6.29}$$

For positive integer $M \ge k \vee (2\beta - 1)^{-1}$, define

$$K^M(x) = \sum_{j=k}^{M} \frac{h_j}{j!} H_j(x).$$

Since K^M is a polynomial,

$$K_\infty^{M(r)}(0) \equiv (K_\infty^M)^{(r)}(0) = EK^M(X_n)H_r(X_n) = h_r, \qquad r \ge 1,$$

and the technical conditions in Corollary 3.3 are easily verified with K^M in place of K. Thus, both the noncentral and central limit theorems there hold with K^M in place of K. Next, recall that

$$\begin{aligned} &E\biggl(\frac{1}{\sqrt{N}} \sum_{n=1}^{N} [K(X_n) - K^M(X_n)]\biggr)^2 \\ &\quad \le \sum_{j=M+1}^{\infty} \frac{h_j^2}{j!} \biggl(\frac{1}{N} \sum_{m,n=1}^{N} |\rho(m-n)|^j\biggr) \\ &\quad \le C\biggl(\sum_{m=-\infty}^{\infty} |\rho(m)|^{M+1}\biggr) \sum_{j=M+1}^{\infty} \frac{h_j^2}{j!} < \infty \end{aligned} \tag{6.30}$$

for some constant C. Since $\sqrt{N} = o(S_{N,k})$ if $k < (2\beta-1)^{-1}$, $\sigma_{N,k}^{-1} \sum_{n=1}^{N} K(X_n)$ has the same limiting distribution as $\sigma_{N,k}^{-1} \sum_{n=1}^{N} K^M(X_n)$, and (5.1) readily follows from the above discussion. Note also that the last sum on the right of (6.30) converges to 0 as $M \to \infty$, so that if $k > (2\beta - 1)^{-1}$ then $N^{-1/2}\sum_{n=1}^{N} K(X_n)$ satisfies the central limit theorem, and so does $N^{-1/2}\sum_{n=1}^{N} K^M(X_n)$. Thus, it remains to verify that the variance σ^2 of the limiting normal distribution is indeed as in (5.3). By (3.6) of Theorem 3.2, we need to show

$$\lim_{\ell\to\infty}\lim_{N\to\infty} N^{-1}E\biggl(\sum_{n=1}^{N} K^M(X_{n,\ell}) - \sum_{r=0}^{k-1} K_\ell^{M(r)}(0)Y_{N,r,\ell}\biggr)^2 = \sum_{j=k}^{M}\frac{h_j^2 c_j}{j!}, \tag{6.31}$$

where

$$c_j = \frac{1}{j!}\lim_{N\to\infty} E\biggl(\frac{1}{\sqrt{N}}\sum_{n=1}^{N} H_j(X_n)\biggr)^2 = \sum_{m=-\infty}^{\infty}\rho^j(m).$$

Denote by $P_j(x)$ the jth Appell polynomial associated with $X_{n,\ell}$ [cf. Avram and Taqqu (1987)], defined recursively by

$$P_j'(x) = j!P_{j-1}(x), \qquad P_0(x) = 1, \qquad EP_j(X_{n,\ell}) = \delta_j. \tag{6.32}$$

It is known that the Appell polynomials for standard normal reduce to the orthogonal Hermite polynomials. Recall that $K_\ell^M(0) = EK^M(X_{n,\ell})$. Write

$$K^M(X_{n,\ell}) - K_\ell^M(0) = \sum_{j=1}^{M}\frac{h_{j,\ell}}{j!}P_j(X_{n,\ell}), \tag{6.33}$$

where $h_{j,\ell} = K_\ell^{M(j)}(0)$ by the recursive relations of (6.32). It is clear that

$$\lim_{\ell\to\infty} b_\ell = 1, \qquad \lim_{\ell\to\infty} h_{j,\ell} = \begin{cases} 0, & \text{if } j < k,\\ h_j, & \text{if } j \ge k,\end{cases} \tag{6.34}$$

and

$$P_1(X_{n,\ell}) = b_\ell \sum_{i=1}^{\ell}(a_i/b_\ell)\varepsilon_{n-i} = b_\ell H_1(X'_{n,\ell}). \tag{6.35}$$

Using formula (3.9) of Avram and Taqqu (1987), we have for fixed $j \ge 2$,

$$\begin{aligned} P_j(X_{n,\ell}) &= b_\ell^j\biggl[j!\sum_{1\le i_1<\cdots<i_j\le\ell}\prod_{s=1}^{j}(a_{i_s}/b_\ell)\varepsilon_{n-i_s}\\ &\qquad + \sum_{\pi\ne(1,\dots,1)}\binom{j}{p_1,\dots,p_t}\sum_{(i)_{t,\ell}}{}'\prod_{s=1}^{t}(a_{i_s}/b_\ell)^{p_s}H_{p_s}(\varepsilon_{n-i_s})\biggr]\\ &= b_\ell^j H_j(X'_{n,\ell}). \end{aligned} \tag{6.36}$$

The t-tuple $\pi = (p_1,\dots,p_t)$ above satisfies $\sum_{i=1}^{t} p_i = j$. The summation $\sum'_{(i)_{t,\ell}}$ requires that $i_s < i_{s+1}$ if $p_s = p_{s+1}$. Combine (6.33), (6.35) and the last identity of (6.36) to get

$$\begin{aligned}S_{N,k-1,\ell} &= \sum_{n=1}^{N} K^M(X_{n,\ell}) - \sum_{r=0}^{k-1} K_\ell^{M(r)}(0) Y_{N,r,\ell}\\ &= \sum_{j=1}^{k-1} h_{j,\ell}\left[\frac{b_\ell^j}{j!}\sum_{n=1}^{N} H_j(X'_{n,\ell}) - Y_{N,j,\ell}\right] + \sum_{j=k}^{M} \frac{h_{j,\ell}}{j!} b_\ell^j \left[\sum_{n=1}^{N} H_j(X'_{n,\ell})\right]\\ &\equiv A_{N,\ell} + B_{N,\ell}.\end{aligned}$$

Since $E[H_i(X)H_j(Y)] = \delta(i-j)i!E(XY)$, for any bivariate normal (X,Y) with standard normal marginals, (6.36) implies $EA_{N,\ell}B_{N,\ell} = 0$,

$$\begin{aligned}&E\left(\frac{b_\ell^i}{i!}\sum_{n=1}^{N} H_i(X'_{n,\ell}) - Y_{N,i,\ell}\right)\\ &\qquad \times \left(\frac{b_\ell^j}{j!}\sum_{n=1}^{N} H_j(X'_{n,\ell}) - Y_{N,j,\ell}\right) = 0, \qquad i \neq j,\end{aligned} \tag{6.37}$$

and, for $j \geq 2$,

$$\begin{aligned}&N^{-1}E\left[\frac{b_\ell^j}{j!}\sum_{n=1}^{N} H_j(X'_{n,\ell}) - Y_{N,j,\ell}\right]^2\\ &\quad = \sum_{n=-N+1}^{N-1}\left(1 - \frac{|n|}{N}\right)\sum_{\pi\neq(1,\dots,1)} \frac{1}{p_1!\cdots p_t!}\\ &\qquad\qquad \times \sum_{(i)_{t,\ell}}{}' \prod_{s=1}^{t} (a_{i_s} a_{i_s+n})^{p_s} I(1 \leq i_s + n \leq \ell)\\ &\quad \leq B \sum_{n=0}^{N-1} \sum_{\pi\neq(1,\dots,1)} \frac{1}{p_1!\cdots p_t!} \sum_{1\leq i_1<\cdots<i_t<\infty} \prod_{s=1}^{t} [i_s(i_s+|n|)]^{-\beta' p_s},\end{aligned} \tag{6.38}$$

where $\beta' \in (1/2, \beta)$ and the constant B is independent of N and ℓ. Note that each t-tuple $\pi = (p_1, \dots, p_t)$ satisfies $t \leq j-1$ and $p_1 + \cdots + p_t = j$. Therefore, the preceding bound in (6.38) can be further bounded by

$$C \sum_{n=0}^{N-1} \sum_{i=1}^{\infty} [i(i+|n|)]^{-j(\beta'-1/2)-1} \leq C_1 \sum_{n=1}^{\infty} |n|^{-j(\beta'-1/2)-1} < \infty, \tag{6.39}$$

using Lemma 6.3. Both constants C and C_1 are independent of N and ℓ. Relations (6.34), (6.35), (6.36) and (6.39) imply

$$\lim_{\ell\to\infty} \lim_{N\to\infty} N^{-1} E A_{N,\ell}^2 = 0.$$

Hence it is enough to consider $B_{N,\ell}$. From (6.29) and (6.34), we have

$$\begin{aligned}
&\lim_{\ell\to\infty}\lim_{N\to\infty} N^{-1}EB_{N,\ell}^2\\
&\quad = \lim_{\ell\to\infty}\lim_{N\to\infty}\sum_{j=k}^{M}\frac{h_{j,\ell}^2}{j!}\biggl(\sum_{n=-\ell+1}^{\ell-1}\biggl(1-\frac{|n|}{N}\biggr)\biggl[\sum_{i=1}^{\ell}a_i a_{i+n} I(n+i\le \ell)\biggr]^j\biggr)\\
&\quad = \sum_{j=k}^{M}\frac{h_j^2}{j!}\biggl(\sum_{n=-\infty}^{\infty}\biggl[\sum_{i=1}^{\infty}a_i a_{i+n}\biggr]^j\biggr)\\
&\quad = \sum_{j=k}^{M}\frac{h_j^2 c_j}{j!},
\end{aligned}$$

which shows (6.31). It is clear that $\sigma^2 \ge c_k > 0$. The proof is complete. □

PROOF OF THEOREM 5.2. Fix $r \ge 1$ and l-tuple $\pi = (p_1, p_2, \dots, p_l)$ of positive integers such that $\sum_{j=1}^{l} p_j = r$. Define

$$T_{n,r,\pi} = \sum_{(i)_l}{}' \prod_{s=1}^{r} a_{i_s}^{p_s} H_{p_i}(\varepsilon_{n-i_s}),$$

where $\sum'_{(i)_l}$ runs over all l-tuples $(i)_l = (i_1, \dots, i_l)$ such that $i_m \ne i_n$ for $m \ne n$, and $i_m < i_{m+1}$ if $p_m = p_{m+1}$. By Theorem 1 of Avram and Taqqu (1987),

$$H_r(X_n) = r! \sum_{1\le j_1<\cdots<j_r<\infty} \prod_{s=1}^{r} a_{j_s}\varepsilon_{n-j_s} + \sum_{\pi\ne(1,\dots,1)} \binom{r}{p_1,\dots,p_l} T_{n,r,\pi}.$$

Thus,

$$\begin{aligned}
\sum_{n=1}^{N} K(X_n) &= \sum_{r=1}^{\infty}\frac{K_\infty^{(r)}(0)}{r!}\sum_{n=1}^{N} H_r(X_n)\\
&= \sum_{r=1}^{\infty} K_\infty^{(r)}(0) Y_{N,r} + \sum_{n=1}^{N}\sum_{r=1}^{\infty}\sum_{\pi\ne(1,\dots,1)}\binom{r}{p_1,\dots,p_l} T_{n,r,\pi}.
\end{aligned}$$

Thus,

$$Y_{N,\infty} = \sum_{n=1}^{N}\sum_{r=1}^{\infty}\sum_{\pi\ne(1,\dots,1)}\binom{r}{p_1,\dots,p_l} T_{n,r,\pi},$$

whose L_2 convergence is guaranteed by that of the Hermite expansion. It suffices to show that for any positive integers n_1, n_2, r, l, any r-tuple $(j_1, \dots, j_r)$ of positive integers such that $j_1 < \cdots < j_r$, any l-tuple $(p_1, \dots, p_l)$ of positive integers not equal to $(1, \dots, 1)$ and any l-tuple $(i_1, \dots, i_l)$ of positive integers,

$$E\biggl[\biggl(\prod_{s=1}^{r}\varepsilon_{n_1-j_s}\biggr)\biggl(\prod_{t=1}^{l} H_{p_t}(\varepsilon_{n_2-i_t})\biggr)\biggr] = 0. \qquad (6.40)$$

Clearly, (6.40) holds if $\{n_1 - j_s\colon 1 \le s \le r\} \ne \{n_2 - j_s\colon 1 \le s \le l\}$. But if $l = r$ and $\{n_1 - j_s\colon 1 \le s \le r\} = \{n_2 - i_t\colon 1 \le t \le r\}$, then

$$E\left[\left(\prod_{s=1}^{r} \varepsilon_{n_1 - j_s}\right)\left(\prod_{t=1}^{l} H_{p_t}(\varepsilon_{n_2 - i_t})\right)\right] = \prod_{i=1}^{r} E\left[H_1(\varepsilon_i) H_{p_i}(\varepsilon_i)\right],$$

which equals 0 since at least one of the p_i is not 1. This concludes the proof. □

The following lemmas handle some technical details required for the proofs above. In the sequel, given any bivariate function $H(x_1, x_2)$, we define

$$H_{d_1, d_2}(x_1, x_2) = \sup_{0 \le |x| \le d_1,\, 0 \le |y| \le d_2} |H(x_1 + x, x_2 + y)|.$$

For a smooth function $H(x_1, x_2)$, let

$$H^{(i,j)}(x_1, x_2) = \frac{\partial^{i,j} H(x_1, x_2)}{\partial x_1^i\, \partial x_2^j}$$

denote its (i, j)th partial derivatives, and let

$$H^{(i,j)}_{d_1, d_2}(x_1, x_2) = (H^{(i,j)})_{d_1, d_2}(x_1, x_2).$$

LEMMA 6.1. *Let $H(x_1, x_2)$ be a function with continuous partial derivatives $H^{(i,j)}(\cdot, \cdot)$, $i = 0, 1, \ldots, q+1$ and $j = 0, 1$. Define*

$$D(x, y, u) = \left| H(x + y, u) - \sum_{i=0}^{q} (i!)^{(-1)} y^i H^{(i,0)}(x, u) \right|^r, \qquad r \ge 1. \tag{6.41}$$

Let μ_i, $i = 1, 2, 3$ be probability measures on $\Re$ and $d\mu(x, y, u) = d\mu_1(x) d\mu_2(y) d\mu_3(u)$. Assume that for all real x,

$$H^{(i,0)}(x, 0) = 0, \qquad i = 0, 1, \ldots, q+1,$$

and there exists $\lambda > 0$ such that the following integrals are finite for $i = 1, \ldots, q+1$:

$$\begin{aligned} I_1^{(i)} &= \int [H^{(i,1)}_{\lambda,\lambda}(x, 0)]^{2r}\, d\mu_1(x), \\ I_2^{(i)} &= \int [H^{(i,0)}_{\lambda,0}(x, u)]^{2r}\, d\mu_1(x)\, d\mu_3(u), \\ I_3 &= \int [H^{(0,1)}_{0,\lambda}(x + y, 0)]^{2r}\, d\mu_1(x)\, d\mu_2(y), \\ I_4 &= \int |H(x + y, u)|^{2r}\, d\mu. \end{aligned} \tag{6.42}$$

Then

$$\int D(x, y, u)\, d\mu(x, y, u) \le A\left[\int y^{2r(q+1)}\, d\mu_2(y) \int u^{2r}\, d\mu_3(u)\right]^{1/2}, \tag{6.43}$$

where, for fixed q, r and λ, A is a finite constant whose value depends only on $\bigvee_{i=1}^{q} \int y^{2ri}\, d\mu_2(y)$ and $(\bigvee_{i=1}^{q+1} I_1^{(i)}) \vee (\bigvee_{i=1}^{q+1} I_2^{(i)}) \vee I_3 \vee I_4$ and can be taken to be increasing in both quantities.

PROOF. Write

$$\begin{aligned}\int D(x, y, u)\, d\mu &= \int_{\{|y|\le\lambda,\, |u|\le\lambda\}} + \int_{\{|y|\le\lambda,\, |u|>\lambda\}} + \int_{\{|y|>\lambda,\, |u|\le\lambda\}} + \int_{\{|y|>\lambda,\, |u|>\lambda\}} D(x, y, u)\, d\mu \\ &\equiv J_1 + J_2 + J_3 + J_4.\end{aligned}$$

We shall show that each J_i, $i = 1, 2, 3, 4$ is bounded by the right-hand side of (6.43). There are two key steps in our argument. The first one is to use Taylor's theorem to obtain for each $i = 0, 1, \dots, q+1$,

$$D(x, y, u) \le |y^{q+1} H_{\lambda,0}^{(q+1,0)}(x, u)|^r/q!, \qquad |y| \le \lambda,\ x, u \in \Re$$

and

$$|H^{(i,0)}(x, u)|^r \le |u|^r [H_{0,\lambda}^{(i,1)}(x, 0)]^r, \qquad |u| \le \lambda,\ x \in \Re.$$

The second step is to observe that, by the Chebyshev inequality,

$$\int_{\{|y|>\lambda,\, |u|>\lambda\}} d\mu_2(y)\, d\mu_3(u) \le \lambda^{-2r(q+1)-2r} \int y^{2r(q+1)}\, d\mu_2 \int u^{2r}\, d\mu_3.$$

Repeatedly using these two steps and Hölder's inequality, we obtain

$$\begin{aligned}
J_1 &\le \Big[\int \big(H_{\lambda,\lambda}^{(q+1,1)}(x, 0)\big)^r\, d\mu_1(x)\Big] \int |y|^{r(q+1)}\, d\mu_2(y) \int |u|^r\, d\mu_3(u), \\
J_2 &\le \int |y|^{r(q+1)} I(|u| > \lambda) \big(H_{\lambda,0}^{(q+1,0)}(x, u)\big)^r\, d\mu \\
&\le \Big[\int (H_{\lambda,0}^{(q+1,0)}(x, u))^{2r}\, d\mu_1(x)\, d\mu_3(u)\Big]^{1/2} \\
&\qquad \times \int |y|^{r(q+1)}\, d\mu_2(y) \Big[\lambda^{-2r} \int |u|^{2r}\, d\mu_3(u)\Big]^{1/2}, \\
J_3 &\le \int I(|y| > \lambda, |u| \le \lambda) |u|^r \\
&\qquad \times \Big[H_{0,\lambda}^{(0,1)}(x+y, 0) + \sum_{i=0}^{q} (i!)^{-1} |y|^i H_{0,\lambda}^{(i,1)}(x, 0)\Big]^r d\mu \\
&\le \lambda^{-r(q+1)} \Big\{\int |y|^{2r(q+1)}\, d\mu_2(y) \int |u|^{2r}\, d\mu_3(u)\Big\}^{1/2} \\
&\qquad \times (q+2)^r \Big\{\int \big[H_{0,\lambda}^{(0,1)}(x+y, 0)\big]^{2r}\, d\mu_1(x)\, d\mu_2(y) \\
&\qquad\qquad + \sum_{i=0}^{q} (i!)^{-2r} \int |y|^{2ri}\, d\mu_2(y) \int [H_{0,\lambda}^{(i,1)}(x, 0)]^{2r}\, d\mu_1(x)\Big\}^{1/2},
\end{aligned}$$

$$J_4 \le \lambda^{-r(q+2)}\Big\{\int |y|^{2r(q+1)}\,d\mu_2(y)\int |u|^{2r}\,d\mu_3(u)\Big\}^{1/2}$$
$$\times (q+2)^r\bigg\{\int |H(x+y,u)|^{2r}\,d\mu + \sum_{i=0}^{q}(i!)^{-2r}\int |y|^{2ri}\,d\mu_2(y)$$
$$\times \int |H^{(i,0)}(x,u)|^{2r}\,d\mu_1(x)\,d\mu_3(u)\bigg\}^{1/2},$$

and (6.43) is then immediate. □

LEMMA 6.2. *If $E\varepsilon_1^8 < \infty$ and for some $\tau \ge 0$ and $\lambda > 0$, condition $C(t,\tau,\lambda)$ holds for $t = 0,\dots,p+2$, then:*

(i) *For all $t = 0,1,\dots,p$ and $k = i$ or $i+1$ where $i \ge \tau$,*

$$E\big[K_i^{(t)}(\tilde X_{n,k}) - K_{i+1}^{(t)}(\tilde X_{n,k+1}) - a_{k+1}\varepsilon_{n-k-1}K_k^{(t+1)}(\tilde X_{n,k+1})\big]^2 \le C(a_k^4 + a_{k+1}^4).$$

(ii) *For all $t = 0,1,\dots,p$ and $k = i$ or $i+1$ with $\tau \le i < i+1 \le \ell - 1$ and $\ell \ge \tau + 1$,*

$$E\big\{\big[K_i^{(t)}(\tilde X_{n,k}) - K_{i+1}^{(t)}(\tilde X_{n,k+1}) - a_{k+1}\varepsilon_{n-k-1}K_k^{(t+1)}(\tilde X_{n,k+1})\big]$$
$$- \big[K_i^{(t)}(\tilde X_{n,k,\ell}) - K_{i+1}^{(t)}(\tilde X_{n,k+1,\ell}) - a_{k+1}\varepsilon_{n-k-1}K_k^{(t+1)}(\tilde X_{n,k+1,\ell})\big]\big\}^2$$
$$\le C\bigg(\sum_{m=\ell+1}^{\infty} a_m^2\bigg)(a_k^4 + a_{k+1}^4).$$

(iii) *For $j \ge \tau + 1$,*

$$E\big[K_j^{(p)}(\tilde X_{n,j}) - K_\infty^{(p)}(0)\big]^2 \le C\sum_{m=j}^{\infty} a_m^2.$$

(iv) *For $\tau + 1 \le j \le \ell$,*

$$E\big\{\big[K_j^{(p)}(\tilde X_{n,j}) - K_\infty^{(p)}(0)\big] - \big[K_j^{(p)}(\tilde X_{n,j,\ell}) - K_\ell^{(p)}(0)\big]\big\}^2 \le C\sum_{m=\ell+1}^{\infty} a_m^2,$$

where C is a universal constant.

In addition, if $E\varepsilon_1^4 < \infty$ and for some $\tau \ge 0$ and $\lambda > 0$, condition $C(t,\tau,\lambda)$ holds for $t = 0, 1$, then:

(v) *For $\tau + 1 \le j \le \ell$,*

$$E\big\{\big[K_{j-1}(\tilde X_{n,j-1}) - K_j(\tilde X_{n,j})\big] - \big[K_{j-1}(\tilde X_{n,j-1,\ell}) - K_j(\tilde X_{n,j,\ell})\big]\big\}^2$$
$$\le C\bigg(\sum_{i=\ell+1}^{\infty} a_i^2\bigg)(a_{j-1}^2 + a_j^2).$$

(vi) *For $j \ge \ell + 1$,*

$$E\big[K_{j-1}(\tilde X_{n,j-1}) - K_j(\tilde X_{n,j})\big]^2 \le C(a_{j-1}^2 + a_j^2),$$

where C is a universal constant.

PROOF. We only prove (i)–(iv) since the proofs of (v) and (vi) are similar.

(i) Taking into account that $\int u\,dG(u) = 0$ and $K_{i+1}^{(t)}(x) = \int K_i^{(t)}(x + a_{i+1}x')\,dG(x')$ (see Lemma 2.1), we write for $k = i+1$,

$$K_i^{(t)}(\tilde{X}_{n,\,i+1}) - K_{i+1}^{(t)}(\tilde{X}_{n,\,i+2}) - a_{i+2}\varepsilon_{n-i-2}K_{i+1}^{(t+1)}(\tilde{X}_{n,\,i+2}) = J_1 - J_2,$$

where

$$J_1 = \int\big[K_i^{(t)}(\tilde{X}_{n,\,i+2} + a_{i+2}\varepsilon_{n-i-2} + a_{i+1}x') - K_i^{(t)}(\tilde{X}_{n,\,i+2} + a_{i+1}x') \\ - a_{i+2}\varepsilon_{n-i-2}K_i^{(t+1)}(\tilde{X}_{n,\,i+2} + a_{i+1}x')\big]\,dG(x')$$

and

$$J_2 = \int\big[K_i^{(t)}(\tilde{X}_{n,\,i+2} + a_{i+2}\varepsilon_{n-i-2} + a_{i+1}x') - K_i^{(t)}(\tilde{X}_{n,\,i+2} + a_{i+2}\varepsilon_{n-i-2}) \\ - a_{i+1}x'K_i^{(t+1)}(\tilde{X}_{n,\,i+2} + a_{i+2}\varepsilon_{n-i-2})\big]\,dG(x').$$

We shall apply Lemma 6.1 to bound $E|J_1|^2$ and $E|J_2|^2$. First, choose $q = 1$, $r = 2$, and

$$H(x_1, x_2) = K_i^{(t)}(x_1)x_2, \qquad t = 0, 1, \ldots, p$$

in (6.41). By writing

$$J_1 = E\big[K_i^{(t)}(\tilde{X}_{n,\,i+2} + a_{i+2}\varepsilon_{n-i-2} + a_{i+1}\varepsilon_{n-i-1}) - K_i^{(t)}(\tilde{X}_{n,\,i+2} + a_{i+1}\varepsilon_{n-i-1}) \\ - a_{i+2}\varepsilon_{n-i-2}K_i^{(t+1)}(\tilde{X}_{n,\,i+2} + a_{i+1}\varepsilon_{n-i-1})\big|\tilde{X}_{n,\,i+2}, \varepsilon_{n-i-1}\big],$$

it is clear by the Cauchy–Schwarz inequality that

$$\begin{aligned} E|J_1|^2 &\le E\big[K_i^{(t)}(\tilde{X}_{n,\,i+2} + a_{i+2}\varepsilon_{n-i-2} + a_{i+1}\varepsilon_{n-i-1}) \\ &\qquad - K_i^{(t)}(\tilde{X}_{n,\,i+2} + a_{i+1}\varepsilon_{n-i-1}) \\ &\qquad - a_{i+2}\varepsilon_{n-i-2}K_i^{(t+1)}(\tilde{X}_{n,\,i+2} + a_{i+1}\varepsilon_{n-i-1})\big]^2 \\ &= \int\int\int\big|H(x+y, u) - H(x, u) - yH^{(1,0)}(x, u)\big|^2 \\ &\qquad \times d\mu_1(x)\mu_2(y)\,d\mu_3(u), \end{aligned} \tag{6.44}$$

where μ_1 is the convolution of the distributions $\tilde{F}_{i+2}$ and G_{i+1}, μ_2 is the distribution of G_{i+2}, and μ_3 is the probability measure concentrated at $\{1\}$. It is straightforward to check that the integral conditions (6.42) in Lemma 6.1 are satisfied, using conditions $C(t, \tau, \lambda)$. For example, with $s = 0, 1, 2$, by the Cauchy–Schwarz inequality,

$$\int[H_{\lambda,\,0}^{(s,0)}(x, u)]^4\,d\mu_1(x)\,d\mu_3(u) = \int[K_{i,\,\lambda}^{(t+s)}(x)]^4\,d\mu_1(x) \\ \le E\big[K_{\tau,\,\lambda}^{(t+s)}(\tilde{X}_{n,\,\tau} - a_{i+2}\varepsilon_{n-(i+2)})\big]^4,$$

which is uniformly bounded for all i, by (ii) of condition $C(t, \tau, \lambda)$. Hence, combining (6.43) and (6.44), we have

$$E|J_1|^2 \le C\Big[\int y^8\, d\mu_2\Big]^{1/2} \le Ca_{i+2}^4$$

for some universal constant C. The proof for J_2 is similar, except for switching the roles of $a_{i+2}\varepsilon_{n-i-2}$ and $a_{i+1}\varepsilon_{n-i-1}$. Specifically, let μ_1 and μ_2 be the distributions of $\tilde{F}_{i+1}$ (i.e., the convolution of $\tilde{F}_{i+2}$ and G_{i+2}) and G_{i+1}, respectively. Then, by the Cauchy–Schwarz inequality and Lemma 6.1 again,

$$E|J_2|^2 \le Ca_{i+1}^4,$$

and (i) is immediate for $k = i+1$. The case $k = i$ is handled by the same argument and the details are omitted.

(ii) We focus on the case $k = i+1$ and write

$$\begin{aligned}
&\big[K_i^{(t)}(\tilde{X}_{n,i+1}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2}) - a_{i+2}\varepsilon_{n-i-2}K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2})\big]\\
&\quad - \big[K_i^{(t)}(\tilde{X}_{n,i+1,\ell}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell}) - a_{i+2}\varepsilon_{n-i-2}K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2,\ell})\big]\\
&\qquad = J_1 - J_2,
\end{aligned}$$

where this time

$$\begin{aligned}
J_1 &= \big[K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell} + \tilde{X}_{n,\ell} + a_{i+2}\varepsilon_{n-i-2}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell} + a_{i+2}\varepsilon_{n-i-2})\big]\\
&\quad - \big[K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell} + \tilde{X}_{n,\ell}) - K_{i+1}^{(t)}(\tilde{X}_{n,i+2,\ell})\big]\\
&\quad - a_{i+2}\varepsilon_{n-i-2}\big[K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2,\ell} + \tilde{X}_{n,\ell}) - K_{i+1}^{(t+1)}(\tilde{X}_{n,i+2,\ell})\big],\\
J_2 &= \int\big\{\big[K_i^{(t)}(\tilde{X}_{n,i+1,\ell} + \tilde{X}_{n,\ell} + a_{i+1}u') - K_i^{(t)}(\tilde{X}_{n,i+1,\ell} + a_{i+1}u')\big]\\
&\qquad - \big[K_i^{(t)}(\tilde{X}_{n,i+1,\ell} + \tilde{X}_{n,\ell}) - K_i^{(t)}(\tilde{X}_{n,i+1,\ell})\big]\\
&\qquad - a_{i+1}u'\big[K_i^{(t+1)}(\tilde{X}_{n,i+1,\ell} + \tilde{X}_{n,\ell}) - K_i^{(t+1)}(\tilde{X}_{n,i+1,\ell})\big]\big\}\, dG(u').
\end{aligned}$$

First of all we need to identify the function $H(x_1, x_2)$ and the probability measures $\mu_i,\ i = 1, 2, 3$. For J_1, set

$$H(x_1, x_2) = K_{i+1}^{(t)}(x_1 + x_2) - K_{i+1}^{(t)}(x_1),$$

and let μ_1 and μ_2 be the distributions of $\tilde{X}_{n,i+2,\ell}$ and $a_{i+2}\varepsilon_{n-i-2}$, respectively. For J_2, set

$$H(x_1, x_2) = K_i^{(t)}(x_1 + x_2) - K_i^{(t)}(x_1),$$

and let μ_1 and μ_2 be the distributions of $\tilde{X}_{n,i+2,\ell}$ and $a_{i+1}\varepsilon_{n-i-1}$, respectively. Let μ_3 be the distribution of $\tilde{X}_{n,\ell}$ for both J_1 and J_2. Applying the Cauchy–

Schwarz inequality and (6.43) with $q = 1$, we obtain

$$E|J_h|^2 \le C\big(E|\tilde{X}_{n,\ell}|^4 E|a_{i+h}\varepsilon_{n-i-2}|^8\big)^{1/2}$$
$$\le C\bigg(\sum_{m\ge \ell+1} a_m^2\bigg)a_{i+h}^4, \qquad h = 1, 2.$$

Again, as in part (i) above, the integral conditions (6.42) of Lemma 6.1 can be verified by using (ii) of condition $C(t, \tau, \lambda)$.

(iii) Write

$$K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0) = \int \big(K_j^{(p)}(\tilde{X}_{n,j}) - K_j^{(p)}(y)\big)\, d\tilde{F}_j(y).$$

Hence

$$E\big[K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0)\big]^2 \le 2\bigg[\int |K_j^{(p)}(y_1) - K_j^{(p)}(0)|^2\, d\tilde{F}_j(y_1)$$
$$+ \int |K_j^{(p)}(y_2) - K_j^{(p)}(0)|^2\, d\tilde{F}_j(y_2)\bigg].$$

Set $q = 0$ and

$$H(x_1, x_2) = K_j^{(p)}(x_1)x_2.$$

Choose the probability measures μ_i, $i = 1, 2, 3$, to be such that $\mu_1(\{0\}) = 1$, $\mu_3(\{1\}) = 1$ and μ_2 is $\tilde{F}_j$ or $\tilde{F}_{j-1}$. Then, by the Cauchy–Schwarz inequality and (6.43) with $r = 2$,

$$E\big[K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0)\big]^2 \le B_1\bigg\{\bigg[\int |y|^4\, d\tilde{F}_j(y)\bigg]^{1/2} + \bigg[\int |y|^4\, d\tilde{F}_j(y)\bigg]^{1/2}\bigg\}$$
$$\le B_2 \sum_{m=j}^{\infty} a_m^2$$

for some constants B_1 and B_2.

(iv) Write

$$\big[K_j^{(p)}(\tilde{X}_{n,j}) - K_\infty^{(p)}(0)\big] - \big[K_j^{(p)}(\tilde{X}_{n,j,\ell}) - K_\ell^{(p)}(0)\big] = J_1 - J_2,$$

where now

$$J_1 = \big[K_j^{(p)}(\tilde{X}_{n,j,\ell} + \tilde{X}_{n,\ell}) - K_j^{(p)}(\tilde{X}_{n,j,\ell})\big],$$
$$J_2 = \int \big[K_\ell^{(p)}(y) - K_\ell^{(p)}(0)\big]\, d\tilde{F}_\ell(y).$$

Let

$$H(x_1, x_2) = K_j^{(p)}(x_1)x_2,$$

and let μ_1 and μ_2 be $\tilde{F}_{j,\ell}$ and $\tilde{F}_\ell$, respectively. Let μ_3 satisfy $\mu_3(\{1\}) = 1$. Then by the Cauchy–Schwarz inequality and (6.43) with $q = 0$ and $r = 2$,

$$EJ_1^2 \le C \int |H(x+y,u) - H(x,u)|^2 \, d\mu_1(x) \, d\mu_2(y) \mu_3(u)$$
$$\le C\Big(\int y^4 \, d\mu_2\Big)^{1/2} \le C \sum_{m=\ell+1}^{\infty} a_m^2$$

for some finite constant C. The bound for EJ_2^2 is the same, where the only changes are: $H(x_1, x_2)$ is defined to be $K_\ell^{(p)}(x_1)x_2$ instead of $K_j^{(p)}(x_1)x_2$ and the probability measure μ_1 is now concentrated at $\{0\}$. □

LEMMA 6.3. *Given constants $\gamma_1, \dots, \gamma_t > 1/2$, and $t \ge 1$, there exists $C < \infty$ such that for all $n, \ell \ge 1$,*

$$\sum_{\ell+1 \le j_1 < j_2 < \cdots < j_t < \infty} \prod_{s=1}^{t} [j_s(n+j_s)]^{-\gamma_s}$$
$$\le C \sum_{j=\ell+1}^{\infty} [j(n+j)]^{-\gamma}$$
$$\le \begin{cases} C(n \vee \ell)^{-2\gamma+1}, & \text{if } \gamma \in (1/2, 1), \\ Cn^{-1}\Big[I(n > \ell) \log\Big(\dfrac{n}{\ell}\Big) + I(n \le \ell)\dfrac{n}{\ell}\Big], & \text{if } \gamma = 1, \\ C(n \vee \ell)^{-\gamma}\ell^{1-\gamma}, & \text{if } \gamma \in (1, \infty), \end{cases}$$

where $\gamma = \sum_{s=1}^{t} \gamma_s - (t-1)/2$.

PROOF. The first inequality follows as in the proof of Lemma 6.4 of Ho and Hsing (1994). We illustrate the proof of the second inequality by considering the case $\gamma \in (1, \infty)$. By elementary arguments,

$$\sum_{j=\ell+1}^{\infty} [j(n+j)]^{-\gamma} \le n^{-2\gamma+1} \int_{\ell/n}^{\infty} (y^{-2\gamma} \wedge y^{-\gamma}) \, dy$$
$$\le \begin{cases} n^{-2\gamma+1}\Big(\displaystyle\int_{\ell/n}^{1} y^{-\gamma} \, dy + \int_{1}^{\infty} y^{-2\gamma} \, dy\Big), & \text{if } n > \ell, \\ n^{-2\gamma+1} \displaystyle\int_{\ell/n}^{\infty} y^{-2\gamma} \, dy, & \text{if } n \le \ell, \end{cases}$$

so that

$$\sum_{j=\ell+1}^{\infty} [j(n+j)]^{-\gamma} \le C(n \vee \ell)^{-\gamma}\ell^{1-\gamma}.$$ □

LEMMA 6.4. *For $\delta \in (1/2, 1)$, there exists a constant C such that for all $n, \ell \ge 1$,*

$$\sum_{j=1}^{\ell} [j(n+j)]^{-\delta} \le C(n \wedge \ell)^{1-\delta} n^{-\delta}.$$

PROOF. As in the previous proof, we have

$$\sum_{j=1}^{\ell}[j(n+j)]^{-\delta} \le \begin{cases} n^{1-2\delta}\int_0^{\ell/n} y^{-\delta}\,dy, & \text{if } n > \ell, \\ n^{1-2\delta}\left(\int_0^1 y^{-\delta}\,dy + \int_1^{\ell/n} y^{-2\delta}\,dy\right), & \text{if } n \le \ell, \end{cases}$$

and hence the conclusion. □

LEMMA 6.5. *Given constants* $\gamma_1, \dots, \gamma_{t-1} \in (1/2, 1)$, *there exists* $C < \infty$ *such that for all* $n, \ell \ge 1$,

$$\sum_{\substack{1 \le j_1 < j_2 < \cdots < j_t, \\ j_t \ge \ell+1}} \prod_{s=1}^{t} [j_s(n+j_s)]^{-\gamma_s} \le C \sum_{k=0}^{t-1} B(k, n, \ell),$$

where, with $\gamma(k) = \sum_{s=k+1}^{t} \gamma_s - (t-k-1)/2$,

$$B(k,n,\ell) = (n \wedge \ell)^{\sum_{s=1}^{k}(1-\gamma_s)} n^{-\sum_{s=1}^{k}\gamma_s} \times \begin{cases} (n \vee \ell)^{-2\gamma(k)+1}, & \text{if } \gamma(k) \in (1/2, 1), \\ n^{-1}\left[I(n > \ell)\log\left(\dfrac{n}{\ell}\right) + I(n \le \ell)\dfrac{n}{\ell}\right], & \text{if } \gamma(k) = 1, \\ (n \vee \ell)^{-\gamma(k)} \ell^{1-\gamma(k)}, & \text{if } \gamma(k) \in (1, \infty). \end{cases}$$

PROOF. Observe that

$$\begin{aligned} &\sum_{\substack{1 \le j_1 < j_2 < \cdots < j_t, \\ j_t \ge \ell+1}} \prod_{s=1}^{t} [j_s(n+j_s)]^{-\gamma_s} \\ &\quad = \sum_{j_1=\ell+1}^{\infty} \sum_{j_2=j_1+1}^{\infty} \cdots \sum_{j_t=j_{t-1}+1}^{\infty} \prod_{s=1}^{t} [j_s(n+j_s)]^{-\gamma_s} \\ &\qquad + \sum_{j_1=1}^{\ell} \sum_{j_2=\ell+1}^{\infty} \cdots \sum_{j_t=j_{t-1}+1}^{\infty} \prod_{s=1}^{t} [j_s(n+j_s)]^{-\gamma_s} \\ &\qquad + \cdots + \sum_{j_1=1}^{\ell} \sum_{j_2=j_1+1}^{\ell} \cdots \sum_{j_k=j_{k-1}+1}^{\ell} \sum_{j_{k+1}=\ell+1}^{\infty} \sum_{j_{k+2}=k_{k+1}+1}^{\infty} \cdots \\ &\qquad\qquad \sum_{j_t=j_{t-1}+1}^{\infty} \prod_{s=1}^{t} [j_s(n+j_s)]^{-\gamma_s} \\ &\qquad + \cdots + \sum_{j_1=1}^{\ell} \sum_{j_2=j_1+1}^{\ell} \cdots \sum_{j_{t-1}=j_{t-2}+1}^{\ell} \sum_{j_t=\ell+1}^{\infty} \prod_{s=1}^{t} [j_s(n+j_s)]^{-\gamma_s}, \end{aligned}$$

where the $(k+1)$th term on the right is bounded by

$$\prod_{s=1}^{k} \sum_{j=1}^{\ell} [j(n+j)]^{-\gamma_s} \sum_{j_{k+1}=\ell+1}^{\infty} \sum_{j_{k+2}=j_{k+1}+1}^{\infty} \cdots \sum_{j_t=j_{t-1}+1}^{\infty} \prod_{s=k+1}^{t} [j_s(n+j_s)]^{-\gamma_s},$$

which, by Lemmas 6.3 and 6.4, is bounded by a constant times $B(k, n, \ell)$. □

LEMMA 6.6. *Let* $(p+1)(2\beta-1)>1$. *Then there exists* $\zeta>0$ *such that*

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}\sum_{n=0}^{N-1}\sum_{\substack{1\le j_1<j_2<\cdots<j_p,\\ j_p\ge\ell+1}}\left(\prod_{s=1}^{p-1}[j_s(n+j_s)]^{-\beta+\zeta}\right)[j_p(n+j_p)]^{-2\beta+1/2+2\zeta}=0.$$

PROOF. First consider the sum for $n=1$ to $N-1$. Use the notation of Lemma 6.5 with $t=p$ and $\gamma_s=\beta-\zeta$ for $1\le s\le p-1$ and $2\beta-1/2-2\zeta$ for $s=p$. Then

$$\begin{aligned}\gamma(k)&=\sum_{s=k+1}^{p-1}(\beta-\zeta)+2\beta-1/2-2\zeta-\frac{p-k-1}{2}\\ &=(p-k+1)(\beta-1/2-\zeta)+\tfrac{1}{2},\qquad 0\le k\le p-1.\end{aligned}$$

Clearly there exists a small $\zeta>0$ such that $\gamma(0)>1$ and

$$\gamma(k)>1\quad\text{or}\quad 1/2<\gamma(k)<1,\qquad k=1,\dots,p-1.$$

If $\gamma(k)>1$ then

$$\begin{aligned}\sum_{n=1}^{N-1}B(n,k,\ell)&\le\sum_{n=1}^{N-1}n^{\sum_{s=1}^{k}(1-2\gamma_s)}(n\vee\ell)^{-\gamma(k)}\ell^{1-\gamma(k)}\\ &\le\sum_{n=1}^{N-1}(n\vee\ell)^{-\gamma(k)}\ell^{1-\gamma(k)}\\ &\le C\ell^{2-2\gamma(k)}.\end{aligned}$$

Hence

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}\sum_{n=1}^{N-1}B(n,k,\ell)=0. \tag{6.45}$$

Now, for $1/2<\gamma(k)<1$, $k=1,\dots,p-1$,

$$\begin{aligned}\sum_{n=1}^{N-1}B(n,k,\ell)&=\sum_{n=1}^{N-1}(n\wedge\ell)^{k(1-\beta+\zeta)}n^{-k(\beta-\zeta)}(n\vee\ell)^{-2[(p-k+1)(\beta-1/2-\zeta)+1/2]+1}\\ &=\ell^{-2[(p-k+1)(\beta-1/2-\zeta)+1/2]+1}\sum_{n=1}^{\ell}n^{-2k(\beta-1/2-\zeta)}\\ &\quad+\ell^{k(1-\beta+\zeta)}\sum_{n=\ell+1}^{N-1}n^{-k(\beta-\zeta)-2[(p-k+1)(\beta-1/2-\zeta)+1/2]+1}\\ &\le C\ell^{1-2(p+1)(\beta-1/2-\zeta)},\end{aligned}$$

since

$$\begin{aligned}&-k(\beta-\zeta)-2[(p-k+1)(\beta-1/2-\zeta)+1/2]+1\\ &\quad<-k(1-\beta)-1+(2p+2-k)\zeta<-1.\end{aligned}$$

Thus, (6.45) holds again. It remains to deal with the summand that corresponds to $n=0$ and show that

$$\lim_{\ell\to\infty}\limsup_{N\to\infty}\sum_{\substack{1\le j_1<j_2<\cdots<j_p,\\ j_p\ge\ell+1}}\left(\prod_{s=1}^{p-1} j_s^{-2(\beta-\zeta)}\right) j_p^{-2(2\beta-1/2-2\zeta)}=0,$$

which is straightforward and the details are omitted. This concludes the proof. □

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