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D. Braess, R. Verfürth

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Abstract

The present paper analyzes a multigrid algorithm for the Crouzeix-Raviart discretization of the Poisson and Stokes equations in two and three dimensions. The central point is the construction of easily computable L2-projections based on suitable quadrature rules for the transfer from coarse to fine grids and vice versa.

MULTIGRID METHODS FOR NONCONFORMING FINITE ELEMENT METHODS*

D. BRAESS† AND R. VERFÜRTH‡

Abstract. The present paper analyzes a multigrid algorithm for the Crouzeix–Raviart discretization of the Poisson and Stokes equations in two and three dimensions. The central point is the construction of easily computable L^2-projections based on suitable quadrature rules for the transfer from coarse to fine grids and vice versa.

Key words. multigrid methods, nonconforming finite element methods

AMS(MOS) subject classifications. 65N30, 65N20

1. Introduction. The so-called Crouzeix–Raviart element [5] is one of the most popular nonconforming finite element methods for the discretization of elliptic systems of second-order partial differential equations in two and three dimensions. When standard multigrid algorithms are applied to the resulting discrete problems, the main difficulty lies in the construction of suitable prolongation and restriction operators for the transfer from coarse to fine grids and vice versa.

Extending the techniques of [8], we overcome this difficulty by constructing easily computable L^2-projections based on suitable quadrature rules. The resulting prolongation and restriction operators are natural extensions of the standard ones.

To simplify the exposition, we give a detailed convergence analysis of the resulting multigrid algorithm for the Poisson equation. Afterwards, we comment on the generalization to the Stokes equations. The generalization also shows how the method may be extended to other nonconforming elements. The convergence analysis follows the framework of [1] and [2] and applies to the W-cycle with Jacobi relaxation as smoothing procedure. For technical reasons we assume that the finite element partitions are uniform. But the definition of the projections and the formulation of the multigrid algorithm immediately carry over to locally refined grids.

2. The Crouzeix–Raviart element. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded, connected polyhedral domain with boundary Γ. $H^m(\Omega)$, $m \in \mathbb{N}$, and $L^2(\Omega) := H^0(\Omega)$ are the usual Sobolev and Lebesgue spaces equipped with the norm

$$\|u\|_m := \left\{ \sum_{|\alpha| \leq m} \int_\Omega |D^\alpha u|^2 \right\}^{1/2}.$$

The inner product of $L^2(\Omega)$ is denoted by

$$(u, v) := \int_\Omega uv.$$

Finally, $H_0^1(\Omega)$ is the subspace of all $H^1(\Omega)$-functions that vanish on Γ.

Let $\mathcal{T}_0$ be a partition of Ω into d-simplices such that any two simplices share at most a whole lower-dimensional simplex. For $k = 1, 2, \cdots$, the partition $\mathcal{T}_k$ is obtained

from $\mathcal{T}_{k-1}$ by subdividing each $T \in \mathcal{T}_{k-1}$ into 2^d d-simplices by joining the midpoints of the edges. Denote by h_k the longest side of all $T \in \mathcal{T}_k$. Obviously, we have $h_k = h_{k-1}/2$ for all $k \geq 1$.

Given two simplices T_i, $T_j \in \mathcal{T}_k$, $k \geq 0$, which share a common face, we denote by x_{ij} the barycentre of this face. Similarly, the points $x_{i0} \in \Gamma$ are defined by formally replacing T_j by $\mathbb{R}^d \setminus \Omega$. Let P_m, $m \geq 0$, be the space of all polynomials of degree less than or equal to m. Then the Crouzeix–Raviart element corresponding to $\mathcal{T}_k$ is defined by (cf. [5])

$$(2.1) \quad S_k := \{u \in L^2(\Omega) : u|_T \in P_1 \forall T \in \mathcal{T}_k, u|_{T_i}(x_{ij}) = u|_{T_j}(x_{ij}) \forall i, j, \text{ and } u(x_{i0}) = 0 \forall i\}.$$

Note that $S_k \not\subset H_0^1(\Omega)$ and that $S_{k-1} \not\subset S_k$, $k \geq 0$.

We define a bilinear form a_k and a seminorm $|.|_{1,k}$ on $S_k \oplus H_0^1(\Omega)$ by

$$(2.2) \qquad a_k(u, v) := \sum_{T \in \mathcal{T}_k} \int_T \nabla u \nabla v \quad \forall u, v \in S_k \oplus H_0^1(\Omega),$$

$$(2.3) \qquad |u|_{1,k} := a_k(u, u)^{1/2} \quad \forall u \in S_k \oplus H_0^1(\Omega).$$

Obviously, $|.|_{1,k}$ provides a norm on S_k.

With this notation, the Crouzeix–Raviart discretization of the Poisson equation

$$(2.4) \qquad -\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma$$

is given by the following.

$$(2.5) \qquad \text{Find } u_k \in S_k \text{ such that } a_k(u_k, v_k) = (f, v_k) \text{ for all } v_k \in S_k.$$

Problem (2.5) has a unique solution u_k^*. Moreover, Theorems 4.2.2 and 4.2.5 and Exercise 4.2.3 in [4] yield the error estimate

$$(2.6) \qquad \|u - u_k^*\|_0 + h_k |u - u_k^*|_{1,k} \leq c h_k^2 \|f\|_0$$

where the constant c is independent of k (compare also the proof of equations (5.11) and (5.12) in [5]). For later use we note that (2.6) remains valid if the right-hand side of (2.5) is evaluated by numerical integration, provided the quadrature rule is exact for linear finite elements and $f \in S_k$ (cf. the proof of Theorem 4.1.1 in [4]).

3. A multigrid algorithm for the Poisson equation. Since the Crouzeix–Raviart element is nonconforming and $S_{k-1} \not\subset S_k$, we have to establish a suitable transfer between S_{k-1} and S_k. To this end, consider a simplex T and denote its measure by $|T|$ and the barycentres of its faces by $b_1, \cdots, b_{d+1}$. Then the quadrature rule

$$Q_T(u) := \frac{|T|}{d+1} \sum_{i=1}^{d+1} u(b_i)$$

is exact for all $u \in P_2$, if $d = 2$, and for all $u \in P_1$, if $d = 3$. Referring to Q_T, we introduce a mesh-dependent L^2-inner product on $S_{k-1} \oplus S_k$ by

$$(3.1) \qquad (u, v)_k := \sum_{T \in \mathcal{T}_k} Q_T(uv) = \sum_{i,j} \frac{|T_i| + |T_j|}{d+1} u(x_{ij}) v(x_{ij}) \quad \forall u, v \in S_{k-1} \oplus S_k$$

where the second sum in (3.1) extends over the barycentres of all interior faces corresponding to $\mathcal{T}_k$. Note that $(u, v)_k = (u, v)$ for all $u, v \in S_{k-1} \oplus S_k$, if $d = 2$ (cf. Chapter 4 of [4]). To simplify extensions of our method we will use only the fact that $(u, v)_k$ and (u, v) induce equivalent norms on $S_{k-1} \oplus S_k$:

$$(3.2) \qquad c^{-1} \|v\|_0^2 \leq (v, v)_k \leq c \|v\|_0^2 \quad \forall v \in S_{k-1} \oplus S_k.$$

The standard basis functions of S_k equal 1 at the midpoint of exactly one edge and vanish at the midpoints of all other edges. From (3.1) it follows that these functions are mutually orthogonal with respect to the inner product $(\cdot, \cdot)_k$. Therefore, we can obtain an easily computable "prolongation" operator $P_{k-1,k} : S_{k-1} \to S_k$ by

$$(3.3) \qquad (P_{k-1,k}v_{k-1}, w_k)_k = (v_{k-1}, w_k)_k \quad \forall v_{k-1} \in S_{k-1}, \quad w_k \in S_k.$$

In practice, the calculation of $P_{k-1,k}v_{k-1}$ results in a suitable averaging of the nodal values of v_{k-1}. To see this, consider two adjacent triangles T_i, $T_j \in \mathcal{T}_{k-1}$ together with the corresponding fine-grid triangles as depicted in Fig. 1.

Using (3.1) and (3.3), an easy calculation yields

$$(3.4a) \qquad P_{k-1,k}v_{k-1}(x_A) = \tfrac{1}{2}\{v_{k-1}(x_1) + v_{k-1}(x_2)\},$$

$$
\begin{aligned}
(3.4b) \qquad P_{k-1,k}v_{k-1}(x_B) &= \frac{1}{2\{|T_i| + |T_j|\}} \\
&\cdot \{|T_i|[2v_{k-1}(x_1) + v_{k-1}(x_2) - v_{k-1}(x_3)] \\
&\quad + |T_j|[2v_{k-1}(x_1) + v_{k-1}(x_4) - v_{k-1}(x_5)]\}.
\end{aligned}
$$

If $|T_i| = |T_j|$, (3.4b) simplifies to

$$
\begin{aligned}
(3.4c) \qquad P_{k-1,k}v_{k-1}(x_B) &= \tfrac{1}{2}\{v_{k-1}(x_1) + \tfrac{1}{2}v_{k-1}(x_2) - \tfrac{1}{2}v_{k-1}(x_3)\} \\
&\quad + \tfrac{1}{2}\{v_{k-1}(x_1) + \tfrac{1}{2}v_{k-1}(x_4) - \tfrac{1}{2}v_{k-1}(x_5)\}.
\end{aligned}
$$

This means that $P_{k-1,k}v_{k-1}(x_B)$ is the arithmetic mean of $v_{k-1}|_{T_i}(x_B)$ and $v_{k-1}|_{T_j}(x_B)$. Moreover, (3.4a–c) reduce to the standard interpolation formulae whenever $v_{k-1} \in C(\bar{T}_i \cup \bar{T}_j)$. A similar expression can be deduced for the three-dimensional case.
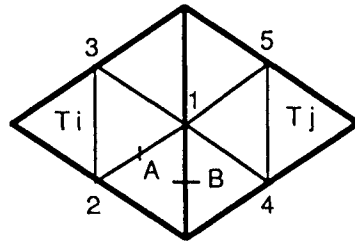

FIG. 1

Using the operators $P_{k-1,k}$, we are now in a position to formulate our multigrid algorithm for the Crouzeix–Raviart discretization of the Poisson equation.

ALGORITHM 3.1 (One interation at level k with m smoothing steps).

1. *Smoothing.* Given $u_k^0 \in S_k$. For $i = 1, \cdots, m$ compute u_k^i from u_k^{i-1} by solving

$$(3.5) \qquad (u_k^i, v_k)_k = (u_k^{i-1}, v_k)_k + \omega_k^{-1}\{(f, v_k) - a_k(u_k^{i-1}, v_k)\} \quad \forall v_k \in S_k.$$

2. *Coarse-grid correction.* Denote by $u_{k-1}^* \in S_{k-1}$ the solution of the coarse-grid problem

$$(3.6) \qquad a_{k-1}(u_{k-1}, v_{k-1}) = (f, P_{k-1,k}v_{k-1}) - a_k(u_k^m, P_{k-1,k}v_{k-1}) \quad \forall v_{k-1} \in S_{k-1}.$$

If $k = 1$, set $\tilde{u}_{k-1} := u_{k-1}^*$. If $k > 1$, compute an approximation $\tilde{u}_{k-1}$ to u_{k-1}^* by applying $\mu = 1$ or 2 iterations of the algorithm at level $k-1$ to problem (3.6) with starting value zero. Put

$$(3.7) \qquad u_k^{m+1} := u_k^m + P_{k-1,k}\tilde{u}_{k-1}.$$

The parameter ω_k in (3.5) has to be chosen greater or equal to the spectral radius of a_k.

The coarse-grid correction has been designed and chosen such that the right-hand side of (3.6) can easily be computed by a suitable averaging of the fine-grid residuals. To see this, consider two adjacent triangles T_i, $T_j \in \mathcal{T}_k$ together with some neighbouring triangles and the corresponding fine-grid triangles as depicted in Fig. 2.
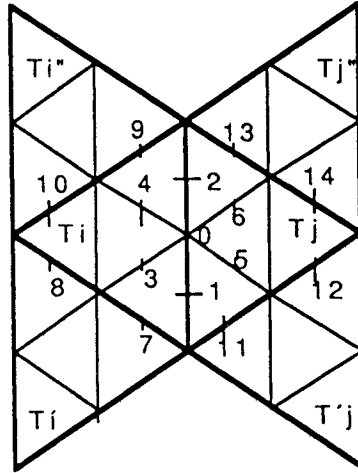

FIG. 2

Denote by $\varphi_j \in S_k$ the basis functions corresponding to the points x_j, $1 \leq j \leq 14$, and by $\psi_0 \in S_{k-1}$ the basis function corresponding to the point x_0. Put

$$r_j := (f, \varphi_j) - a_k(u_k^m, \varphi_j), \qquad 1 \leq j \leq 14.$$

Using (3.1) and (3.3), an easy calculation then yields

(3.8)

$$(f, P_{k-1,k}\psi_0) - a_k(u_k^m, P_{k-1,k}\psi_0)$$

$$= \frac{1}{2}\{2r_1 + 2r_2 + r_3 + r_4 + r_5 + r_6\} + \frac{|T_i|}{2\{|T_i| + |T_i'|\}}\{r_7 - r_8\} + \frac{|T_i|}{2\{|T_i| + |T_i''|\}}\{r_9 - r_{10}\}$$

$$+ \frac{|T_j|}{2\{|T_j| + |T_j'|\}}\{r_{11} - r_{12}\} + \frac{|T_j|}{2\{|T_j| + |T_j''|\}}\{r_{13} - r_{14}\}.$$

A similar expression can be deduced for the three-dimensional case.

We note that we cannot do without a projection and that we cannot replace the right-hand side of (3.6) by the simpler expression

$$(f, v_{k-1}) - a_k(u_k^m, v_{k-1}).$$

To understand this, assume that u_k^m is the finite-element solution in S_k. The expression above may be nonzero for some $v_{k-1} \in S_{k-1}$ since $S_{k-1} \not\subset S_k$. In this case, a nonzero coarse-grid correction would be the result, which obviously weakens the solution.

Therefore some projection is needed, and the analysis in the next section shows that the projection in the L^2-norm (or an equivalent norm) leads to reasonable multigrid algorithms.

4. Convergence analysis of the multigrid algorithm. We will provide a convergence analysis of Algorithm 3.1 in the framework of [1]. From a *smoothing property* (cf. (4.5)) and an *approximation property* (cf. (4.11)), we deduce the convergence of the two-level algorithm (cf. (4.13)) and the multigrid algorithm (cf. (4.14)). When we compare the analysis of [1] for conforming methods, only the proof of the approximation property will not be standard.

Denote by u_k^* the solution of the discrete problem at level k and put

$$e_k^i := u_k^* - u_k^i.$$

Since $a_k(\cdot, \cdot)$ and $(\cdot, \cdot)_k$ are symmetric, positive definite bilinear forms on S_k, there is a complete set of eigenvalues $0 < \lambda_{k,1} \leq \cdots \leq \lambda_{k,N_k}$ and of eigenfunctions $\varphi_{k,1}, \cdots, \varphi_{k,N_k}$, $N_k := \dim S_k$. In particular, the latter may be chosen to be mutually orthogonal with respect to the inner product $(\cdot, \cdot)_k$, i.e.,

$$(4.1) \qquad \begin{aligned} a_k(\varphi_{k,i}, v_k) &= \lambda_{k,i}(\varphi_{k,i}, v_k)_k \quad \forall v_k \in S_k, \quad 1 \leq i \leq N_k, \\ (\varphi_{k,i}, \varphi_{k,j})_k &= \delta_{ij} \qquad\qquad \forall 1 \leq i, \quad j \leq N_k. \end{aligned}$$

Using the uniformity of the triangulation $\mathcal{T}_k$, we conclude from a standard inverse estimate that

$$(4.2) \qquad\qquad \lambda_{k,N_k} \leq c h_k^{-2}$$

where c is independent of k. In the sequel, we assume for simplicity that $\omega_k = \lambda_{k,N_k} = \max_j \lambda_{k,j}$. Using the representation $v_k = \sum_i c_i \varphi_{k,i}$ of functions $v_k \in S_k$, we define a scale of mesh-dependent norms by

$$(4.3) \qquad\qquad |\!|\!|v_k|\!|\!|_{s,k} := \{\Sigma_i \lambda_{k,i}^s c_i^2\}^{1/2} \quad \forall v_k \in S_k.$$

Formulae (4.1), (4.3), and (3.2) imply that

$$(4.4) \qquad c^{-1} \|v_k\|_0 \leq |\!|\!|v_k|\!|\!|_{0,k} \leq c \|v_k\|_0, \, |\!|\!|v_k|\!|\!|_{1,k} = |v_k|_{1,k} \quad \forall v_k \in S_k.$$

With the same arguments as in [1], we conclude that the *smoothing property*

$$(4.5) \qquad\qquad |\!|\!|e_k^m|\!|\!|_{2,k} \leq \frac{c_1}{m+1} h_k^{-2} \|e_k^0\|_0$$

holds with a constant c_1 which is independent of k.

The error of the coarse-grid correction will be estimated by a duality argument. We recall that in the analysis of conforming elements, u_{k-1}^* is directly treated as the approximation of e_k^m in S_{k-1}. Here, the technique will be modified, and u_{k-1}^* and e_k^m will be considered as the finite element solutions of an auxiliary problem in S_{k-1} and S_k, respectively. To this end define the function $r_k \in S_k$ by

$$(4.6) \qquad\qquad (r_k, v_k)_k = a_k(e_k^m, v_k) \quad \forall v_k \in S_k.$$

Formally, r_k may be understood as the Riesz–Fisher representation of the residual in S_k endowed with the inner product $(\cdot, \cdot)_k$. From the definition of e_k^m we have

$$a_k(e_k^m, v_k) = (f, v_k) - a_k(u_k^m, v_k).$$

Equations (3.3) and (3.6) then imply that

$$(4.7) \qquad \begin{aligned} a_{k-1}(u_{k-1}^*, v_{k-1}) &= a_k(e_k^m, P_{k-1,k} v_{k-1}) = (r_k, P_{k-1,k} v_{k-1})_k \\ &= (r_k, v_{k-1})_k \quad \forall v_{k-1} \in S_{k-1}. \end{aligned}$$

Let $w \in H_0^1(\Omega)$ be the unique weak solution of

$$-\Delta w = r_k \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \Gamma.$$

Since Ω is convex, we have $w \in H^2(\Omega)$ and $\|w\|_2 \leqq c'\|r_k\|_0$. From (4.6) and (4.7) we conclude that e_k^m and u_{k-1}^* are the finite element approximations of w. Recalling the error estimate (2.6), we have

$$(4.8) \qquad \|e_k^m - u_{k-1}^*\|_0 \leqq \|e_k^m - w\|_0 + \|w - u_{k-1}^*\|_0 \leqq c'' h_k^2 \|r_k\|_0.$$

From the definition of r_k and of the norms $\|\|\cdot\|\|_{s,k}$ and from (3.2) we conclude that

$$\|r_k\|_0^2 = a_k(e_k^m, r_k) \leqq \|\|e_k^m\|\|_{2,k} \|\|r_k\|\|_{0,k} = c\|\|e_k^m\|\|_{2,k}\|r_k\|_0$$

and thus

$$(4.9) \qquad \|r_k\|_0 \leqq c\|\|e_k^m\|\|_{2,k}.$$

Finally, we obtain from the definition (3.3) of $P_{k-1,k}$ that

$$\|\|e_k^m - P_{k-1,k}u_{k-1}^*\|\|_{0,k}^2 = (e_k^m - P_{k-1,k}u_{k-1}^*, e_k^m - u_{k-1}^*)_k$$
$$\leqq \|\|e_k^m - P_{k-1,k}u_{k-1}^*\|\|_{0,k}\|\|e_k^m - u_{k-1}^*\|\|_{0,k}$$

and hence

$$(4.10) \qquad \|\|e_k^m - P_{k-1,k}u_{k-1}^*\|\|_{0,k} \leqq \|\|e_k^m - u_{k-1}^*\|\|_{0,k}.$$

Combining (4.8)–(4.10) with (3.2), we immediately obtain the *approximation property*

$$(4.11) \qquad \|e_k^m - P_{k-1,k}u_{k-1}^*\|_0 \leqq c_2 h_k^2 \|\|e_k^m\|\|_{2,k}.$$

The smoothing property (4.5) and the approximation property (4.11) imply

$$(4.12) \qquad \|\|u_k^{m+1} - u_k^*\|\|_{0,k} \leqq \frac{c_1 c_2}{m+1} \|\|u_k^0 - u_k^*\|\|_{0,k}.$$

This means that in each cycle of the two-level algorithm the $\|\|\cdot\|\|_{0,k}$-error is reduced by a factor

$$(4.13) \qquad \delta_{TL} \leqq \frac{c_3}{m+1}$$

where $c_3 = c_1 c_2$. The extension to the multilevel algorithm follows the ideas in [1]. Denote the convergence rate of Algorithm 3.1 at level k by δ_k. Then $\delta_1 = \delta_{TL}$ and we have the recursion relation

$$\delta_k \leqq (1 + \delta_{k-1}^\mu) \frac{c_3}{m+1} + \delta_{k-1}^\mu.$$

If $\mu = 2$ and $\delta_1 \leqq \frac{1}{5}$ this finally yields the multigrid convergence rate

$$(4.14) \qquad \delta_k \leqq \frac{5c_3}{2(m+1)} \leqq \frac{1}{2} \quad \forall k \geqq 0.$$

5. Convergence analysis with respect to the energy norm. In the preceding section convergence was established under the assumption that the number of smoothing steps is sufficiently large. On the other hand, following [2] we obtain a two-level convergence rate with respect to the energy norm

$$(5.1) \qquad \delta_{TL} \leqq \left(\frac{c_4^2}{c_4^2 + 2m} \right)^{1/2}$$

with c_4 being independent of k. This yields convergence of the two-level algorithm in the energy norm if there is (at least) one smoothing step.

In order to get this result, we have to add a third step to Algorithm 3.1. In that step an appropriate steplength parameter is computed and the coarse-grid correction is multiplied with that parameter. This extra device is introduced not only for theoretical reasons. Numerical experiences with another nonconforming method show that the convergence rate is substantially improved in this way [6].

ADDITION TO ALGORITHM 3.1.

3. *Steplength control.* Set $w_k := P_{k-1,k}\tilde{u}_{k-1}$ and determine

$$(5.2) \qquad \alpha_k = \frac{(f, w_k) - a_k(u_k^m, w_k)}{a_k(w_k, w_k)}.$$

Set

$$u_k^{m+2} = u_k^m - \alpha_k w_k.$$

The parameter α_k has been chosen such that the functional

$$\tfrac{1}{2} a_k(u_k^{m+2}, u_k^{m+2}) - (f, u_k^{m+2})$$

becomes minimal.

In this framework we estimate the effect of the smoothing by Lemma 4.3 in [2]

$$(5.3) \qquad \|\!|\!|e_k^i|\!|\!\|_{1,k} \leqq \rho^i \|\!|\!|e_k^0|\!|\!\|_{1,k} \quad \text{for } i = 1, 2, \cdots, m$$

with

$$(5.4) \qquad \rho = 1 - \frac{1}{\lambda_{N_k}} \frac{\|\!|\!|e_k^m|\!|\!\|_{2,k}^2}{\|\!|\!|e_k^m|\!|\!\|_{1,k}^2}.$$

There is no change since (5.3) depends only on the algebraic properties of positive definite matrices.

For estimating the approximation property, we replace (4.8) by the corresponding $|\cdot|_{1,k}$-norm estimate

$$\|\!|\!|e_k^m - u_{k-1}^*|\!|\!\|_{1,k} = |e_k^m - u_{k-1}^*|_{1,k}$$
$$\leqq |e_k^m - w|_{1,k} + |w - u_{k-1}^*|_{1,k} \leqq c_5 h_k \|r_k\|_0.$$

From this and (4.9) it follows that

$$(5.5) \qquad \|\!|\!|e_k^m - u_{k-1}^*|\!|\!\|_{1,k} \leqq c_6 h_k \|\!|\!|e_k^m|\!|\!\|_{2,k}.$$

Inequalities (5.5) and (4.2) imply

$$\|\!|\!|e_k^m|\!|\!\|_{2,k}^2 = \lambda_{N_k}(1 - \rho)\|\!|\!|e_k^m|\!|\!\|_{1,k}^2 \leqq c h_k^{-2}(1 - \rho)\|\!|\!|e_k^m|\!|\!\|_{1,k}^2.$$

This and (5.5) yield

$$(5.6) \qquad \|\!|\!|e_k^{m+1}|\!|\!\|_{1,k} \leqq c_4 \sqrt{1 - \rho}\, \|\!|\!|e_k^m|\!|\!\|_{1,k}.$$

The choice of the parameter α_k implies that

$$\|\!|\!|e_k^{m+2}|\!|\!\|_{1,k} \leqq \min\{\|\!|\!|e_k^m|\!|\!\|_{1,k}, \|\!|\!|e_k^{m+1}|\!|\!\|_{1,k}\}.$$

This together with (5.3) and (5.6) yields the estimate

$$(5.7) \qquad \|\!|\!|e_k^{m+2}|\!|\!\|_{1,k} \leqq \rho^m \min\{1, c_4\sqrt{1 - \rho}\}\|\!|\!|e_k^0|\!|\!\|_{1,k}$$

where $0 \leqq \rho < 1$. If $c_4^2 \leqq 2m + 1$, then $\rho^m c_4 \sqrt{1 - \rho} \leqq c_4/\sqrt{2(2m+1)}$; otherwise the factor on the right-hand side of (5.7) attains its maximum at $\rho = 1 - c_4^{-2}$. In both cases, we obtain the bound as stated in (5.1).

Although it follows from (5.1) that we need only one smoothing step in the two-level case, we do not yet have an analogous result for the multilevel method. This is due to the fact that for nonconforming methods the error of the coarse-grid correction is no longer orthogonal to S_{k-1}.

6. Application to the Stokes problem. Combined with piecewise constant pressures, the Crouzeix-Raviart element yields a stable, nonconforming, mixed finite element discretization of the Stokes equations:

$$(6.1) \qquad -\Delta\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \qquad \nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \qquad \mathbf{u} = 0 \quad \text{on } \Gamma.$$

More precisely, we define

$$X_k := (S_k)^d, \qquad Y_k := \left\{ p \in L^2(\Omega): p|_T \in P_0 \forall T \in \mathcal{T}_k, \int_\Omega p = 0 \right\}$$

and introduce a bilinear form $\mathcal{L}_k$ on $X_k \times Y_k$ by

$$\mathcal{L}_k([\mathbf{u}, p]; [\mathbf{v}, q]) := \sum_{T \in \mathcal{T}_k} \int_T \{\nabla\mathbf{u}\nabla\mathbf{v} - p\nabla\cdot\mathbf{v} - q\nabla\cdot\mathbf{u}\}.$$

Then the Crouzeix-Raviart discretization of problem (6.1) is given by (cf. [5])
    Find $\mathbf{u}_k \in X_k$, $p_k \in Y_k$ such that

$$(6.2) \qquad \mathcal{L}_k([\mathbf{u}_k, p_k]; [\mathbf{v}_k, q_k]) = (\mathbf{f}, \mathbf{v}_k) \quad \forall\mathbf{v}_k \in X_k, \quad q_k \in Y_k.$$

Problem (6.2) has a unique solution $\mathbf{u}_k^*$, p_k^*. Moreover, equations (5.11) and (5.12) in [5] and the stability of (6.2) in the sense of [3] imply the error estimate

$$(6.3) \qquad \|\mathbf{u} - \mathbf{u}_k^*\|_0 + h_k\{|\mathbf{u} - \mathbf{u}_k^*|_{1,k} + \|p - p_k^*\|_0\} \leq ch_k^2\|\mathbf{f}\|_0.$$

Problem (6.2) fits into the abstract framework of [7] for multigrid algorithms applied to mixed problems with the exception of the nonconformity. Generalizing the techniques of [8], this drawback can be overcome by applying the projection operator $P_{k-1,k}$ to the velocity components when passing from a coarse to a fine grid and vice versa. With these modifications, the multigrid algorithm of [7] can be applied to problem (6.2). The convergence analysis of [7] carries over, provided the proof of the approximation property is modified as shown in §4 for the Poisson equation.

REFERENCES

[1] R. E. BANK AND T. DUPONT, *An optimal order process for solving finite element equations*, Math. Comput., 36 (1981), pp. 35–51.

[2] D. BRAESS AND W. HACKBUSCH, *A new convergence proof for the multi-grid method including the V-cycle*, SIAM J. Numer. Anal., 20 (1983), pp. 967–975.

[3] F. BREZZI, *On the existence, uniqueness, and approximation of saddle-point problems arising from Lagrangian multipliers*, RAIRO Anal. Numér. 8 (1974), pp. 129–151.

[4] P. G. CIARLET, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, New York, 1978.

[5] M. CROUZEIX AND P. A. RAVIART, *Conforming and non-conforming finite element methods for solving the stationary Stokes equations*, RAIRO Anal. Numér., 7 (1973), pp. 33–76.

[6] P. PEISKER AND D. BRAESS, *A conjugate gradient method and a multi-grid algorithm for Morley's finite element approximation of the biharmonic equation*, Numer. Math., 50 (1987), pp. 567–586.

[7] R. VERFÜRTH, *A multilevel algorithm for mixed problems*, SIAM J. Numer. Anal., 21 (1984), pp. 264–271.

[8] ———, *Multilevel algorithms for mixed problems. II. Treatment of the mini-element.* SIAM J. Numer. Anal., 25 (1988), pp. 285–293.