ON SOME ASPECTS IN THE SPECIAL THEORY OF GRADIENT ELASTICITY

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ABSTRACT

In this paper a special form of gradient-dependent elasticity is considered. The motivation for considering higher-order gradients of strains in elasticity is discussed. Equilibrium equations and boundary conditions are discussed. The relationship between the special form of gradient elasticity adopted in this study and mixture or nonlocal theories is considered. Solutions to certain problems including the propagation of harmonic waves, the longitudinal vibrations of a beam, and the displacement field in an infinite medium weakened by a line crack are given.

1. INTRODUCTION

Although the basic idea of taking into account not only the first but also the higher gradients of the displacement field in the expression for the strain energy function can be traced all the way back to Bernoulli and Euler a corresponding formulation did not attract the attention of scientists for a long time. After Voigt [1] briefly indicated the role of the gradients of rotation in elasticity, E. and F. Cosserat [2] gave the first systematic treatment of the rotation gradients and the associated *couple-stresses*. The Cosserats were drawn to the general concept of a continuous medium each point of which has six degrees of freedom (three displacements and three rotations) similar to rigid bodies. This concept was already known in various theories of rods and shells, and they extended this notion in a rigorous way to three-dimensional continuous media. The novel feature in their theory was the appearance of couple-stresses in the equations of motion. As a consequence of the Cosserat theory, the stress tensor is not symmetric as in the classical theory of elasticity.

For almost fifty years, not much attention was given to such generalizations in continuum mechanics. Hellinger [3] and Von Heun [4] drew attention to

the problem of asymmetric stress of the Cosserat medium. Jaramillo [5] constructed a generalization of the classical theory of infinitesimal elastic deformations based on the assumption that the strain energy density was a quadratic function of the second- order spatial derivatives of the displacement field, as well as the first- order spatial derivatives and velocity components. Since Jaramillo kept the stresses in their classical form, thus disregarding couple stresses, he went on to impose some unnatural restrictions on the dependence of the strain energy density upon the second- order spatial derivatives of the displacement field. Truesdell [6] elaborated upon the balance equations for the Cosserat continuum. Ericksen & Truesdell [7] developed a purely kinematical description of Cosserat continua emphasizing the cases of rods and shells. They also suggested a natural generalization of the Cosserat continuum. The orientation of a volume element can be represented mathematically by three mutually perpendicular unit vectors. The Cosserats formulated their theory by assuming that this triad (directors) is rigid. In Ericksen & Truesdell's generalization of Cosserat continuum, the orientation vectors were stretchable and did not remain mutually orthogonal. An interesting connection between the kinematics of a Cosserat continuum and the theory of continuous distribution of dislocations was pointed out by Guenther [8]. A modern treatment of a continuum of grade 2 (i.e. a material whose strain energy density is a function of the second-order spatial derivatives of the displacement field, in addition to the first- order spatial derivatives) was given by Truesdell and Toupin [9a]. They also discussed the indeterminacy of the couple-stress tensor in the Cosserat theory. Grioli [10] gave the first general and correct treatment of elastic materials of grade 2, whose strain energy function was of the same form as the Cosserats' strain energy function. Toupin [11] has derived the associated constitutive equations for finite deformation of perfectly elastic materials. Upon linearization, Toupin's results are identical with those which were obtained, for example, by Aero and Kuvshinskii [12]. In his study, Toupin [11] also reviewed the foundations of the theory of grade 2 elastic materials, corrected the formula for the couple-stresses given by Truesdell and Toupin [9a], pointed out that the Cosserat continuum was a peculiar subclass of the grade 2 elastic materials, and studied the propagation of plane sound waves. Schaeffer [13] solved some explicit boundary value problems for a two-dimensional Cosserat medium so as to illustrate some of the novel features of the theory. Mindlin & Tiersten [14] gave an extensive analysis on the derivations of the finite and linearized equations for the Cosserat continuum and also discussed previous derivations in detail. Moreover, they extended many of the classical results on uniqueness theorems, stress functions, fundamental solutions, propagation of plane waves (they showed that the propagating waves were accompanied by non-propagating waves in Cosserat continuum), thickness-shear vibrations of an infinite plate, stress concentrations and singularities, stresses around spherical and cylindrical cavities in an infinite body under tension, nuclei of strains, *etc.* They also provided explicit

solutions to certain boundary value problems illustrating the novel features of the theory. Mindlin [15] derived a linear generalized Cosserat theory for a three-dimensional elastic continuum, in which the constitutive equations were identical to those obtained by Toupin [11], and studied the propagation of plane waves. Green & Rivlin [16] developed a more general theory by considering higher-order surface and body force multipoles. They studied the kinematics and the nature of higher-order force multipoles extensively, and gave the constitutive equations of a generalized elasticity, by also employing an appropriate energy equation and an entropy production inequality. In a subsequent paper, Green & Rivlin [17] developed a general theory for multipolar displacement and velocity fields with corresponding multipolar body and surface forces, as well as multipolar stresses. They accomplished this by using an energy principle, an entropy production inequality, and invariance conditions under superposed rigid body motion. They also showed that their previous work (Green & Rivlin [16]) is special case of that developed in [17]. Toupin [18] reviewed the models developed for continuous media with couple-stresses, identified the concepts and principles of continuum mechanics common to all models and devised a mathematical machinery for easy and precise expression of the basic ideas and assumption pertaining to each model. In addition, Toupin [18] pursued quite another direction which also leads to a modification of the familiar concept of stress. Instead of introducing rigid or deformable material points, he expressed the relative position vector of a material point $\underset{\sim}{x}'$ in the neighborhood of the material point $\underset{\sim}{x}$ in terms of the successively higher-order gradients of the displacement vector at the point x. Furthermore, he argued that it is a quite natural generalization to assume that the strain energy density depends on not only the first but also on the higher gradients of the displacement field. Toupin [18] also showed that a stress-free configuration (*natural state*) for materials of grade 2 is an exception which is a rule in the classical theory of elasticity.

The boundary layer effect in crystals was known for a long time and observed by low-energy electron diffraction experiments (see, for example, Germer, MacRae & Hartman [19]). Toupin & Gazis [20] illustrated the relation between the strain-gradient elasticity and atomic lattice (with nearest neighbor and next nearest neighbor interaction) theories and explored the consequences of an initial, homogeneous, self-equilibrating stress field. Later, Gazis & Wallis [21] modeled the free-surface of a crystal by considering a semi-infinite, one-dimensional, monatomic lattice with nearest and next nearest neighbor interactions including a harmonic interaction between nearest neighbors at the end. They showed that the particles near the free-surface must move to a new equilibrium position, and that the force constants characterizing small oscillations of these particles will be different from those of the infinite crystals. Mindlin [22] formulated a linearized theory for an elastic solid in which the strain energy density is a function of the strain and its first and second gradients. He

showed that cohesive force and surface-tension were intrinsically included in this theory. Also, an explicit solution for the strain and surface-tension, resulting from the separation of a solid along a plane was given; and a comparison was made with an analogous lattice model.

Higher-order gradients of constitutive variables have also been employed in other branches of continuum mechanics. In 1901, Korteweg formulated a constitutive equation for the Cauchy stress that included density gradients, in order to model the fluid capilarity effect. Theories of Korteweg's type have also been employed to analyze the structure of liquid-vapor interfaces by Aifantis & Serrin [24,25]. Motivated by the success of this approach for fluid interface problems, Triantafyllidis & Aifantis [26] formulated a nonlinear theory for hyperelastic materials by adding the second deformation gradient into the strain energy function to analyze the pre-and post-localization behavior of deformation. It was shown that the width and direction of the localized deformation zone could be described (without the occurrence loss of ellipticity in the governing equations), in contrast to the classical results. higher-order gradients of strain or other constitutive variables had been already considered for analyzing dislocation patterns, microvoids and other material microstructures in solids by Aifantis and his co-workers [27-33]. These theories provided a means to account for internal length scale and size effects in inelastic material behavior in contrast to standard theories which could not capture these and other pattern-forming instabilities effects.

In this paper, a special form of gradient-elasticity, which is based on the linear version of the constitutive equations obtained by Triantafyllidis & Aifantis [24], is employed. Our purpose is to discuss non-classical implications of the simplest possible gradient elasticity theory and, thus, our motivation is completely different than the fundamental continuum mechanics works reviewed above. In the following section, the derivation of the field equations and the boundary conditions of the special form of gradient-elasticity considered in this study is briefly introduced. In the subsequent sections the relationship between the special form of gradient-elasticity, the nonlocal elasticity, and the mixture theory are pointed out. The three basic modes of cracks are formulated and solved. Then, the propagation of plane waves in an infinite medium is considered. In the last section, natural frequencies and modes during the longitudinal vibration of a bar are discussed. Some of the results reviewed in the present paper have discussed by the authors separately in previous publications. It was decided, however, to include a summary of these results here for completeness and for the convenience of the reader who is not very familiar with the field.

2. FIELD EQUATIONS AND BOUNDARY CONDITIONS

A customary approach in obtaining a constitutive equation in elasticity is to assume the existence of a strain energy density, which is taken as a function of the symmetric part of the first gradient of the displacement field

$$w = w(\varepsilon_{ij}) \tag{1}$$

where ε_{ij} is the symmetric part of the displacement field

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

where u_i is the Cartesian component of the displacement vector and indices following a comma, as usual, denote partial derivatives with respect to the space coordinates. In gradient elasticity, the strain energy density function is assumed to depend not only on the first gradients but also on the second gradients of the displacement field

$$w = w(\varepsilon_{ij}, \varepsilon_{ij,k}) \tag{3}$$

Since we are dealing with single valued displacement fields one can easily establish a one-to-one correspondence between $\varepsilon_{ij,k}$ and $u_{i,jk}$ (see Mindlin & Eshel [23]). The most general form of the strain energy density function for a linear, isotropic, gradient-dependent elastic material is

$$\begin{aligned} w = {} & \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + \\ & c_1\varepsilon_{ij,j}\varepsilon_{ik,k} + c_2\varepsilon_{ii,k}\varepsilon_{kj,j} + c_3\varepsilon_{ii,k}\varepsilon_{jj,k} + c_4\varepsilon_{ij,k}\varepsilon_{ij,k} + c_5\varepsilon_{ij,k}\varepsilon_{kj,i} \end{aligned} \tag{4}$$

For the special form of gradient elasticity that we consider here we assume that c_3 and c_4 are the only non-vanishing gradient coefficients. More specifically, we take the strain energy density function as

$$w = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + c(\frac{1}{2}\lambda\varepsilon_{ii,k}\varepsilon_{jj,k} + \mu\varepsilon_{ij,k}\varepsilon_{ij,k}) \tag{5}$$

where c denotes a newly introduced strain gradient parameter which is the only non-standard coefficient of the theory.

Next, we consider that the following expression for the variation of virtual work (for a body occupying region B bounded smooth surface S)

$$\int_B \delta w dv - \int_S (t_i \delta u_i + \tau_i D\delta u_i) da = 0 \tag{6}$$

holds for every variation of the displacement field (δu_i) where f_i is the force, t_i is the surface traction, τ_i is a "hypertraction" and $D\delta u_i = \delta u_{i,j} n_j$ represents the normal derivative of the variation δu_i. It was shown by Ru and Aifantis [34] that the variational statement (6) leads to the following system of equlibrium equations and boundary conditions

$$\sigma_{ij,j} = 0 \tag{7}$$

$$t_i = \sigma_{ij} n_j + cL_j\{\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\}_{,k}\ n_k \tag{8}$$

$$\tau_i = c\{\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\}_{,k}\ n_k n_j \tag{9}$$

where

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} - c\{\lambda_1\varepsilon_{kk}\delta_{ij} + \mu_1\varepsilon_{ij}\}_{,mm} \tag{10}$$

and L_j is a differential operator defined by

$$L_i\{v_i\} = n_{k,k}\{v_i\} n_j - \{v_{i,j}\} + v_{i,l} n_l n_j$$

Although these results are obtained for static cases there is no essential difficulty to their dynamic counterpart.

Boundary value problems can now be defined in terms of determining an ordered triplet $\{u_i, \varepsilon_{ij}, t_{ij}\}$ which satisfies the displacement - strain relation (2), the balance equation (7), the stress - strain relations (10) and the boundary conditions

$$u_i = \overline{U}_i \quad \text{or} \quad t_i = \sigma_{ij} n_j + cL_j\{\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\}_{,k}\ n_k = \overline{T}_i \tag{11a}$$

and

$$u_{i,j}\, n_j = \overline{E}_i \quad \text{or} \quad \tau_i = c\{\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\}_{,k}\ n_k n_j = \overline{S}_i \tag{11b}$$

where $\overline{U}_i$, $\overline{T}_i$, $\overline{E}_i$, $\overline{S}_i$ are prescribed functions on appropriate portions of the boundary.

2.1. A Uniqueness Theorem

Next, we sketch the proof of a uniqueness result for the equilibrium equations (7) with the constitutive equations (10), and the boundary conditions (11). On assuming the existence of two solutions $\{u_i^1, \varepsilon_{ij}^1, t_{ij}^1\}$, $\{u_i^2, \varepsilon_{ij}^2, t_{ij}^2\}$it can be easily seen that the *difference solution* defined by

$$u_i = u_i^1 - u_i^2 \quad , \quad \varepsilon_{ij} = \varepsilon_{ij}^1 - \varepsilon_{ij}^2 \quad , \quad \sigma_{ij} = \sigma_{ij}^1 - \sigma_{ij}^2 \tag{12}$$

satisfies the balance equations

$$\sigma_{ij,j} = 0 \tag{13}$$

and the homogeneous boundary conditions.

$$u_i = 0 \quad \text{or} \quad t_i = \sigma_{ij} n_j + cL_j\{\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\}_{,k}\, n_k = 0 \tag{14a}$$

and

$$u_{i,j}\, n_j = 0 \quad \text{or} \quad \tau_i = c\{\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}\}_{,k}\, n_k n_j = 0 \tag{14b}$$

It can be shown then that if

$$3\lambda + 2\mu > 0 \quad , \quad \mu > 0 \quad , \text{ and } \quad c > 0 \tag{15}$$

$\{u_i, \varepsilon_{ij}, \sigma_{ij}\}$ is identically zero and, thus, the two solutions $\{u_i^1, \varepsilon_{ij}^1, \sigma_{ij}^1\}$ and $\{u_i^2, \varepsilon_{ij}^2, \sigma_{ij}^2\}$ corresponding to the same boundary data are identical. Indeed, by multiplying (13) by u_i and integrating over the domain B we can write

$$\int_B \sigma_{ij}\varepsilon_{ij} dV = \int_B (\sigma_{ij} u_i)_{,j}\, dV = \int_S \sigma_{ij} u_i n_j ds \tag{16}$$

where the divergence theorem is used. The surface integral on the right hand side of this equation is zero because of the boundary conditions (14a). By employing the constitutive equation (10), and the boundary conditions (14b), after some manipulations it is arrived at

$$\int_B (\lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij})\varepsilon_{ij} dvV + c\int_B (\lambda\varepsilon_{ll}\delta_{ij} + 2\mu\varepsilon_{ij})_{,k}\, \varepsilon_{ij,k} dV = 0 \tag{17}$$

The integrants of both integrals are quadratic forms. Since this relation should hold true for an arbitrary volume B, for an arbitrary strain field ε_{ij}, and strain gradient field $\varepsilon_{ij,k}$, the conditions given by (15) should hold in order uniqueness to be hold. The first two conditions are the same as those met in classical elasticity. If the gradient parameter c is positive, then uniqueness of the boundary value problems is assured without any further consideration. On the other hand, there is some evidence that the gradient parameter c should be negative in certain circumstances in order to capture realistic material behavior. In such cases, it is clear that uniqueness fails and, thus, existence and stability of solutions require further care. On the other hand experiences on the elliptic differential equations shows that if uniqueness fails, existence and stability of the solutions require further care. In other words, the solution space of boundary value problems in the special form of gradient elasticity should be defined carefully to ensure the uniqueness, existence, and stability of the solutions when $c<0$. Alternatively, one may allow for the inclusion of an additional higher-order spatial derivative in the constitutive equation such that standard uniqueness results can be established even for the case $c<0$.

3. CONNECTION WITH OTHER GENERALIZED THEORIES

The special form of gradient elasticity suggests a modification of the constitutive equation for elastic bodies by adding the Laplacian of Hookean stress. With this modification, the constitutive equation of elasticity relates closely to the constitutive equation of in nonlocal elasticity and mixture theory. These relations are displayed below.

3.1 Relation to Nonlocal Elasticity

Nonlocal elasticity is based on the assumption that the forces between material points can be at long-range in character, thus reflecting the long-range character of interatomic forces. Kroener [35] indicated the relation between nonlocal theory and the theory of continuous distribution of dislocations. Beran [36] showed (see also Levin [37]) that, the relation between moving averages or ensemble averages of the stress and strain in a statistically non-homogeneous medium is of a nonlocal form. Later, Eringen & Edelen [38] provided a formal thermomechanical and variational derivation of the constitutive equations of nonlocal elasticity. Rogula [39] and his co-workers investigated the mathematical structure of nonlocal elasticity, proposed different types of nonlocal relations between stress and strain, and applied it to various critical problems in continuum mechanics. Kunin [40] collected his works on the physical background of nonlocal elasticity in a book, and studied various problems in Fourier space.

The most familiar form of nonlocal relationship between stress and strain reads.

$$\sigma_{ij} = \int_B k(|\underset{\sim}{x} - \underset{\sim}{x}'|)\{\lambda\varepsilon_{kk}(\underset{\sim}{x}')\delta_{ij} + 2\mu\varepsilon_{ij}(\underset{\sim}{x}')\}dv' \tag{18}$$

where

$$\tau_{ij}(x', y') = \lambda\varepsilon_{kk}(\underset{\sim}{x}')\delta_{ij} + 2\mu\varepsilon_{ij}(\underset{\sim}{x}')$$

is the conventional Hookean stress and $k(|\underset{\sim}{x} - \underset{\sim}{x}'|)$ is the *interaction (or nonlocality) kernel*. More details about this equation and some fundamental aspects on the boundary value problems can be found, among others, in Altan [41-45].

Let us consider the special form of the nonlocality kernel

$$k(|\underset{\sim}{x} - \underset{\sim}{x}'|) = \frac{\beta^2}{2\pi}K_0\left\{\beta\sqrt{(x-x')^2 + (y-y')^2}\right\} \tag{19}$$

where K_0 is the modified Bessel's function of the first kind and $\beta > 0$ is the nonlocality parameter for a two-dimensional infinite domain. The nonlocal kernel given by (19) was proposed by Ari [46] to match the dispersion relation obtained within the framework of lattice dynamics for the propagation of two-dimensional waves. We first assume that the expansion

$$\tau_{ij}(x', y') = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{\partial^{k+l}\tau_{ij}(x, y)}{\partial x^k \partial y^l}\frac{(x'-x)^k}{k!}\frac{(y'-y)^l}{l!} \tag{20}$$

is valid for every pair of x, y and throughout the entire domain. Next, we define

$$I_{kl} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(\beta\sqrt{(x-x')^2 + (y-y')^2})(x'-x)^k(y'-y)^l dx'dy' \tag{21}$$

which can be written in the equivalent form

$$I_{kl} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K_0(\beta\sqrt{\xi^2 + \eta^2})\xi^k\eta^l d\xi d\eta \tag{22}$$

where $\xi = x' - x$, $\eta = y' - y$. It can easily be shown that

$$I_{kl} = 0 \qquad (k, l = 1, 3, 5, \ldots) \tag{23}$$

The remaining integrals can be handled in polar coordinates ($\xi = \rho\cos\theta$ $\eta = \rho\sin\theta$) more conveniently. We then have

$$I_{2k,\,2l} = 4\left(\int_0^{\pi/2} cos^{2k}\theta sin^{2l}\theta d\theta\right)\left(\int_0^{\infty} \rho^{2(k+l)+1} K_o(\beta\rho)d\rho\right) \tag{24}$$

Since

$$\int_0^{\pi/2} cos^{2k}\theta sin^{2l}\theta d\theta = \frac{1}{2^{k+l}}\frac{(2k-1)!!(2l-1)!!}{(l+k)!}\frac{\pi}{2} \tag{25}$$

(Gradshteyn & Ryzhik [64], 2.511.1~2) and

$$\int_0^{\infty} \rho^{2(l+k)+1} K_o(\beta\rho)d\rho = 2^{2(k+l)}\beta^{-2(k+l+1)}[(k+l)!]^2 \tag{26}$$

(Gradshteyn & Ryzhik [64], 6.561.16) we arrive at

$$I_{2k,\,2l} = 2^{(k+l)}\beta^{-2(k+l+1)}[(2k-1)!!][(2l-1)!!][(k+l)!]\frac{\pi}{2} \tag{27}$$

from (18) - (27) we have

$$\sigma_{ij}(x, y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{2}{\beta^2}\right)^{k+l}\frac{[(2k-1)!!][(2l-1)!!][(k+l)!]}{[(2k)!][(2l)!]}\frac{\partial^{2(k+l)}\tau_{ij}(x, y)}{\partial x^{2k}\partial y^{2l}} \tag{28}$$

Considering the identity

$$\frac{(2k-1)!!}{(2k)!} = \frac{1}{2^k(k!)} \tag{29}$$

expression (28) is radically simplified to read

$$\sigma_{ij}(x, y) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\beta^{-2(k+l)}\frac{(k+l)!}{(k!)(l!)}\frac{\partial^{2(k+l)}\tau(x, y)}{\partial x^{2k}\partial y^{2l}} \tag{30}$$

Additional simplifications can be achieved by taking into account the following identities

$$\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A_{kl} = \sum_{m=0}^{\infty} \sum_{n=0}^{m} A_{m-n,n} \tag{31}$$

and

$$\sum_{n=0}^{m} \frac{m!}{(m-n!)(n!)} \frac{\partial^{2m}\tau(x,y)}{\partial x^{2(m-n)}\partial y^{2n}} = \Delta^m \tau \tag{32}$$

indicating that the Laplacian Δ

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{33}$$

with $\Delta^m = \Delta\Delta\ldots\Delta$ (m times) obeys a binomial expansion. Introducing (31) and (32) into (28) we obtain the following gradient expression for the stress

$$\sigma_{ij}(x,y) = \sum_{m=0}^{\infty} \beta^{-2m}\Delta^m \tau_{ij}(x,y) = \tau_{ij} + \frac{1}{\beta^2}\tau_{ij,kk} + \frac{1}{\beta^4}\tau_{ij,kkll} + \ldots \tag{34}$$

This result indicates that the nonlocal constitutive equation of the form (18) is a gradient type relation containing all order Laplacians of the strain field. By recalling the Green's function formalism we may also deduce that a constitutive relation of the type (18) also contains the boundary conditions which should be imposed on the strain field.

In concluding, we wish to point out the following interesting consequence of expression (34). Note that

$$\beta^{-2}\Delta\sigma_{ij}(x,y) = \sum_{m=0}^{\infty} \beta^{-2(m+1)}\Delta^{m+1}\tau_{ij}(x,y) = \beta^{-2}\Delta\tau_{ij} + \beta^{-4}\Delta^2\tau_{ij} + \ldots = \sigma_{ij} - \tau_{ij} \tag{35}$$

from which it follows that

$$\tau_{ij} = \sigma_{ij} - \beta^2\Delta\sigma_{ij} \tag{36}$$

which can be written equivalently as

$$\varepsilon_{ij}(x, y) = S_{ij} - \beta^2 \Delta S_{ij} \tag{37}$$

where

$$S_{ij} = \frac{1}{2\mu(3\lambda + 2\mu)}\{-\lambda\sigma_{kk}\delta_{ij} + (3\lambda + 2\mu)\sigma_{ij}\} \tag{38}$$

Expression (37) which is equivalent to (18) (with the assumption of (19)) suggests that the nonlocal elasticity may be viewed as a gradient elasticity not in strain space but in stress space. In other words, if we wish to produce a gradient-dependent stress-strain relation by following the procedure outlined in the previous section but choosing stress (instead of strain) as independent constitutive variable, then we would arrive at the constitutive relation of the form given by (37).

Finally, we to point out that if the nonlocality kernel is chosen as

$$k(|\underset{\sim}{x} - \underset{\sim}{x}'|) = \left(\frac{\beta}{2}\right)^2 exp\{|x - x'| + |y - y'|\} \tag{39}$$

the stress-strain relation can be written in the form

$$\sigma_{ij}(x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{\beta^{2m}} \frac{\partial^{2m}\tau_{ij}(x, y)}{\partial x^{2(m-n)}\partial y^{2n}} \tag{40}$$

or equivalently

$$\varepsilon_{ij}(x, y) = S_{ij} - \beta^{-2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \beta^{-2}\frac{\partial^4}{\partial x^2 \partial y^2}\right)S_{ij} \tag{41}$$

where s_{ij} is defined by (38).

3.2. Relation to Mixture Theory

Motivated by Maxwell's kinetic theory of gases, Truesdell & Toupin [9b] presented an axiomatic mixture theory for interacting continua based on the premise that each point of space is simultaneously occupied by all constituents of the mixture. In the 60's and 70's some special mixture theories were derived to investigate the mechanical behavior of composite materials. Bedford and

Stern [47], and Stern and Bedford [48] developed such a theory and applied it to laminated composites. Fiber-reinforced composites were considered by Bedford, Sutherland, and Linge [49] for investigating wave propagation in heterogeneous media. Another binary mixture theory was developed by Hegemier, Gurtman and Nayfeh [50] for modeling wave propagation in laminated and unidirectional fibrous composites. This theory was equipped with microstructure such that it is possible to obtain some information on the distribution of displacements and stresses within individual constituents. This theory was applied for both transversely and horizontally polarized shear waves propagating parallel to laminates by Nayfeh and Gurtman [51]. Another type of mixture theory has been developed by McNiven and Mengi [52-54] for modeling wave propagation in periodically structured composites. A general theory for two phase materials is developed in detail by McNiven and Mengi [52]. This theory was applied to laminated composites by McNiven and Mengi [53, 54] who also studied transient wave propagation in laminated composites.

A suggestion for producing higher-order strain gradients in the constitutive equation for an elastic nanostructured material was given by Aifantis [29b] by considering it as a mixture of two phases: the "bulk" and the "grain boundary" regions. We further elaborate upon this suggestion and show that the constitutive equation of the present gradient elasticity theory, Eq. (10), can be obtained by considering the total stress for a mixture of two elastic continua with each constituent obeying the classical Hooke's law. To this end, we consider a mixture of two elastic materials indicated by superscripts 1 and 2. The displacement of each constituent is denoted by u_i^1 and u_i^2. Each constituent obeys its own equilibrium equation of the form

$$\sigma^1_{ij,j} = \alpha(u_i^1 - u_i^2) \quad \text{and} \quad \sigma^2_{ij,j} = -\alpha(u_i^1 - u_i^2) \tag{42}$$

where the terms in the right hand side account for the interaction between the constituents in the form of an internal body force.

The constitutive equations for each constituent are the same as those in classical elasticity, i.e.

$$\sigma^1_{ij} = \lambda_1 u^1_{k,k}\delta_{ij} + \mu_1(u^1_{i,j} + u^1_{j,i}) \tag{43}$$

and

$$\sigma^2_{ij} = \lambda_2 u^2_{k,k}\delta_{ij} + \mu_2(u^2_{i,j} + u^2_{j,i}) \tag{44}$$

The balance equations (42) then give

$$\sigma^1_{ij,j} = (\lambda_1 + \mu_1)u^1_{k,ki} + \mu_1 u^1_{i,kk} = \alpha(u^1_i - u^2_i) \tag{45}$$

and

$$\sigma^2_{ij,j} = (\lambda_2 + \mu_2)u^2_{k,ki} + \mu_2 u^2_{i,kk} = -\alpha(u^1_i - u^2_i) \tag{46}$$

In the following we show that if the average displacement of a mixture is represented by the arithmetic average of the displacements of each constituent and the average stress as the sum of the stresses of each constituent, then the resulting constitutive equation for the mixture as a whole is of a gradient type. To this end, we define the *average* u_i and the *difference* v_i displacements by

$$u_i = (u^1_i + u^2_i)/2 \quad \text{and} \quad v_i = (u^1_i - u^2_i)/2 \tag{47a}$$

such that

$$u^1_i = u_i + v_i \quad \text{and} \quad u^2_i = u_i - v_i \tag{47b}$$

and assume that the total stress for the mixture considered as a whole is given by

$$\sigma_{ij} = \sigma^1_{ij} + \sigma^2_{ij} \tag{48}$$

Next, we express u^2_i in terms of u^1_i and its spatial derivatives by using (45)

$$u^2_i = u^1_i - \frac{\lambda_1 + \mu_1}{\alpha} u^1_{k,ki} - \frac{\mu_1}{\alpha} u^1_{i,kk} \tag{49}$$

Similarly, from (46) we have

$$u^1_i = u^2_i - \frac{\lambda_2 + \mu_2}{\alpha} u^2_{k,ki} - \frac{\mu_2}{\alpha} u^2_{i,kk} \tag{50}$$

On substituting (49) in (46) we obtain

$$[(\lambda_2 + \mu_2) + (\lambda_1 + \mu_1)]\, u^1_{k,ki} + (\mu_2 + \mu_1)u^1_{i,kk} -$$
$$\frac{1}{\alpha}\{\mu_1\mu_2\, u^1_{i,kkll} + [\lambda_1\lambda_2 + 2\lambda_1\mu_2 + 2\lambda_2\mu_1 + 3\mu_1\mu_2]u^1_{k,klli}\} = 0 \tag{51}$$

and a similar equation by means of (50) and (45) for u_i^2

$$[(\lambda_2+\mu_2)+(\lambda_1+\mu_1)]\, u^2_{k,ki}+(\mu_2+\mu_1)u^2_{i,kk}-$$
$$\frac{1}{\alpha}\{\mu_1\mu_2\, u^2_{i,kkll}+[\lambda_1\lambda_2+2\lambda_1\mu_2+2\lambda_2\mu_1+3\mu_1\mu_2]u^2_{k,klli}\} = 0 \tag{52}$$

Moreover, substitution of (49) into (44) gives

$$\sigma^2_{ij} = \lambda_2 u^1_{k,k}\delta_{ij}+\mu_2(u^1_{i,j}+u^1_{j,i})-$$
$$\frac{1}{\alpha}\{\lambda_2(\lambda_1+2\mu_1)u^1_{k,kll}\delta_{ij}+\mu_1\mu_2(u^1_{i,jkk}+u^1_{j,ikk})\}-\frac{2}{\alpha}\{\mu_2(\lambda_1+\mu_1)u^1_{k,kij}\} \tag{53}$$

while substitution of (50) into (43) gives

$$\sigma^1_{ij} = \lambda_1 u^2_{k,k}\delta_{ij}+\mu_1(u^2_{i,j}+u^2_{j,i})-$$
$$\frac{1}{\alpha}\{\lambda_1(\lambda_2+2\mu_2)u^2_{k,kll}\delta_{ij}+\mu_1\mu_2(u^2_{i,jkk}+u^2_{j,ikk})\}-\frac{2}{\alpha}\{\mu_1(\lambda_2+\mu_2)u^2_{k,kij}\} \tag{54}$$

In view of (53) and (54) the total stress of the mixture can be written in terms of the average u_i and difference displacements v_i as

$$\sigma_{ij} = \sigma^1_{ij}+\sigma^2_{ij} =$$
$$(\lambda_2+\lambda_1)u_{k,k}\delta_{ij}+(\mu_2+\mu_1)(u_{i,j}+u_{j,i})-$$
$$\frac{1}{\alpha}\{[\lambda_2(\lambda_1+2\mu_1)+\lambda_1(\lambda_2+2\mu_2)]u_{k,kll}\delta_{ij}+2\mu_1\mu_2(u_{i,jkk}+u_{j,ikk})\}-$$
$$\frac{2}{\alpha}[\mu_2(\lambda_1+\mu_1)+\mu_1(\lambda_2+\mu_2)]u_{k,kij}+$$
$$(\lambda_2-\lambda_1)v_{k,k}\delta_{ij}+(\mu_2-\mu_1)(v_{i,j}+v_{j,i})-$$
$$\frac{1}{\alpha}[\lambda_2(\lambda_1+2\mu_1)-\lambda_1(\lambda_2+2\mu_2)]v_{k,kll}\delta_{ij}-$$
$$\frac{2}{\alpha}[\mu_2(\lambda_1+\mu_1)-\mu_1(\lambda_2+\mu_2)]v_{k,kij} \tag{55}$$

Since the stress for each constituent satisfies (42) the total stress satisfies the following balance equation

$$\sigma_{ij,j} = \overset{1}{\sigma}_{ij,j} + \overset{2}{\sigma}_{ij,j} = 0 \tag{56}$$

By summing (51) and (52) side by side we obtain

$$[(\lambda_2+\mu_2)+(\lambda_1+\mu_1)]\, u_{k,ki} + (\mu_2+\mu_1)u_{i,kk} - \frac{1}{\alpha}\{\mu_1\mu_2\, u_{i,kkll} + [\lambda_1\lambda_2 + 2\lambda_1\mu_2 + 2\lambda_2\mu_1 + 3\mu_1\mu_2]u_{k,klli}\} \tag{57}$$

On the other hand, by using (55), we obtain from (56) the relation

$$[(\lambda_2+\mu_2)+(\lambda_1+\mu_1)]\, u_{k,ki} + (\mu_2+\mu_1)u_{i,kk} - \frac{2}{\alpha}\{\mu_1\mu_2\, u_{i,kkll} + [\lambda_1\lambda_2 + 2\lambda_1\mu_2 + 2\lambda_2\mu_1 + 3\mu_1\mu_2]u_{k,klli}\} + [(\lambda_2+\mu_2)-(\lambda_1+\mu_1)]v_{k,ki} + (\mu_2-\mu_1)v_{i,kk} = 0 \tag{58}$$

On comparing (57) and (58) we conclude

$$\frac{1}{\alpha}\{\mu_1\mu_2\, u_{i,kkll} + [\lambda_1\lambda_2 + 2\lambda_1\mu_2 + 2\lambda_2\mu_1 + 3\mu_1\mu_2]u_{k,klli}\} - [(\lambda_2+\mu_2)-(\lambda_1+\mu_1)]v_{k,ki} + (\mu_2-\mu_1)v_{i,kk} = 0 \tag{59}$$

Next, we define

$$S_{ij} = (\lambda_2-\lambda_1)v_{k,k}\delta_{ij} + (\mu_2-\mu_1)(v_{i,j}+v_{j,i}) \tag{60}$$

and

$$\Sigma_{ij} = \frac{1}{\alpha}\{[\lambda_1\lambda_2 + 2\lambda_1\mu_2 + 2\lambda_2\mu_1 + 2\mu_1\mu_2]u_{k,kll}\delta_{ij} + \mu_1\mu_2(u_{i,jll} + u_{j,ill})\} \tag{61}$$

and observe that the relation

$$S_{ij} = \Sigma_{ij} \tag{62}$$

implies that, condition (59) is satisfied. From (62), the symmetric part of the gradient of the "difference displacement" is found as

$$v_{i,j} + v_{j,i} = \frac{1}{2(\mu_2-\mu_1)}\left[\Sigma_{ij} - \frac{\lambda_2-\lambda_1}{3(\lambda_2-\lambda_1)+2(\mu_2-\mu_1)}\Sigma_{kk}\delta_{ij}\right] \tag{63}$$

In terms of the average displacement u_i and the difference displacement v_i, the constitutive equations of each constituent (43) and (44)) read

$$\sigma_{ij}^{1} = \lambda_1 u_{k,k}\delta_{ij} + \mu_1(u_{i,j} + u_{j,i}) + \lambda_1 v_{k,k}\delta_{ij} + \mu_1(v_{i,j} + v_{j,i}) \tag{64}$$

$$\sigma_{ij}^{2} = \lambda_2 u_{k,k}\delta_{ij} + \mu_2(u_{i,j} + u_{j,i}) - \lambda_2 v_{k,k}\delta_{ij} - \mu_2(v_{i,j} + v_{j,i}) \tag{65}$$

So that the expression for the total stress σ_{ij} becomes

$$\sigma_{ij} = \sigma_{ij}^{1} + \sigma_{ij}^{2} = (\lambda_1 + \lambda_2)u_{k,k}\delta_{ij} + (\mu_1 + \mu_2)(u_{i,j} + u_{j,i}) + (\lambda_1 - \lambda_2)v_{k,k}\delta_{ij} + (\mu_1 - \mu_2)(v_{i,j} + v_{j,i}) \tag{66}$$

Upon substitution of (63) into (67) we finally arrive at the following gradient-dependent expression for the total stress.

$$\sigma_{ij} = (\lambda_1 + \lambda_2)u_{k,k}\delta_{ij} + (\mu_1 + \mu_2)(u_{i,j} + u_{j,i}) - \{[\lambda_1\lambda_2 + 2\lambda_1\mu_2 + 2\lambda_2\mu_1 + 2\mu_1\mu_2]u_{k,k,ll}\delta_{ij} + \mu_1\mu_2(u_{i,j} + u_{j,i})\}_{,ll} \tag{67}$$

This is identical to the constitutive equation of the special theory of gradient elasticity as introduced on purely phenomenological grounds.

4. CRACK PROBLEMS

In this section, solutions to the three basic modes of crack problems will be given by employing gradient theory of elasticity. As is well known, linear elasticity predicts infinite stress and strain at the crack tip and, thus, fails to give an accurate description of the state of affairs in that region. Since the singularity in both stress and strain at the crack tip is not realistic, it prohibits the use of any fracture criteria based on stresses or strains. Various alternatives have been introduced to circumvent this difficulty (see, for example, Unger [55]). Although these concepts, such as stress intensity factor, the J-integral, and other fracture toughness parameters have often been proven useful for engineering purposes, they do not provide any information about the structure of the crack tip. In this connection, we would like to mention two models previously proposed in order to explain the structure of the crack tip. Elliot [56] proposed an atomistic model which is basically a discretized version of the

continuum. An important result of in this study is that the adjacent atomic planes defining the crack surface displace with respect to each other beyond the crack tip in contrast to the results of classical elasticity. In his celebrated work, Barrenblatt [57] has introduced a small cohesive zone (and corresponding interatomic forces) ahead of the "physical" crack tip whose size is explicitly determined by requiring the cancellation of singularity at the tip of the cohesive zone (or the tip of the "effective" crack). However, in this model, the slope of the crack opening displacement at the physical crack/cohesive zone tip becomes infinite, even though a smooth closure of the crack faces is assured. Having recognized the importance of the interatomic forces, Eringen *et.al.* [58] have attacked the crack problem by using nonlocal elasticity. Their work seems to indicate that nonlocal elasticity eliminates the stress singularity at the crack tip. However, the solution seems to be approximate, in the sense that the stress boundary condition at the crack surface is not satisfied exactly.

Recently, crack problems are investigated by the special form of gradient elasticity. Altan and Aifantis [59] solved mode III crack problem within the framework of the special form of gradient elasticity. In this study which will be summarized here, the off-plane component of the classical surface traction and the second derivative of the off-plane displacement are set equal to zero on the crack surface. The interesting features of this solution is that the strain is finite everywhere (including the crack tip) and the displacement is discontinuous not only on the crack surface but also on the crack plane outside the crack. Ru and Aifantis [34] developed a method for reducing traction boundary value problems in gradient elasticity to corresponding problems of classical elasticity. They obtained an expression for the crack opening displacement by requiring it to vanish at the tip of the mathematical crack. Unger and Aifantis [60] considered the mode III crack problem by searching for the "small scale yielding" analogue of the classical solution within the structure of gradient elasticity. Their solution exhibits a smooth closure at the crack tip and can also lead to oscillatory crack profiles. Vardoulakis, Exadaktylos & Aifantis [61] considered a modified form of gradient elasticity to solve the mode III crack problem. This form motivated by the work of Casal [62] and considers an additional term in the strain energy function to account for surface energy effects. They formulated the mode III crack problem in such a way that the solution of the problem was reduced to the solution of a successive system of dual integral equations. By using the Riemann-Liouville fractional integral representation, the system of successive dual integral equations were transformed into a Fredholm integral equation of the first kind and then this equation was transformed to a Fredholm integral equation of the second kind by employing a generalized Delta function. Based upon this procedure, they found that the crack tip forms a cusp of the first kind and of zero enclosed angle with zero first derivative of the displacement at the crack tip.

In what follows, the solution of the mode III crack problem within the framework of the special form of gradient elasticity is outlined and the solution of the other two basic modes is also provided. We consider an infinite medium weakened by a line crack which is located at $l \le x \le l$, y=0. The crack is viewed as an interior surface, and therefore is treated as a boundary of the body. Since the theory we employ requires additional boundary conditions, the following extra boundary condition $\partial^2 u/\partial y^2$ is adopted on the crack surface where u is the appropriate component of the displacement field according to the type of the crack mode considered.

4.1 Formulation of Mode III Crack

As in the classical case, we assume that the displacement field for a crack loaded in the off-plane direction in the mode III configuration is given by

$$u_x = 0, \quad u_y = 0, \quad u_z = w(x, y) \tag{68}$$

The non-vanishing components of the strain and stress fields are

$$\varepsilon_{xz} = \frac{1}{2}\frac{\partial w}{\partial x}, \quad \varepsilon_{yz} = \frac{1}{2}\frac{\partial w}{\partial y} \tag{69}$$

and

$$\sigma_{xz} = \mu\left\{\frac{\partial w}{\partial x} - c\left(\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right)\right\} \quad , \quad \sigma_{yz} = \mu\left\{\frac{\partial w}{\partial y} - c\left(\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{\partial^3 w}{\partial y^3}\right)\right\} \tag{70}$$

The stress components are obtained from (10). Introducing the stresses into the balance equation (7) the following fourth order differential equation is obtained

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - c\left(\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial x^4} + \frac{\partial^4 w}{\partial y^4}\right) = 0 \tag{71}$$

The Fourier transform of this equation on x-coordinate is

$$c\frac{d^4 \bar{w}}{dy^4} - (1 + 2c\xi^2)\frac{d^2 \bar{w}}{dy^2} + \xi^2(1 + c\xi^2)\bar{w} = 0 \tag{72}$$

where $\bar{w}(\xi, y)$ is the Fourier transform of the displacement and is defined as

$$\bar{w}(\xi, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} w(x, y)e^{i\xi x}dx \tag{73}$$

The characteristic equation of (72) is $[cD^2 - (1 + c\xi^2)][D^2 - \xi^2] = 0$

and, thus, the solution of (72) for $y \geq 0$ reads

$$\bar{w}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{-y\sqrt{(1+c\xi^2)/c}} \tag{74}$$

where the condition that the displacement vanishes as for $y \to \infty$ was used. In view of the fact that the displacement field is symmetric with respect to x, i.e. $w(x, y) = w(-x, y)$ for $-\infty < y < \infty$ it follows that

$$w(x, y) = \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^{\infty}\left[A(\xi)e^{-\xi y} + B(\xi)e^{-y\sqrt{(1+c\xi^2)/c}}\right]\cos(x\xi)d\xi \tag{75}$$

The Fourier transform of the stress components are

$$\begin{aligned} \sigma_{yz}(x, y) &= -\mu\sqrt{\frac{2}{\pi}}\int_0^{\infty}\xi A(\xi)e^{-\xi y}\cos(x\xi)d\xi \\ \sigma_{xz}(x, y) &= -\mu\sqrt{\frac{2}{\pi}}\int_0^{\infty}\xi A(\xi)e^{-\xi y}\sin(x\xi)d\xi \end{aligned} \tag{76}$$

It is interesting to note that the stress field is independent of the extra unknown function $B(\xi)$. As discussed in detail by Altan and Yelkenci [63] it is not necessary to define the cracks as a mixed boundary value problem. Accordingly, we seek a solution for the mode III crack corresponding to the following boundary conditions

$$\sigma_{yz}(x, 0) = \tau_0 \quad \text{and} \quad \frac{\partial^2}{\partial y^2}w(x, 0) = 0 \quad \text{for} \quad |x| \leq l \tag{77}$$

Note that the boundary conditions are imposed only on the crack surface. By following the procedure detailed in [63] we consider the Neumann series expansion of the unknown function $A(\xi)$, i.e.

$$\xi A(\xi) = \sum_{n=0}^{\infty} a_n J_n(l\xi) \tag{78}$$

which, upon substitution in $(76)_1$ gives

$$\sigma_{yz}(x, y) = -\mu\sqrt{\frac{2}{\pi}}\int_0^\infty\left(\sum_{n=0}^{\infty} a_n J_n(l\xi)\right) e^{-\xi y}\cos(x\xi)d\xi$$

In order to find the unknown coefficients a_n's the boundary condition $(77)_2$ is used, i.e.

$$\sum_{n=0}^{\infty} a_n\int_0^\infty J_n(l\xi)(\cos(x\xi)d\xi) = -\sqrt{\frac{\pi}{2}}\frac{\tau_0}{\mu} \tag{79}$$

where (Gradshteyn & Ryzhik [64](6.671.2)),

$$\int_0^\infty J_n(l\xi)\cos(x\xi)d\xi = \cos[n\ arc\sin(x/l)]/\sqrt{l^2 - x^2} \quad , \quad |x| \le l \tag{80}$$

It can also be shown that

$$\begin{aligned} &\cos[(2k+1)arc\sin(x/l)]/\sqrt{l^2 - x^2} = (-1)^k l^{-1} U_{2k}(x/l) \\ &\cos[2k\ arc\sin(x/l)]/\sqrt{l^2 - x^2} = (-1)^k T_{2k}(x/l)/\sqrt{l^2 - x^2} \end{aligned} \tag{81}$$

where U_{2k} and T_{2k} are Chebyshev polynomials of the second and the first kind, respectively. By inserting (80) and (81) in (79) we arrive at

$$\sum_{k=0}^{\infty}\left\{a_{2k+1}(-1)^k l^{-1} U_{2k}(x/l) + a_{2k}\frac{(-1)^k}{\sqrt{l^2 - x^2}} T_{2k}(x/l)\right\} = -\sqrt{\frac{\pi}{2}}\frac{\tau_0}{\mu} \quad \text{for} \quad |x| \le l \tag{82}$$

As is clearly seen from this result, the only possibility to satisfy the stress boundary condition $(77)_1$ is to take

$$a_1 = -\sqrt{\frac{\pi}{2}}\frac{\tau_0}{\mu} \quad , \quad a_k = 0 \quad k = 0, 2, \ldots \tag{83}$$

from which we conclude that

$$A(\xi) = -\sqrt{\frac{\pi}{2}}\frac{l\tau_0}{\mu}\frac{J_1(l\xi)}{\xi} \tag{84}$$

The unknown function $A(\xi)$ is thus the same as in the classical elasticity.

In order to find the other unknown function $B(\xi)$ we employ the boundary condition $(77)_2$ by following similar procedure. The boundary condition $(77)_2$ leads to

$$\int_0^\infty \frac{1+c\xi^2}{c} B(\xi)\cos(x\xi)d\xi = -\sqrt{\frac{\pi}{2}}\frac{\tau_o l}{\mu}\frac{d}{dx}\int_0^\infty J_1(l\xi)\sin(x\xi)d\xi = -\sqrt{\frac{\pi}{2}}\frac{\tau_o l}{\mu}\left\{1-\left(\frac{x}{l}\right)\right\}^{-3/2}$$

$$|x/l| \le 1 \qquad (85)$$

(Gradshteyn & Ryzhik [64](4.451.4)). Upon substitution of the Neumann series expansion for the unknown function $B(x)$

$$\frac{1+c\xi^2}{c} B(\xi) = \xi \sum_{n=1}^{\infty} b_n J_n(l\xi) \qquad (86)$$

in (85) we obtain

$$\sum_{n=1}^{\infty} b_n \left\{\frac{d}{dx}\int_0^\infty J_n(l\xi)\sin(x\xi)d\xi\right\} = -\sqrt{\frac{\pi}{2}}\frac{\tau_o l}{\mu}\left\{1-\left(\frac{x}{l}\right)\right\}^{-3/2} \qquad (87)$$

Since

$$\int_0^\infty J_n(l\xi)\sin(x\xi)d\xi = \sin\{n\cdot arc\sin(x/l)\}/\sqrt{l^2-x^2} \quad , \quad |x| \le l \qquad (88)$$

the boundary condition $(77)_2$ is satisfied by

$$b_1 = -\sqrt{\frac{\pi}{2}}\frac{\tau_o l}{\mu} \quad , \quad b_k = 0 \quad , \quad k = 2, 3, \ldots \qquad (89)$$

It is interesting to note that (89) suggest that the boundary condition $(77)_2$ is satisfied not only for $|x| < l$ but on the whole x-axis. In conclusion, the boundary conditions $(77)_1$ and $(77)_2$ are satisfied by

$$A(\xi) = -\sqrt{\frac{\pi}{2}}\frac{l\tau_0}{\mu}\frac{J_1(l\xi)}{\xi} \qquad \text{and} \qquad B(\xi) = -\sqrt{\frac{\pi}{2}}\frac{\tau_0}{\mu}\frac{c\xi}{1+c\xi^2}J_1(l\xi) \qquad (90)$$

4.2 Formulation of Mode I and II Cracks

The procedure for solving the mode I and mode II crack problems is similar to the one employed for the mode III crack. We assume that the displacement field for these problems are of the form

$$u = u(x, y), \quad v = v(x, y), \quad w = 0 \tag{91}$$

The components of the strain tensor corresponding to this displacement field are

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \varepsilon_z = 0, \quad \gamma_{xz} = 0, \quad \gamma_{yz} = 0 \tag{92}$$

and the corresponding components of the stress tensor are

$$\begin{aligned}
\sigma_{xx} &= (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y} - c\nabla^2\left[(\lambda + 2\mu)\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right] \\
\sigma_{yy} &= (\lambda + 2\mu)\frac{\partial v}{\partial y} + \lambda\frac{\partial u}{\partial x} - c\nabla^2\left[(\lambda + 2\mu)\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right] \\
\sigma_{xy} &= \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - c\mu\left[\nabla^2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \\
\sigma_{zz} &= \lambda\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - c\lambda\left[\nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] \qquad , \qquad \sigma_{zx} = 0 \quad , \quad \sigma_{zy} = 0
\end{aligned} \tag{93}$$

Introducing these stress expressions into the balance equation (7) we obtain

$$\begin{aligned}
&(\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial^2 v}{\partial x \partial y} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right) - \\
&c\nabla^2\left\{(\lambda + \mu)\frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial^2 v}{\partial x \partial y} + \mu\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y}\right)\right\} = 0
\end{aligned} \tag{94}$$

and

$$\begin{aligned}
&(\lambda + 2\mu)\frac{\partial^2 v}{\partial y^2} + \lambda\frac{\partial^2 u}{\partial x \partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right) - \\
&c\nabla^2\left\{(\lambda + \mu)\frac{\partial^2 v}{\partial y^2} + \lambda\frac{\partial^2 v}{\partial x \partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y}\right)\right\} = 0
\end{aligned} \tag{95}$$

For solving these coupled partial differential equations we use the Fourier transform technique. In the Fourier space (94) and (95) become

$$-(\lambda+2\mu)(1+c\xi^2)\xi^{-2}u+[(\lambda+2\mu)(c\xi^2)+\mu(1+c\xi^2)]\frac{d^2\bar{u}}{dy^2}-$$
$$-\mu c\frac{d^4\bar{u}}{dy^2}-(i\xi)(\lambda+\mu)\left[(1+c\xi^2)\frac{d\bar{v}}{dy}-c\frac{d^3\bar{u}}{dy^3}\right]=0 \tag{96}$$

and

$$-\mu(1+c\xi^2)\xi^2\bar{v}+[(\lambda+\mu)(1+c\xi^2)+\mu(c\xi^2)]\frac{d^2\bar{v}}{dy^2}-$$
$$(\lambda+2\mu)c\frac{d^4\bar{v}}{dy^2}-(i\xi)(\lambda+\mu)\left[(1+c\xi^2)\frac{d\bar{u}}{dy}-c\frac{d^3\bar{u}}{dy^3}\right]=0 \tag{97}$$

where $\bar{u}(\xi, y)$ and $\bar{v}(\xi, y)$ are the Fourier transforms of $u(x, y)$ and $v(x, y)$

$$\bar{u}(\xi, y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}u(x, y)e^{i\xi x}dx \quad , \quad \bar{v}(\xi, y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}v(x, y)e^{i\xi x}dx \tag{98}$$

The characteristic equations of these differential equations are

$$[cD^2-(1+c\xi^2)]\{[\mu D^2-(\lambda+2\mu)\xi^2]\bar{u}+(i\xi)(\lambda+\mu)D\bar{v}\}=0 \tag{99}$$
$$[cD^2-(1+c\xi^2)]\{[\mu D^2-(\lambda+2\mu)\xi^2]\bar{u}+(i\xi)(\lambda+\mu)D\bar{v}\}=0 \tag{100}$$

With the definitions

$$[cD^2-(1+c\xi^2)]\bar{u}=\bar{U} \quad , \quad [cD^2-(1+c\xi^2)]\bar{v}=\bar{V} \tag{101}$$

we obtain from (99) and (100) the following set of coupled ordinary linear differential equations for the determination of $\bar{V}$ and $\bar{U}$

$$[\mu D^2-(\lambda+2\mu)\xi^2]\bar{U}+(i\xi)(\lambda+\mu)D\bar{V}=0 \tag{102}$$
$$[(\lambda+2\mu)D^2-\mu\xi^2]\bar{V}+(i\xi)(\lambda+\mu)D\bar{U}=0 \tag{103}$$

Upon elimination of $\bar{V}$ or $\bar{U}$ between these two equations we obtain

$$(D^2-\xi^2)^2\bar{U}=0\ ,\ (D^2-\xi^2)^2\bar{V}=0 \tag{104}$$

It follows that the characteristic equations for $\bar{u}$ and $\bar{v}$ are

$$[cD^2-(1+c\xi^2)](D^2-\xi^2)^2\bar{u}=0 \quad \text{and} \quad [cD^2-(1+c\xi^2)](D^2-\xi^2)^2\bar{v}=0 \tag{105}$$

From these results, the solution for $\bar{u}$ and $\bar{v}$ can be written as follows

$$\bar{u}(\xi, y) = \{A_1(\xi)+yB_1(\xi)\}e^{-|\xi|y}+C_1(\xi)e^{-y\sqrt{(1+c\xi^2)/c}} \qquad y\geq 0 \tag{106}$$

and

$$\bar{v}(\xi, y) = \{A_2(\xi)+yB_2(\xi)\}e^{-|\xi|y}+C_2(\xi)e^{-y\sqrt{(1+c\xi^2)/c}} \qquad y\geq 0 \tag{107}$$

Since the original equations for $\bar{u}$ and $\bar{v}$ are (99) and (100), the unknown coefficients appearing in (106) and (107) should satisfy the following relations

$$iB_2 = \operatorname{sgn}(\xi)B_1 \quad , \quad iA_2 = \operatorname{sgn}(\xi)A_1+\xi\frac{\lambda+3\mu}{\lambda+\mu}B_1 \tag{108}$$

4.2.1 Mode I

Before we list the boundary conditions for mode I we note the symmetry properties for the displacement field $u(x, y) = -u(-x, y)$ and $v(x, y) = v(-x, y)$ imply the following conditions for the unknown functions entering in the displacement expressions (106) and (107)

$$A_1(\xi) = -A_1(-\xi) \quad , \quad B_1(\xi) = -B_1(-\xi) \quad , \quad C_1(\xi) = -C_1(-\xi)$$

and (109)

$$A_2(\xi) = A_2(-\xi) \quad , \quad B_2(\xi) = B_2(-\xi) \quad , \quad C_2(\xi) = C_2(-\xi)$$

For the mode I crack we have the following boundary conditions.

$$\sigma_{yy} = -\sigma_0 \quad , \quad \frac{\partial^2 v}{\partial y^2} = 0 \quad , \quad \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{for } y = 0 \ , \ |x|\leq l$$

and (110)

$$\sigma_{xy} = 0 \quad \text{for} \quad y = 0$$

The boundary condition $(110)_2$ imply

$$-|\xi|A_1(\xi)+(1-2c\xi^2)B_1(\xi)-i|\xi|A_2(\xi)-2c\xi^2B_2(\xi) = 0$$

which upon combination with (108) gives

$$A_2(\xi) = \frac{1}{\xi}\left(\frac{\lambda+2\mu}{\lambda+\mu} - 2c\xi^2\right)B_2(\xi) \quad \text{for} \quad \xi \geq 0 \tag{111}$$

and

$$A_1(\xi) = -\frac{i}{\xi}\left(\frac{\mu}{\lambda+\mu} + 2c\xi^2\right)B_2(\xi) \quad \text{for} \quad \xi \geq 0 \tag{112}$$

On the other hand, in view of (93), (107-110), (111) and (112) we have

$$\bar{\sigma}_{yy}(\xi, 0) = -2\mu B_2(\xi) \quad \text{for} \quad \xi \geq 0 \tag{113}$$

By following the procedure which already outlined for mode III, we conclude that

$$B_2(\xi) = \sqrt{\frac{\pi}{2}}\,\frac{l\sigma_0}{2\mu}\,J_1(l\xi) \tag{114}$$

With the results (108), (112-113) and (114) at hand we have (for $y \geq 0$ and $\xi \geq 0$)

$$\begin{aligned}
\bar{u}(\xi, y) &= -i\sqrt{\frac{\pi}{2}}\,\frac{l\sigma_0}{2\mu}\left\{\left(\frac{\mu}{\lambda+\mu} + 2c\xi^2\right) + \xi y\right\}\frac{J_1(l\xi)}{\xi}e^{-\xi y} + C_1(\xi)e^{-y\sqrt{(1+c\xi^2)/c}} \\
\bar{v}(\xi, y) &= \sqrt{\frac{\pi}{2}}\,\frac{l\sigma_0}{2\mu}\left\{\left(\frac{\lambda+2\mu}{\lambda+\mu} - 2c\xi^2\right) + \xi y\right\}\frac{J_1(l\xi)}{\xi}e^{-\xi y} + C_2(\xi)e^{-y\sqrt{(1+c\xi^2)/c}}
\end{aligned} \tag{115}$$

In order to find the unknown coefficients $C_1(\xi)$ and $C_2(\xi)$ we employ the extra boundary conditions on the displacement field. The inverse transform of the u component of the displacement field can be written as

$$u(x, y) = -i\sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}(\xi, y)\sin(x\xi)d\xi$$

and, thus, the boundary condition $\dfrac{\partial^2 u}{\partial y^2} = 0 \quad y = 0 \;,\; |x| < l$ implies

$$i\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1+c\xi^2}{c}\,C_1(\xi)\,\sin(x\xi)d\xi = \frac{l\sigma_0}{2\mu}\int_0^\infty \xi^2\left\{\frac{2\lambda+\mu}{\lambda+\mu} - 2c\xi^2\right\}\frac{J_1(l\xi)}{\xi}\sin(x\xi)d\xi \tag{116}$$

By a similar procedure as for the mode III problem we find

$$C_1(\xi) = -i\sqrt{\frac{\pi}{2}}\frac{\sigma_0}{2\mu}\frac{c\xi}{1+c\xi^2}\left\{\frac{2\lambda+\mu}{\lambda+\mu}-2c\xi^2\right\}J_1(l\xi) \quad , \quad \xi \geq 0 \tag{117}$$

The inverse transform for the v component of the displacement field is

$$v(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{v}(\xi, y)(\cos(x\xi)\, d\xi$$

and, thus, the boundary condition $\frac{d^2 v}{dy^2} = 0 \quad y = 0 \ , \ |x| \leq l$ implies

$$\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(1+c\xi^2)}{c} C_2(\xi)\cos(x\xi)d\xi = \frac{l\sigma_0}{2\mu}\int_0^\infty \xi^2\left\{\frac{\lambda}{\lambda+\mu}+2c\xi^2\right\}\frac{J_1(l\xi)}{\xi}\cos(x\xi)d\xi \tag{118}$$

By a similar procedure as for the mode III problem, we find

$$C_2(\xi) = -\sqrt{\frac{\pi}{2}}\frac{\sigma_0}{2\mu}\frac{c\xi}{1+c\xi^2}\left\{\frac{\lambda}{\lambda+\mu}+2c\xi^2\right\}J_1(l\xi) \quad , \quad \xi \geq 0 \tag{119}$$

4.2.3. Mode II

The displacement field for the mode II crack satisfies the symmetry properties $u(x, y) = u(-x, y)$ and $v(x, y) = -v(-x, y)$ which imply

$$A_1(\xi) = A_1(-\xi) \ , \ B_1(\xi) = B_1(-\xi) \ , \ C_1(\xi) = C_1(-\xi)$$

and (120)

$$A_2(\xi) = -A_2(-\xi) \ , \ B_2(\xi) = -B_2(-\xi) \ , \ C_2(\xi) = -C_2(-\xi)$$

The relevant boundary conditions are

$$\sigma_{xy} = -\sigma_0 \ , \ \frac{\partial^2 v}{\partial y^2} = 0 \ , \ \frac{\partial^2 u}{\partial y^2} = 0 \ \text{ for } \ y = 0 \ , \ |x| \leq l$$

and (121)

$$\sigma_{yy} = 0 \ \text{ for } \ y = 0$$

The boundary condition $(121)_2$ imposes the following condition on the unknown coefficients.

$$-i\lambda\xi A_1(\xi) - 2i\lambda c\xi^2 B_1(\xi) - (\lambda+2\mu)\xi A_2(\xi) + (\lambda+2\mu)(1-2c\xi^2)B_2(\xi) = 0$$

which upon combination with (108) gives

$$A_2(\xi) = -\frac{i}{\xi}\left(\frac{\mu}{\lambda+\mu} - 2c\xi^2\right)B_1(\xi) \quad \text{for} \quad \xi \geq 0 \tag{122}$$

and

$$A_1(\xi) = -\frac{1}{\xi}\left(\frac{\lambda+2\mu}{\lambda+\mu} + 2c\xi^2\right)B_1(\xi) \quad \text{for} \quad \xi \geq 0 \tag{123}$$

On the other hand, in view of (93), (107-109), (120), (122) and (123) we have

$$\bar{\sigma}_{xy}(\xi, 0) = -2\mu B_1(\xi) \quad \text{for} \quad \xi \geq 0 \tag{124}$$

By following the same procedure as for the mode III problem, we conclude that

$$B_1(\xi) = \sqrt{\frac{\pi}{2}}\,\frac{l\sigma_0}{2\mu}\,J_1(l\xi) \tag{125}$$

With the results (108), (123-124) and (125) at hand we have (for $y \geq 0$ and $\xi > 0$)

$$\bar{u}(\xi, y) = -\sqrt{\frac{\pi}{2}}\,\frac{l\sigma_0}{2\mu}\left\{\frac{\lambda+2\mu}{\lambda+\mu} + 2c\xi^2 - \xi y\right\}J_1\frac{(l\xi)}{\xi}e^{-\xi y} + C_1(\xi)e^{-y\sqrt{(1+c\xi^2)/c}}$$

and (126)

$$\bar{v}(\xi, y) = -\,i\sqrt{\frac{\pi}{2}}\,\frac{l\sigma_0}{2\mu}\left\{\frac{\mu}{\lambda+\mu} - 2c\xi^2 + \xi y\right\}J_1\frac{(l\xi)}{\xi}e^{-\xi y} + C_2(\xi)e^{-y\sqrt{(1+c\xi^2)/c}}$$

In order to find the unknown coefficients $C_1(\xi)$ and $C_2(\xi)$ we employ the extra boundary conditions for the displacement field. The inverse transform of the u component of the displacement field can be written as

$$u(x, y) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}(\xi, y)\cos(x\xi)d\xi$$

from the boundary condition $\dfrac{\partial^2 u}{\partial y^2}$ $y = 0$, $|x| \leq l$ we arrive at

$$i\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1+c\xi^2}{c} C_1(\xi)\cos(x\xi)d\xi =$$
$$-\frac{l\sigma_0}{2\mu}\int_0^\infty \xi^2\left\{\frac{\lambda}{\lambda+\mu}-2c\xi^2\right\}\frac{J_1(l\xi)}{\xi}\cos(x\xi)d\xi = \quad , \quad |x|\le l \tag{127}$$

By a similar procedure that we have followed for mode III problem we find

$$C_1(\xi) = -\sqrt{\frac{\pi}{2}}\frac{\sigma_0}{2\mu}\frac{c\xi}{1+c\xi^2}\left\{\frac{\lambda}{\lambda+\mu}-2c\xi^2\right\}J_1(l\xi) \tag{128}$$

The inverse transform of the v component of the displacement field is

$$v(x,y) = -i\sqrt{\frac{2}{\pi}}\int_0^\infty \bar{v}(\xi,y)\sin(x\xi)\,d\xi$$

From the boundary condition $\frac{d^2v}{dy^2} = 0 \quad y = 0 \;,\; |x|\le l$ we arrive at

$$i\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1+c\xi^2}{c} C_2(\xi)\sin(x\xi)d\xi = \frac{l\sigma_0}{2\mu}\int_0^\infty \xi^2\left\{\frac{2\lambda+\mu}{\lambda+\mu}+2c\xi^2\right\}\frac{J_1(l\xi)}{\xi}\sin(x\xi)d\xi \tag{129}$$

By a similar procedure as for the mode III problem, we find

$$C_2(\xi) = -i\sqrt{\frac{\pi}{2}}\frac{\sigma_0}{2\mu}\frac{c\xi}{1+c\xi^2}\left\{\frac{\lambda}{\lambda+\mu}+2c\xi^2\right\}J_1(l\xi) \quad , \quad \xi\ge 0 \tag{130}$$

5. PROPAGATION OF PLANE WAVES

This section is devoted to some immediate consequences of gradient elasticity on wave propagation phenomena. Propagation of a disturbance in an infinite medium is investigated. Dispersion of harmonic waves and the attenuation effect is also discussed.

Wave propagation is generally dispersive, i.e. the frequency or the speed of propagation (phase velocity) is a function of the wavelength (or wave number), a result more pronounced at the range of small wavelengths and/or at large travelling distances of waves. Studies on dispersive elastic wave propagation have

been motivated by the problems of seismic wave propagation and the nature of the seismograms recorded from earthquakes (see, for example, Postma [65] and the references cited therein). On the other hand, the importance of dispersion in the application of acoustic waves in mining and non-destructive testing of materials is well-known. Moreover, as the use of composite materials increases, a simple approach to modeling dispersion will become quite useful since it is a well-accepted fact that wave propagation in composites is dispersive (see, for example, Herrmann & Achenbach [66] or Drumheller & Bedford [67]).

In order to indicate the relation between dispersive wave propagation and higher-order strain gradients, we refer to a paper by Kohn [68] in which one-dimensional wave propagation in a periodically-structured medium is considered. It is assumed that a harmonic motion in a periodically-structured medium can be represented as follows

$$u(x,t) = v(x,k)e^{i(kx-\omega t)} \tag{131}$$

where $v(x,k)$ is a strictly periodic function, i.e. $v(x,k) = v(x+a,k)$. By considering the following series expansion

$$v(x,k) = 1 + (ik)v_1(x) + (ik)^2 v_2(x) + \ldots$$

and a corresponding polynomial expression for the dispersion relation

$$\omega^2(k) = c^2k^2 - \beta^2k^4 + \ldots$$

a consecutive set of equations for $v_1(x), v_2(x), \ldots$ are obtained. It is shown that "*the envelope function*" U(x,t) which is defined by

$$u(x,t) = \left\{1 + v_1(x)\frac{\partial}{\partial x} + v_2(x)\frac{\partial^2}{\partial x^2} + \ldots\right\}U(x,t)$$

satisfies the differential equation

$$\frac{\partial^2}{\partial t^2}U(x,t) = \left\{c^2\frac{\partial^2}{\partial x^2} - \beta^2\frac{\partial^4}{\partial x^4} + \ldots\right\}U(x,t) \tag{132}$$

Finally, it is shown that the envelope function U(x,t) alone can describe all mechanical quantities, such as strain and stress in the medium. The impor-

tance of this study as related to the gradient elasticity is Eq. (132). Even though conventional elasticity was assumed, the existence of dispersion in the medium resulted in a field equation (132) for the envelope function which contains higher-order derivatives with respect to the space variable. As will be shown in the next section, the equation of motion of the gradient elasticity is exactly the same as (132) for one-dimensional problems.

For the sake of simplicity we will deal with propagation of one-dimensional longitudinal waves. As usual, we consider the following displacement field

$$u_1(\underset{\sim}{x}, t) = u(x, t), \qquad u_2 = 0, \qquad u_3 = 0 \tag{133}$$

for which the strains are

$$\varepsilon_{11} = \varepsilon(x, t) \qquad \text{others} \quad \varepsilon_{ij} = 0 \tag{134}$$

and the stresses are given by

$$\sigma_{11} = \sigma_x = (\lambda + 2\mu)\left\{\varepsilon - c\frac{\partial^2 \varepsilon}{\partial x^2}\right\}, \ \sigma_{22} = \sigma_{33} = \lambda\left\{\varepsilon - c\frac{\partial^2 \varepsilon}{\partial x^2}\right\} \ \text{others} \ \sigma_{ij} = 0 \tag{135}$$

For one-dimensional problems the following boundary values should be prescribed for a finite domain $(0,l)$.

$$u(0, t) \quad \text{or} \quad \sigma_x(0, t), \quad ; \quad u(l, t) \quad \text{or} \quad \sigma_x(l, t)$$

and (136)

$$\varepsilon(0, t) \quad \text{or} \quad c(\lambda + 2\mu)\left\{\frac{\partial \varepsilon}{\partial x}\right\}_{x=0} \quad ; \quad \varepsilon(l, t), \quad \text{or} \quad c(\lambda + 2\mu)\left\{\frac{\partial \varepsilon}{\partial x}\right\}_{x=l}$$

The relevant initial conditions are the same as in classical elasticity, i.e. the values

$$u(x, 0) \quad \textit{and} \quad \dot{u}(x, 0) \tag{137}$$

should be supplied when dealing with initial value problems.

Before we proceed, it is pointed out that the equation of motion for the gradient elasticity reads

$$(\lambda + 2\mu)\left\{\frac{\partial^2 u}{\partial x^2} - c\frac{\partial^4 u}{\partial x^4}\right\} = \rho\frac{\partial^2 u}{\partial t^2} \tag{138}$$

which is identical to (132)arrived at by a different argument. It is thus not unreasonable to assert that an elastic medium in which the wave propagation is dispersive, could be viewed as obeying an elastic gradient-dependent constitutive relation.

5.1 Dispersion Relation

In this section, we first show that wave propagation in gradient elasticity is dispersive and then discuss the structure and consequences of the dispersion relation. The dispersion relation is obtained by looking for a solution of the governing equations in the form

$$u(x, t) = Ae^{i(kx - \omega t)} \tag{139}$$

Equation (138) admits a solution of the form of (139) if

$$\omega^2 = v_L^2 k^2 (1 + ck^2) \tag{140}$$

This is the dispersion relation of gradient elasticity where

$$v_L^2 = (\lambda + 2\mu)/\rho \tag{141}$$

denotes the velocity of wave propagation (phase velocity) for longitudinal waves in conventional elasticity. Next, the structure of the dispersion relation (140) is discussed for different signs of the gradient parameter c.

i) For c = 0, the dispersion relation (140) reduces to the classical result

$$\omega = v_L k \tag{142}$$

for which the phase velocity defined by

$$v_p = \frac{\omega}{k} = v_L \tag{143}$$

and the group velocity defined by

$$v_g = \frac{d\omega}{dk} = v_L \tag{144}$$

are same.

ii) For c > 0 we have

$$\omega = v_L k\sqrt{1 + ck^2} \tag{145}$$

for which the frequency increases monotonically with increasing wave numbers. A natural result of this property is that the group velocity

$$v_g = \frac{d\omega}{dk} = v_L \frac{1 + 2ck^2}{\sqrt{1 + ck^2}} \tag{146}$$

is larger than the phase velocity

$$v_p = \frac{\omega}{k} = v_L\sqrt{1 + ck^2} \tag{147}$$

for every value of k, as it can easily be seen from the relation

$$v_g = v_p + v_L \frac{ck^2}{\sqrt{1 + ck^2}} \tag{148}$$

The case $v_g > v_p$ is called "anomalous dispersion" in the literature (Achenbach [69]).

iii) For c < 0 we have

$$\omega = v_L k\sqrt{1 - ck^2} \tag{149}$$

Eqn. (149) is quite close to the result obtained by using dynamic lattice theory (Brilluoin [70]), as shown in Fig.1.

Following Brilluoin [70], we find the cut-off frequency and the corresponding wave number as

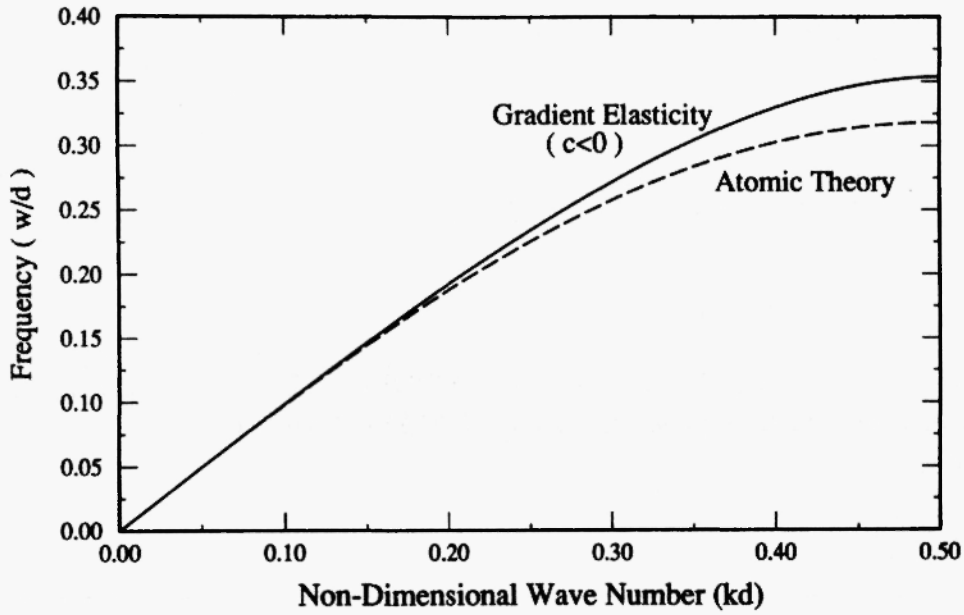

Figure 1: Dispersion relation in one Brillouin Zone

$$\omega_c = \frac{v_L}{2\sqrt{c}} \quad and \quad k_c = \frac{1}{\sqrt{2c}} \tag{150}$$

It follows that waves with frequency $\omega > \omega_c$ cannot propagate with real k. The imaginary part of k is known to be the *attenuation factor*. This factor can be found by solving k in terms of real ω's from (149)

$$\Omega^2 = (\omega/v_L)^2 = k^2(1-ck^2) \quad , \quad k_{1,2}^2 = \frac{1 \mp \sqrt{1-4c\Omega^2}}{2c} \tag{151}$$

It is quite clear that gradient elasticity is capable of modeling the attenuation effect, which is also an important issue in composite materials (see, for example, Christensen [71]). This phenomenon will be studied extensively in connection with composite materials. It is noted in a future publication in more detail, however, that the uniqueness of corresponding boundary value problems is not unconditionally assured if c<0.

5.2 Propagation of a Disturbance

In this section, we discuss the general solution of the one-dimensional ini-

tial-value problem in gradient elasticity. Consider an infinite medium which is at rest for t < 0. Let a one-dimensional disturbance defined by

$$u(x, o) = U(x) \quad and \quad \dot{u}(x, o) = V(x) \tag{152}$$

be introduced into the medium at t=0. The governing differential equation is still given by (138). Application of Fourier transform on x upon this equation yields

$$\frac{d^2\bar{u}}{dt^2} + v_L^2 k^2 (1 + ck^2)\bar{u} = 0 \tag{153}$$

where $\bar{u}$ is the Fourier transform of u given by

$$\bar{u}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx} u(x, t) dx \tag{154}$$

i) c > 0

In this case the general solution of (153) reads

$$\bar{u}(k, t) = A(k) \cos \omega(k) t + B(k) \sin \omega(k) t \tag{155}$$

where ω(k) is the frequency defined by (145) and A(k), B(k) are unknown coefficients to be determined by the initial conditions (152). In fact, if $\bar{U}$ and $\bar{V}$ denote respectively Fourier transforms of the initial displacement U and the initial velocity V, respectively. it can easily be shown that

$$A(k) = \bar{U}(k) \quad and \quad B(k) = \bar{V}(k)/\omega(k) \tag{156}$$

Thus, we arrive at

$$\bar{u}(k, t) = \bar{U}(k) \cos \omega(k) t + \{\bar{V}(k)/\omega(k)\} \sin \omega \tag{157}$$

which by inverse Fourier transform gives

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \{\bar{U}(k) \cos \omega(k) t + [\bar{V}(k)/\omega(k)] \sin \omega(k) t\} e^{-ikx} dx \tag{158}$$

Notice that if $\bar{V}(k) = 0$ and $\omega(k) = v_0 k$ (classical case), the solution can be written in the form

$$u(x,t) = \tfrac{1}{2}\{U(x - v_o t) + U(x - v_o t)\} \tag{159}$$

which is the well known D'Alembert solution of classical one-dimensional elastodynamics. In the present case, the form of an initial disturbance is subject to change while it is propagating, because of the dispersive properties exhibited by the gradient elasticity and therefore The D'Alembert solution is not applicable in gradient elasticity model.

ii) c < 0

In this case the corresponding differential equation becomes

$$v_L^2\left(\frac{\partial^2 u}{\partial x^2} + c\frac{\partial^4 u}{\partial x^4}\right) = \frac{\partial^2 u}{\partial t^2} \tag{160}$$

whose Fourier transform is

$$\frac{d^2\bar{u}}{dt^2} + v_L^2 k^2(1 - ck^2)\bar{u} = 0 \tag{161}$$

The first complication with this equation arises in the sign of the coefficient of $\bar{u}$, that is

$$v_L^2 k^2(1 - ck^2) > 0 \quad \text{for} \quad k < 1/\sqrt{c} \tag{162}$$

$$v_L^2 k^2(1 - ck^2) < 0 \quad \text{for} \quad k > 1/\sqrt{c} \tag{163}$$

indicating that the solution of (160) can be written in the form

$$\bar{u}(k,t) = A(k)\cos\omega(k)t + B(k)\sin\omega(k)t \quad \text{for} \quad k < 1/\sqrt{c} \tag{164}$$

$$\bar{u}(k,t) = E(k)e^{-\omega^*(k)t} + F(k)e^{\omega^*(k)t} \quad \text{for} \quad k > 1/\sqrt{c} \tag{165}$$

where

$$\omega(k) = v_L k\sqrt{1 - ck^2} \quad \text{for} \quad k < \sqrt{c} \tag{166}$$

$$\omega^*(k) = v_L k\sqrt{ck^2 - 1} \quad \text{for} \quad k > \sqrt{c} \tag{167}$$

Note that the condition F(k) = 0 ensures finite displacements for increasing t. It can easily be verified that the initial conditions (152) are satisfied if

$$A(k) = U(k) \quad , \quad B(k) = V(k)/\omega(k) \quad , \quad k < 1/\sqrt{c} \tag{168}$$

$$E = \tfrac{1}{2}\{U - V/\omega^*\} \quad , \quad F = \tfrac{1}{2}\{U + V/\omega^*\} \quad , \quad k > 1/\sqrt{c} \tag{169}$$

and the solution can then be obtained by applying the inverse transform. It should be noted, however, that if

$$F(k) \neq 0 \rightarrow U(k) \neq -V(k)/w^*(k) \tag{170}$$

the solution grows unboundedly with increasing time. Note that uniqueness of the initial-boundary value problems is not unconditionally assured in this case.

From the above discussion we conclude that the role of the sign of the gradient parameter c on wave propagation studies may be different than in uniqueness studies, Uniqueness is always assured by the positive definiteness of the strain energy which, in turn, implies a positive c. On the other hand, wave propagation results similar to the lattice theory are established for negative c. Mindlin [70] encountered the same difficulty in discussing the dispersion relations for a simple cubic Bravais lattice. On the other hand, Beran and McCoy [71] concluded (by comparing the solutions obtained by using gradient and nonlocal elasticity) that it was necessary to give up the positive definiteness of the strain energy density in order to obtain the correct solution of the problem of point force in an infinite medium.

Bedford and Stern [47] developed a "multi-continuum" theory which is a special mixture theory in which the constituents interact by means of intrinsic body forces which depend on the constituent relative motion. The theory was applied to wave propagation in elastic laminates (Stern & Bedford [48]) and in fiber-reinforced elastic material (Bedford, Sutherland & Linge [49]). The dispersion of plane harmonic waves propagating in a direction normal to the fibers was found to obey the relation

$$\frac{v_p^2}{v_L^2} = \frac{\varpi^2 - \varpi}{\beta\omega^2 - \alpha} \tag{171}$$

where $\omega(k) = v_0 k$ is the phase velocity, v_L is defined by (141) with the elastic coefficients determined by the effective properties of the composite, α and β are constants related to the material and geometric properties of the composite. Next, we wish to compare the dispersion relations (171) and (166). It can be easily shown that the phase velocity predicted by the gradient elasticity can be arranged as using (151))

$$v_p = v_L \sqrt{\frac{1}{2}\left(1 + \sqrt{1 - \frac{4c}{v_L^2}\omega^2}\right)} \tag{172}$$

Predictions of both expressions (171) and (172) are quite close for $\alpha = 16$, $\beta = .852$, $c = 0.5625$ in a wide range of frequency. Dispersion relations obtained from both approach are shown in the Fig.2

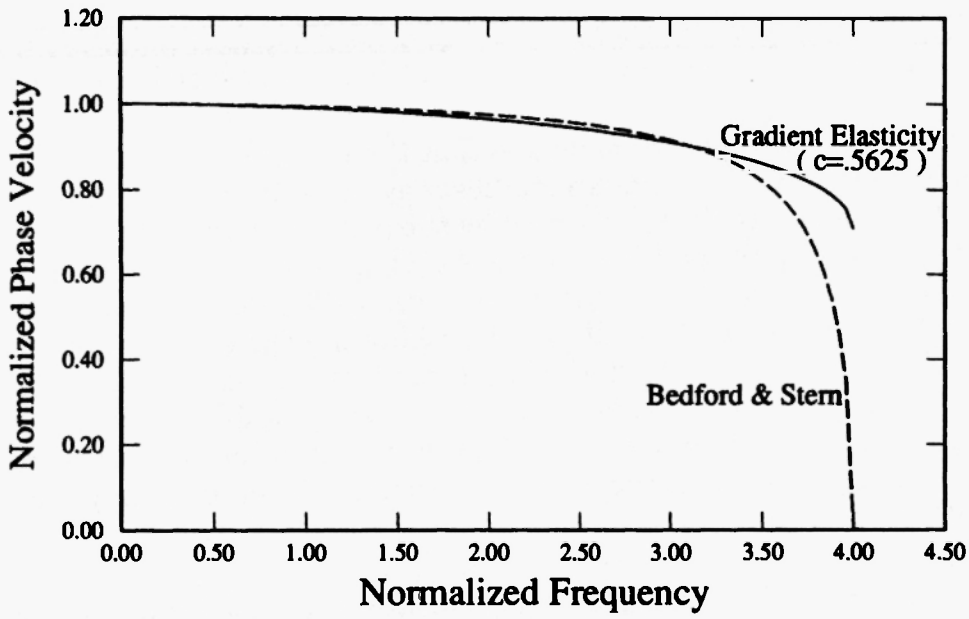

Figure 2 Dispersion relations in gradient elasticity and a "composites" theory

6. TENSILE BAR

In this section we analyze the displacement field in a straight beam loaded by a homogeneous axial stress. the corresponding one-dimensional displacement field is

$$u_1 = u(x) \quad , \quad u_2 = u_3 = 0 \tag{173}$$

The corresponding strain and stress components are

$$\varepsilon_{11} = \varepsilon_x = \varepsilon = \frac{\partial u}{\partial x} \quad \text{all} \quad \textit{others} \quad \text{zero} \tag{174}$$

$$\sigma_{11} = \sigma = (\lambda + 2\mu)\left\{\varepsilon - c\frac{\partial^2 \varepsilon}{\partial x^2}\right\} \quad , \quad \sigma_{22} = \sigma_3 = \lambda\left\{\varepsilon - c\frac{\partial^2 \varepsilon}{\partial x^2}\right\} \tag{175}$$

The equation of motion is

$$\frac{\partial \sigma}{\partial x} = \rho \ddot{u} \tag{176}$$

and the standard boundary conditions are (for a finite domain $0 \le x \le l$)

$$u(0) = U_1 \quad \text{or} \quad \sigma(0) = T_1 \quad ; \quad u(l) = U_2 \quad \text{or} \quad \sigma(l) = T_2 \tag{177}$$

while the non-standard ones are

$$\varepsilon(0) = \bar{E}_1 \quad \text{or} \quad \varepsilon'(0) = S_1 \quad ; \quad \varepsilon(0) = E_1 \quad \text{or} \quad \varepsilon'(l) = S_2 \tag{178}$$

The initial conditions are identical to those of classical elasticity.

Next, we would like to demonstrate the effect of the extra boundary conditions (e.g. the boundary conditions given by (178)) on the displacement of a tensile bar. To this end, we note first that the stress field in the beam is symmetric with respect to the mid-point of the beam, and the same is true for $\varepsilon(x)$ and $e''(x)$. This property, which implies that

$$\varepsilon'(l/2) = 0 \tag{179}$$

simplifies the analysis on the effect of the extra boundary condition.

i) *Strains prescribed at the ends of the bar*

Here, we consider the following boundary conditions

$$\sigma(0) = \sigma(l) = T \quad , \quad \varepsilon(0) = K \quad , \quad \varepsilon'(l/2) = 0 \tag{180}$$

where T and K are constants. Then the relevant differential equation for the strain field

$$\varepsilon' - c\varepsilon''' = 0 \tag{181}$$

obtained by combination of (175) and (176) for the static case of $\ddot{u} = 0$. The general solution of (181) is

$$\varepsilon(x) = A + Be^{-x/\sqrt{c}} + Ce^{x/\sqrt{c}} \tag{182}$$

and the corresponding stress field is

$$\sigma = (\lambda + 2\mu)A \tag{183}$$

i.e. a constant. By using the remaining boundary conditions we find the coefficient A as

$$A = T/(\lambda + 2\mu) \tag{184}$$

By using the other boundary conditions we obtain the solution as

$$\varepsilon(x) = A + \frac{K - A}{ch(l/2\sqrt{c})} ch[(2x - l)/(2\sqrt{c})] \tag{185}$$

The variation of this strain field is displayed in Fig.3 for

$$A = 2 \quad , \quad \text{K=1} \quad \text{and} \quad L = l/\sqrt{c} = 2 \; ; 20 \tag{186}$$

ii) *Strain gradient prescribed at the ends of the bar*

$$\sigma(0) = \sigma(l) = T \quad \varepsilon'(0) = S \quad \varepsilon'(l/2) = 0 \tag{187}$$

The solution can easily be obtained from (182) as

$$\varepsilon(x) = A - \frac{S\sqrt{c}}{sh(l/2\sqrt{c})} ch\{(2x - l)/2\sqrt{c}\} \tag{188}$$

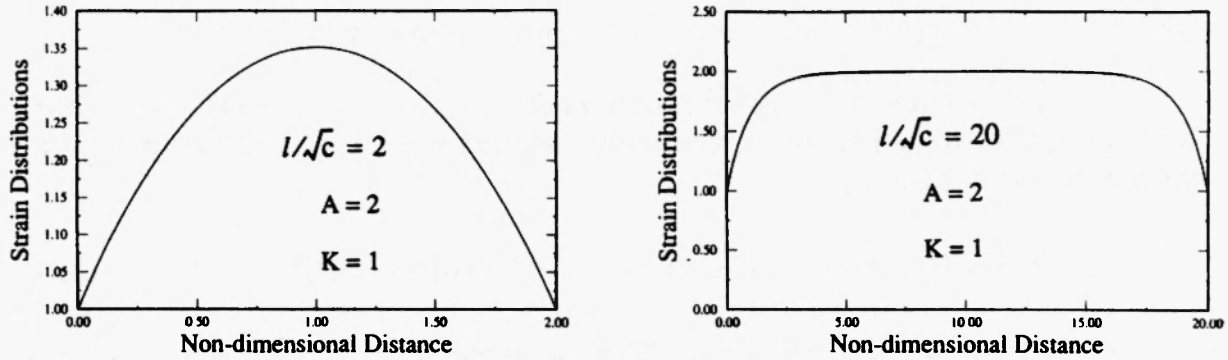

Figure 3: Strain distribution when the strain assigned at the ends of the bar

The variation of this strain field is displayed in Fig.4 for

$$A = 2 \quad , \quad S\sqrt{c} = 1 \quad \text{and} \quad L = l/\sqrt{c} = 2\ ;\ 20 \tag{189}$$

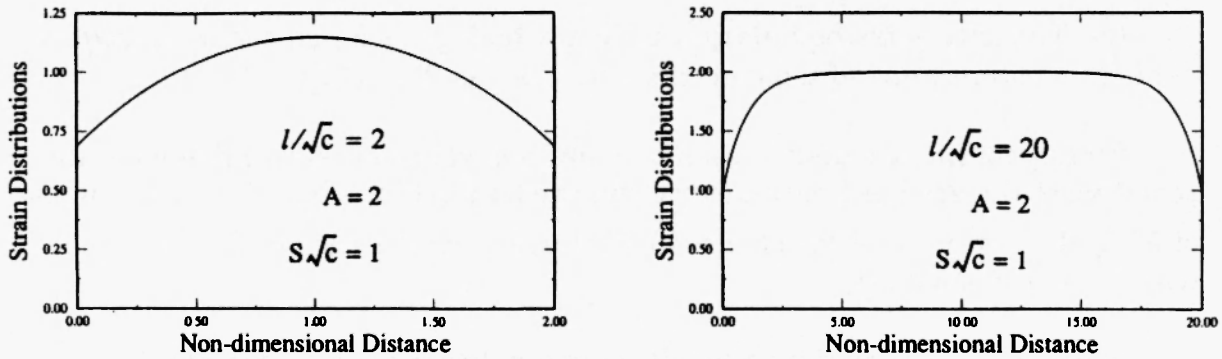

Figure 4: Strain distribution: gradient of strain described at the end of the beam

The following observations can be made on the basis of (185) and (188). From these results the following point are interesting.

(i)If $\varepsilon(0) = K \neq 0$ (extra bc) the strain distribution is generally not homogeneous even if the tension (standard bc) is zero. The strain field is given by

$$\varepsilon(x) = Kch\{(2x-l)/(2\sqrt{c})\}/ch\{l/(2\sqrt{c})\} \tag{190}$$

which can be interpreted as a stress-free deformation in a tensile bar.

(ii) If $\varepsilon(0) = 0$ (extra bc) the strain field does not vanish throughout the bar. Instead, if the applied stress T is constant (standard bc) the strain field is inhomogeneous and given by

$$\varepsilon(x) = [T/(\lambda + 2\mu)]\{1 - ch[(2x - l)/(2\sqrt{c})]/ch[l/(2\sqrt{c})]\} \tag{191}$$

(iii) If $\varepsilon'(0) = S \neq 0$ the strain field corresponding to a homogeneous tension in the bar is not homogeneous even if the tension is zero. The strain distribution is given by

$$\varepsilon(x) = S\sqrt{c}\{ch[(2x - l)/(2\sqrt{c})]/sh[l/(2\sqrt{c})]\} \tag{192}$$

which is also a stress-free deformation in a tensile bar (please note the similarity between (190) and (192))

(iv) The only extra boundary condition which gives a homogeneous strain field for constant applied tension (classical case) is that $\varepsilon'(0) = 0$.

(v) In general, the distribution of a non-homogeneous strain field is quite sensitive to the gradient parameter c and the length of the bar. For large values of $l/\sqrt{c}$ the variation of the strain field is large near the ends of the bar, otherwise is almost constant.

From these observations the following conclusions can be drawn:

a) The gradient theory of elasticity is capable of predicting end effects unless $\varepsilon'(0) = 0$. The "surface effect" in solids seems thus to be included in the theory presented here.

b) The strain distribution corresponding to an applied constant tension is not homogeneous and reaches its maximum value in the middle of the bar. This property is a clear indication that the gradient elasticity can be employed as a potential model to explain the necking phenomenon or the strain localization phenomenon in linear elastic phases. This property may also be used to establish the gradient parameter c through a carefully performed experiment.

c) The strain distribution is strongly dependent on the length l of the bar,

especially when the ratio $l/\sqrt{c}$ is small. This suggests that the gradient theory presented in this study may be quite suitable of modeling the behavior of thin films.

7. LONGITUDINAL VIBRATIONS OF A BAR

The main purpose of this section is to display the consequences of the present gradient elasticity vibration problems. The longitudinal vibration of a bar under homogeneous boundary conditions is analyzed. It is shown that the vibration modes contain not only travelling waves but also exponential end-effect terms.

Consider the natural frequencies of the longitudinal free vibrations of a straight beam whose ends are fixed. The equation of motion is

$$\frac{\partial^2 u}{\partial x^2} - c\frac{\partial^4 u}{\partial x^4} = \frac{1}{C_1^2}\frac{\partial^2 u}{\partial t^2} \qquad C_1^2 = \rho/(\lambda + 2\mu) \tag{193}$$

For wave type solutions of the form $u(x,t) = Ae^{i(kx - \omega t)}$ the relevant dispersion relation is given by

$$\omega^2 = C_1^2 k^2 (1 + ck^2) \tag{194}$$

as discussed earlier by using a somewhat different notation. On considering solutions of (193) of the form

$$u(x,t) = U(x)T(t) \tag{195}$$

it can easily be shown that the functions U and T satisfy the following differential equations

$$\begin{aligned} &cU'''' - U'' - k^2(1 + ck^2)U = 0 \\ &T'' - C_1^2 k^2 (1 + ck^2)T = 0 \end{aligned} \tag{196}$$

The general solution of $(196)_1$ is

$$U(x) = A(k)\cos(kx) + B(k)\sin(kx) + C(k)e^{-\kappa x} + D(k)e^{\kappa x} \tag{197}$$

where

$$\sqrt{\{1 + ck^2\}/c} = \kappa \tag{198}$$

The following homogeneous boundary conditions are considered

$$U(0) = 0 \ , \quad U(l) = 0 \ , \quad \varepsilon(0) = 0 \ , \quad \varepsilon(l) = 0 \tag{199}$$

leading to the characteristic equation

$$(1/c)\sin(kl)sh(l\kappa) + 2k\kappa - 2k\kappa\cos(kl)ch(l\kappa) = 0 \tag{200}$$

The roots of this transcendental equation describe to the natural frequencies and the corresponding natural modes of the beam. For computational purposes, it is convenient to express (200) in the following non-dimensional form

$$F(\lambda, L) = 2\lambda\sqrt{L^2 + \lambda^2}(\cos\lambda ch\sqrt{L^2 + \lambda^2} - 1)/L^2 - \sin\lambda sh\sqrt{L^2 + \lambda^2} \tag{201}$$

where the dimensionless number λ (not to be confused with the elastic constant λ) and L defined as follows

$$\lambda = kl \ , \quad L = l/\sqrt{c} \tag{202}$$

Some roots of this transcendental equation are displayed along with their classical counterparts (obtained for c=0 as the roots of $\sin\lambda = 0$) in Table 1. The frequencies corresponding to each mode are found by

$$\frac{\omega}{C_1} = k\sqrt{1 + ck^2} = \frac{\lambda}{l}\sqrt{1 + \lambda^2/L^2} \tag{203}$$

i.e. they are dependent on the gradient parameter and this dependence is stronger at higher frequencies. The non-dimensional frequencies ($\Omega = \lambda\omega/c_1$) are tabulated in TABLE 1, along with the corresponding frequencies of the classical case (c=0)

$$\Omega_c = \lambda_c \quad , \quad \Omega_g = \lambda_g\sqrt{1 + (\lambda_g \backslash L)^2} \tag{204}$$

TABLE 1. Comparison of normalized natural modes (λ) natural frequencies (Ω)

mode no.	λ, Ω classic	λ (gradient) L=1	L=20	L=500	L=5000	Ω (gradient) L=1	L=20	L=500	L=5000
1	3.142	4.707	3.481	3.154	3.143	22.646	3.534	3.154	3.143
2	6.283	7.845	6.915	6.308	6.286	62.045	7.317	6.309	6.286
3	9.425	10.992	10.283	9.463	9.429	121.312	11.562	9.464	9.429
4	12.566	14.135	13.591	12.617	12.571	200.288	16.431	12.621	12.571
5	15.708	17.277	16.853	15.771	15.714	298.997	22.038	15.779	15.714
6	18.850	20.419	20.083	18.925	18.857	417.442	28.460	18.939	18.857
7	21.991	23.561	23.289	22.079	22.000	555.623	35.747	22.101	22.000
8	25.133	26.703	26.480	25.234	25.143	713.541	43.935	25.266	25.143
9	28.274	29.845	29.659	28.388	28.286	891.198	53.048	28.433	28.286

The corresponding mode shapes are found as

$$X_\lambda(Z) = A\{\cos(\lambda Z) - B(\lambda)\sin(\lambda Z) - C(\lambda)e^{-\kappa z} - D(\lambda)e^{\kappa z}\}/(2\lambda sh\kappa - \kappa\sin\lambda) \quad (205)$$

where

$$\kappa = \sqrt{L^2+\lambda^2}\ , \quad Z = x/l\ , \quad B(\lambda) = 2\kappa(ch\kappa - \cos\lambda)$$
$$C(\lambda) = \lambda e^{\kappa} - \kappa\sin\lambda - \lambda\cos\lambda\ , \quad D(\lambda) = -\lambda e^{-\kappa} - \kappa\sin\lambda + \lambda\cos\lambda \quad (206)$$

The results obtained above clearly show that the predicted mode shapes of a longitudinally vibrating beam are composed of both traveling (sin and cos) and non-traveling (exponential or hyperbolic) terms, in contrast to the classical elasticity which predicts only travelling modes. The following features are interesting to discuss:

(i) The wave numbers and especially the frequencies are strongly dependent on the gradient parameter c. If the ratio $l\sqrt{c}$ is small, the wave numbers and frequencies drastically deviate from the conventional results. For large values of the ratio $l/\sqrt{c}$, the wave numbers and frequencies asymptotically

approach the conventional results. In fact, for the first twenty modes the difference between the conventional and the gradient results are almost negligible for $l/(\sqrt{c} \sim 5000$. For smaller values of the ratio $l/\sqrt{c}$, the difference between the conventional and the gradient results (especially for the frequencies) remain large for all modes.

(ii) The frequencies are dependent on the wavelength. Gradient elasticity is capable of predicting a dispersion relationship. If the gradient parameter is positive then the frequencies increase monotonically. If the gradient parameter is negative the frequencies become imaginary and mode transition occur. Instead of travelling wave, an evanescent wave appears which decays as it travels away from the boundaries.

(iii) The mode shapes given by (205) contain oscillatory terms and exponential terms. Every propagating wave is accompanied by a non-propagating (evanescent) component in gradient elasticity, although this form is not predicted in the conventional theory.

The these observations the following conclusions can be drawn:

a) The wave numbers and the frequencies are quite different from the conventional results, especially for small values of $l/\sqrt{c}$. This phenomenon should be checked by experiments which may yield a realistic estimation of the gradient parameter. Gradient elasticity appears to be more suitable than conventional elasticity for describing wave propagation in thin layers, for example, laminated composites.

b) If the characteristic length of the structure is large compared to characteristic length $\sqrt{c}$ associated with the gradient parameter, and small compared to the wavelength then the results obtained from gradient and classical elasticities coincide.

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