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Poisson-Nijenhuis structures

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ABSTRACT. — We study the deformation, defined by a Nijenhuis operator, and the dualization, defined by a Poisson bivector, of the Lie bracket of vector fields on a manifold and, more generally, of the Lie bracket on a differential Lie algebra over a commutative algebra. Requiring that the two processes commute, one obtains hierarchies of pairwise compatible Lie brackets on the module and on its dual. Each differential Lie algebra-structure on a module gives rise to a cohomology operator on the algebra of forms over the module, as well as to a graded Lie algebra-structure on the algebra of multivectors (the Schouten algebra). We study the deformation and the dualization of the derivations of the algebra of forms and of the Schouten bracket of multivectors, thus obtaining generalizations of the preceding differential geometric results together with new proofs. Section 2 comprises the study of the Nijenhuis operators on the twilled Lie algebras, an "N-matrix version" of the Kostant-Symes theorem, and an application to Hamiltonian systems of Toda type on semisimple Lie algebras.

RÉSUMÉ. — Nous étudions la déformation, définie par un tenseur de Nijenhuis, et la dualisation, définie par un bivecteur de Poisson, du crochet de Lie des champs de vecteurs sur une variété et, plus généralement, du crochet de Lie sur une algèbre de Lie différentielle sur une algèbre commutative. En imposant la commutativité des deux opérations, on obtient, sur le module et sur son dual, des hiérarchies de crochets de Lie deux à deux compatibles. Toute structure d'algèbre de Lie différentielle sur un module correspond d'une part à un opérateur de cohomologie sur l'algèbre des formes sur le module et, d'autre part, à une structure d'algèbre de Lie graduée sur l'algèbre des multivecteurs sur le module (algèbre de Schouten). Nous étudions la déformation et la dualisation des dérivations de l'algèbre des formes et du crochet de Schouten des multivecteurs, obtenant ainsi des généralisations de résultats précédents de géométrie différentielle, ainsi que des démonstrations nouvelles. Au paragraphe 2, nous étudions les opérateurs de Nijenhuis sur les algèbres de Lie bicroisées, une « version N-matrice » du théorème de Kostant-Symes, et une application à des systèmes hamiltoniens du type de Toda sur les algèbres de Lie semi-simples.

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INTRODUCTION

A *Poisson-Nijenhuis manifold* ([30], [31]) is a manifold equipped with both a Poisson structure, defined by a bivector P whose Schouten bracket vanishes, and a (1,1)-tensor N whose Nijenhuis torsion vanishes, to be called a Nijenhuis tensor, which satisfy a compatibility condition to be discussed below. The Poisson-Nijenhuis structures on manifolds and, in particular, on duals of Lie algebras constitute a natural framework for the theory of completely integrable systems. In work in progress, we shall study the relationships between the theory of Poisson-Nijenhuis structures and that of Poisson-Lie groups, bigebras and the modified Yang-Baxter equation as expounded in [21]. This article can be considered as Part 0 of the series of papers [21]. Here, we shall develop the theory of Poisson-Nijenhuis structures in a slightly more general sisuation than that of smooth manifolds since, in section 1, we study the Nijenhuis structures on real or complex Lie algebras, and we show, in section 6, that the rest of the theory developed in sections 3, 4 and 5 on smooth manifolds can be extended to the case of the modules over a communitative algebra which are "differential Lie algebras". (*See* subsection 6.1 for definitions and historical remarks.)

In section 1, we consider an infinitesimal deformation of a Lie algebra-strucure on a vector space E, and we prove that, when such a deformation is defined by an endomorphism N of E whose Nijenhuis torsion vanishes, all the brackets obtained by iteration are Lie brackets which are pairwise compatible. Section 2 studies the relationship between the Nijenhuis operators and the twilled Lie algebra-structures, involution theorems on the dual of a Lie algebra and, as an application, some Hamiltonian systems of Toda type on semisimple Lie algebras.

In section 3, we show how to construct a bracket on the space of 1-forms from the usual Lie bracket of vector fields and a bivector P, and we prove that this bracket is a Lie bracket if and only if the Schouten bracket, [P, P], vanishes. The Lie brackket of 1-forms on a Poisson manifold was introduced by Magri in [30], [31] and independently by several authors. (*See* subsection 3.2.)

For a (1,1)-tensor N, we have used the generally accepted definition of the *Nijenhuis torsion*, formula (1.1), and we have denoted it by $[N, N]_\mu$. (We remark that the bracket originally defined by Frölicher and Nijenhuis is twice this quantity.) For a bivector P, we have considered the *Schouten bracket* $[P, P]_\mu$ defined by (3.1). We have incorporated the factor 2 in the definition in order for this Schouten bracket to coïncide with the one considered in subsection 6.3. In both cases, the index μ refers to the Lie algebra-structure on the space of vector fields. Apart from the trivial question of normalization, the two objects play roles which are strikingly similar: both measure the failure of a mapping to be a morphism of brackets, as shown by formulae (1.6) and (3.5). When $[N, N]_\mu$ vanishes, $N.\mu$ is a Lie bracket and N is a morphism from $(E, N.\mu)$ to (E, μ). When $[P, P]_\mu$ vanishes, $\nu(\mu, P)$ defined by formula (3.2) is a Lie bracket on E^* and P defines a morphism from $(E^*, \nu(\mu, P))$ to (E, μ).

In section 4, we combine the process of deformation of the Lie algebra-structure of the linear space E of vector fields by means of a Nijenhuis tensor N, and that of the dualization of the Lie bracket on E by means of a Poisson bivector P. Requiring that the two processes commute introduces the compatibility condition linking N and P. The deformation process can be iterated, and, in section 5, we show that, when the compatibility condition is satisfied, a sequence of Lie brackets on the linear space E^* of 1-forms is obtained in this way. We study the hierarchy of Lie brackets on E and on E^* obtained from a Poisson-Nijenhuis structure on E, and we prove that the iterated deformations commute with the dualizations and that both the Lie brackets on E and the Lie brackets on E^* are pairwise compatible.

In section 6, we define and study the differential Lie algebras over a commutative algebra A, of which the $C^\infty M$-module of vector fields on a manifold M, with the usual Lie bracket, is the typical example. In subsections 6.1 and 6.2, we show how each differential Lie algebra structure on an A-module E defines a graded differential algebra-structure on $\Lambda(E^*)$, generalizing the de Rham cohomology operator on the exterior algebra of forms, and a graded Lie algebra-structure on ΛE, generalizing the Schouten bracket on the exterior algebra of multivectors.

In subsection 6.4, we show that, given an A-linear mapping N from E to E, the derivation of $\Lambda(E^*)$ associated with the deformed bracket $N.\mu$ on E is the Lie derivation (in the generalized sense) with respect to the E-valued 1-form N on E. We use this fact to give an alternate, less computational proof of a theorem of section 1: If the Nijenhuis torsion of N vanishes, then $N.\mu$ is a Lie bracket. We also study the deformed Schouten bracket on ΛE.

In subsection 6.5, we show that the dualization of the Lie bracket of vector fields described in section 3 can be carried out more generally on a differential Lie algebra E over a commutative algebra A. Given an A-linear, antisymmetric mapping, **P**, from E*, the A-dual of E, to E, there is an associated bracket on E* which is a Lie bracket if and only if the Schouten bracket of **P**, considered as a bivector, vanishes. We prove (proposition 6.2) that the derivation on multivectors associated with an A-linear, antisymmetric mapping **P** from E* to E is the operator [P, .], where [,] denotes the Schouten bracket and P is the bivector defined by **P**. When P is a Poisson bivector on a manifold, this derivation is, up to a sign, the G-cohomology operator introduced by Lichnerowicz [27]. We use this proposition to obtain a generalization together with an alternate proof of the theorem of section 3: On a nondegenerate differential Lie algebra, the bracket on E* associated with P is a Lie bracket if and only if the Schouten bracket of P vanishes. The Schouten bracket on Λ (E*), which extends the Lie bracket on E* defined by the Poisson bivector P, is the bracket introduced by Koszul in [23].

Since morphisms of differential Lie algebras correspond to morphisms of graded differential algebras between the algebras of forms, we obtain a proof of the fact that the exterior powers of the Poisson mapping intertwine the de Rham and the Lichnerowicz cohomology operators. Moreover, morphisms of differential Lie algebras correspond to morphisms of Schouten algebras and, in particular, when P is a Poisson bivector, Λ**P** is a morphism of Schouten algebras.

In conclusion, we stress the fact that the theory of Poisson-Nijenhuis structures applies to the case of the differential Lie algebras, and we outline some applications of this theory.

Many results presented here were previously formulated in various contexts. For instance, the work of de Barros ([10], [11]) is a comprehensive algebraization of the infinitesimal structures on a manifold under more general assumptions than ours and includes, as an application, the consideration of what we have called the deformed bracket $N.\mu$ and the proof that the vanishing of the Nijenhuis torsion of N implies the Jacobi identity for the deformed bracket $N.\mu$. This fact was also proved in [25]. It appears in a short announcement by Dorfman [12] with applications to the theory of integrable systems. We realized while writing this article that the connection between the Lie bracket of differential 1-forms and the Lichnerowicz cohomology operator on a Poisson manifold had been previously noted in [3], [4], and a preprint by Huebschmann [18] came to our attention in which a very general algebraic version of the theory (but not the facts about the existence of the Schouten Lie algebra of a differential Lie algebra) is developed. Many previously unrelated results from various areas are proved in this article and used as the building blocks of the theory of Poisson-Nijenhuis structures.

The main results of the theory had been proved in [30], but the methods of proof used here are new and less computational. The contents of sections 1, 3, 4 and 5 of this article were announced in various lectures [31 *a*] [21 *a*]. Some examples of Poisson-Nijenhuis structures appear in [31 *a*]. The connection between solutions of the modified Yang-Baxter equation and bihamiltonian structures on the dual of a Lie algebra (*see* subsection 6.7), which is due to Magri, was announced in a lecture by Magri (E.N.S., Paris, May 1988) and in [21 *a*].

1. DEFORMATION OF LIE BRACKETS BY MEANS OF A NIJENHUIS OPERATOR

In this section, we construct a hierarchy of pairwise compatible Lie brackets on a Lie algebra by means of a Nijenhuis operator, and we prove that the images of the center of the Lie algebra under the powers of the Nijenhuis operator are Abelian subalgebras with respect to each of these Lie algebra brackets.

1.1. Constructing deformed brackets from Nijenhuis operators

Let E be a vector space over the field of real or complex numbers. In the applications to differential geometry, E will be the vector space TM of smooth vector fields over M or its complexification, or the vector space T^*M of smooth differential 1-forms over M or its complexification.

Let μ be an E-valued 2-form on E that defines a Lie algebra-structure on E. The Lie bracket defined by μ will be denoted by $[\ ,\ ]_\mu$ or simply by $[\ ,\ ]$ when no confusion is possible.

Let N be a linear map from E to itself. The *Nijenhuis torsion* $[N, N]_\mu$ of N, with respect to the Lie algebra-structure μ, is the E-valued 2-form defined by

$$[N, N]_\mu (x, y) = [Nx, Ny]_\mu - N([Nx, y]_\mu + [x, Ny]_\mu) + N^2 [x, y]_\mu, \quad (1.1)$$

for x and y in E.

DEFINITION 1.1. — *If the Nijenhuis torsion of* N *vanishes, we shall say that* N *is a* Nijenhuis operator.

Let $N.\mu$ denote the E-valued 2-form on E defined by

$$(N.\mu)(x, y) = \mu(Nx, y) + \mu(x, Ny) - N(\mu(x, y)),$$

for x and y in E. The bracket defined by $N.\mu$ will be denoted by $[\ ,\ ]_{N.\mu}$. Thus, by definition,

$$[x, y]_{N.\mu} = [Nx, y]_\mu + [x, Ny]_\mu - N([x, y]_\mu). \quad (1.2)$$

The various interpretations of $N.\mu$ to be given below, as well as the examples and applications, will justify the introduction of the bracket

$[\ ,\ ]_{N.\mu}$, which we shall call, somewhat improperly, the *deformed bracket*. In the sense of the theory of deformations of Lie algebras [35], $N.\mu$ is a *trivial infinitesimal deformation of* μ since

$$[x, y]_{N.\mu} = \frac{d}{dt} \mu_t (x, y)\big|_{t=0},$$

where, for any t in $\mathbb{R}$,

$$\begin{aligned} \mu_t (x, y) &= \exp(-t N) (\mu (\exp(t N) x, \exp(t N) y)) \\ &= \mu (x, y) + t (N.\mu)(x, y) + \ldots, \end{aligned}$$

and μ_t defines a Lie algebra-structure on E that is isomorphic to μ.

In general, $N.\mu$ need not define a Lie bracket, even if μ itself does.

In corollary 1.1 below we prove however that $N.\mu$ does define a Lie bracket in the special case where N is a *Nijenhuis operator*. We also prove, that μ and $N.\mu$ are then *compatible* Lie algebra-structures in the following sense.

DEFINITION 1.2. — *Two Lie brackets* μ *and* μ' *are called* compatible *if their sum is also a Lie bracket.*

If μ and μ' are compatible Lie brackets, then for each t in $\mathbb{R}$ or $\mathbb{C}$, $\mu + t\mu'$ is a Lie bracket.

It follows from the general theory of deformations [35] that if μ' is an infinitesimal deformation of μ which is also a Lie algebra-structure, then for each t in $\mathbb{R}$ or $\mathbb{C}$, $\mu + t\mu'$ is a Lie algebra-structure and therefore μ and μ' are compatible. The one-parameter family of Lie algebra-structures $\mu + t\mu'$ is called an *actual deformation* of μ, and the Lie brackets $\mu + t\mu'$ are the *deformed brackets*. It is only by an abuse of language that we call $\mu' = N.\mu$ itself, when N is a Nijenhuis operator, a deformed Lie bracket.

To prove the properties of the deformed brackets, we shall make use of the Richardson-Nijenhuis bracket of E-valued forms on E.

1.2. Richardson-Nijenhuis bracket of E-valued forms on a vector space E and the properties of the deformed bracket

Let E be a vector space over the field $K = \mathbb{R}$ or $\mathbb{C}$. Let α be an E-valued K-multilinear form of degree a on E. To α there corresponds a derivation i_α of degree $a-1$ of the graded algebra of K-multilinear forms on E with values in an arbitrary vector space F, defined as follows. For each b-form β with values in F, if $b \geqq 1$, we define the $(a+b-1)$-form $i_\alpha \beta$ with values in F by

$$(i_\alpha \beta)(x_1, \ldots, x_{a+b-1}) = \sum_{\substack{\sigma(1) < \ldots < \sigma(a) \\ \sigma(a+1) < \ldots < \sigma(a+b-1)}} (-1)^{|\sigma|} \beta (\alpha (x_{\sigma(1)}, \ldots, x_{\sigma(a)}), x_{\sigma(a+1)}, \ldots, x_{\sigma(a+b-1)}),$$

and we set $i_\alpha \beta = 0$ for $b=0$, *i.e.*, when β is a function. We have denoted a permutation of $1, \ldots, a+b-1$ by σ, and its signature by $|\sigma|$. The x_i's, $i=1, \ldots, a+b-1$, denote elements of E. We remark that the form $i_\alpha \beta$ is denoted by $\beta \barwedge \alpha$ in [14].

Let $\mathbf{[\ ,\ ]}$ denote the commutator in the graded sense on the vector space of derivations of the space of E-valued forms on E. The formula

$$\mathbf{[}i_\alpha, i_\beta\mathbf{]} = i_{[\alpha,\ \beta]}$$

defines a bracket $[\ ,\ ]$ on the space of E-valued forms on E which coïncides with the one defined by Nijenhuis and Richardson in [35]. We have denoted this bracket by $[\ ,\ ]$ (small, light type face) to distinguish it form the Lie bracket $[\ ,\ ]$ (light type face), the bracket $\big[\ ,\ \big]$ (large, light type face) used to denote the Nijenhuis torsion, and from the commutator $\mathbf{[\ ,\ ]}$ (normal, heavy type face). If α is an E-valued form of degree a, and if β is an E-valued form of degree b, then

$$[\alpha, \beta] = i_\beta \alpha - (-1)^{(a-1)(b-1)} i_\alpha \beta. \tag{1.3}$$

The Richardson-Nijenhuis bracket is anticommutative (in the graded sense), and satisfies the Jacobi identity (in the graded sense).

It is known [35] that μ defines a Lie algebra-structure on E if and only if the Richardson-Nijenhuis bracket of μ, $[\mu, \mu]$, vanishes.

Let N be a linear mapping from E to E, considered as an E-valued 1-form on E. By the definition of $[N, \mu] = i_\mu N - i_N \mu = -[\mu, N]$, we see that

$$N . \mu = [\mu, N].$$

Moreover,

LEMMA 1.1. – *Let* $\big[N, N\big]_\mu$ *be the Nijenhuis torsion of* N *with respect to the Lie algebra-structure* μ *considered as an* E*-valued 2-form on* E. *Then*

$$[N . \mu, N . \mu] + 2\,[\mu, \big[N, N\big]_\mu] = 0. \tag{1.4}$$

Proof. – The proof is a straightforward computation, using the definition

$$(1/2)\,[N . \mu, N . \mu]\,(x_1, x_2, x_3) = \oint (N . \mu)((N . \mu)(x_1, x_2), x_3),$$

for x_1, x_2, x_3 in E, where $\oint$ denotes the sum over the cyclic permutations over the indices 1, 2 and 3.

The deformed structure $N . \mu$ is a Lie algebra-structure if and only if $[N . \mu, N . \mu] = 0$. Now, the preceding lemma shows that this condition is satisfied if and only if $[\mu, \big[N, N\big]_\mu]$ vanishes. This quantity has a simple cohomological interpretation.

Let us set, for any E-valued form α of degree a,

$$\delta_\mu \alpha = (-1)^{a-1}\,[\mu, \alpha]. \tag{1.5}$$

It follows from the graded Jacobi identity for the Richardson-Nijenhuis bracket that δ_μ is a cohomology operator. It is easy to check [35] that δ_μ is the coboundary operator of the cochain complex of E-valued forms on E, where E acts on itself by the adjoint action defined by μ. Clearly,

$$\mathrm{N}.\mu=\delta_\mu \mathrm{N}$$

and

$$[\mu, [\mathrm{N}, \mathrm{N}]_\mu]=-\delta_\mu([\mathrm{N}, \mathrm{N}]_\mu).$$

Thus $\mathrm{N}.\mu$ defines a Lie algebra bracket on E if and only if the Nijenhuis torsion of N is a 2-cocycle.

The Lie algebra-structures μ and $\mathrm{N}.\mu$ satisfy

$$[\mu, \mathrm{N}.\mu]=-\delta_\mu(\mathrm{N}.\mu)=-(\delta_\mu)^2\mathrm{N}=0.$$

It follows from this fact and from lemma 1.1, that, when the Nijenhuis torsion of N is a 2-cocycle, the expression

$$[\mu+t\,\mathrm{N}.\mu, \mu+t\,\mathrm{N}.\mu]=[\mu, \mu]+t\,([\mu, \mathrm{N}.\mu]+[\mathrm{N}.\mu, \mu])+t^2\,[\mathrm{N}.\mu, \mathrm{N}.\mu]$$

vanishes identically, for t in $\mathbb{R}$ or $\mathbb{C}$, which expresses the fact that μ and $\mathrm{N}.\mu$ are compatible.

In conclusion:

PROPOSITION 1.1. — *Let* μ *be a Lie algebra-structure on* E, *and let* N *be a linear map from* E *to* E. *Then* $\mathrm{N}.\mu=\delta_\mu\mathrm{N}$ *is a Lie algebra-structure on* E *if and only if the Nijenhuis torsion* $[\mathrm{N}, \mathrm{N}]_\mu$ *of* N *is* δ_μ*-closed, in which case* $[\ ,\ ]_\mu$ *and* $[\ ,\ ]_{\mathrm{N}.\mu}$ *are compatible.*

The following corollary will be important in the sequel.

COROLLARY 1.1. — *Let* μ *be a Lie algebra-structure on* E, *and let* N *be a linear map from* E *to* E. *Assume that the Nijenhuis torsion of* N *with respect to* μ *vanishes. Then*

(i) $\mathrm{N}.\mu$ *defines a Lie bracket,*

(ii) *the Lie bracket* $[\ ,\ ]_{\mathrm{N}.\mu}$ *is compatible with the Lie bracket* $[\ ,\ ]_\mu$,

(iii) N *is a Lie algebra-morphism from* $(\mathrm{E}, \mathrm{N}.\mu)$ *to* (E, μ).

Proof. — (i) and (ii) are a special case of proposition 1.1. (An alternate proof of (i) will moreover be given in section 6.) The last statement follows immediately from the identity

$$\mathrm{N}([x, y]_{\mathrm{N}.\mu})-[\mathrm{N}x, \mathrm{N}y]_\mu=-[\mathrm{N}, \mathrm{N}]_\mu(x, y). \tag{1.6}$$

We have shown that $\mathrm{N}.\mu$ is a Lie bracket compatible with μ under the assumption that the Nijenhuis torsion of N, with respect to μ, be closed in the Lie algebra-cohomology. In order to define the iterated brackets and to study their properties, we shall make use of the stronger hypothesis that the Nijenhuis torsion of N vanishes.

1.3. Properties of the iterated brackets

Let us assume, as in corollary 1.1, that N is a Nijenhuis operator. The identity

$$[\mathrm{N}, \mathrm{N}]_{\mathrm{N}\,.\,\mu}=[[\mathrm{N}, \mathrm{N}]_{\mu}, \mathrm{N}]$$

is easy to prove and shows that, when the torsion of N with respect to μ vanishes, the torsion of N with respect to $\mathrm{N}\,.\,\mu$ also vanishes. Therefore the deformation of μ into $\mathrm{N}\,.\,\mu$ can be iterated, and we shall prove below some properties of the brackets obtained from $\mathrm{N}\,.\,\mu$ by iteration.

The following lemma is well-known. (*See* e. g. note 5 of [26].)

LEMMA 1.2. – *If* $[\mathrm{N}, \mathrm{N}]_{\mu}=0$, *then, for each integer* $k \geqq 1$, $[\mathrm{N}^k, \mathrm{N}^k]_{\mu}=0$.

This property implies that we can deform the given Lie bracket not only by means of N but by any power N^k of N. The following lemma expresses the associativity of the deformation process.

LEMMA 1.3. – *Let* μ *and* N *be as in corollary* 1.1. *Then for each integer* $k \geqq 0$,

$$\mathrm{N}^{k+1}\,.\,\mu=\mathrm{N}\,.\,(\mathrm{N}^k\,.\,\mu).$$

Proof. – In fact, let

$$\Phi_{\mathrm{N}}^k=\mathrm{N}^{k+1}\,.\,\mu-\mathrm{N}\,.\,(\mathrm{N}^k\,.\,\mu).$$

Then

$$\begin{aligned}\Phi_{\mathrm{N}}^k(x, y)=&[\mathrm{N}^{k+1} x, y]+[x, \mathrm{N}^{k+1} y]-\mathrm{N}^{k+1}[x, y]\\ &-([\mathrm{N}^{k+1} x, y]+[\mathrm{N} x, \mathrm{N}^k y]-\mathrm{N}^k[\mathrm{N} x, y])\\ &-([\mathrm{N}^k x, \mathrm{N} y]+[x, \mathrm{N}^{k+1} y]-\mathrm{N}^k[x, \mathrm{N} y])\\ &+\mathrm{N}([\mathrm{N}^k x, y]+[x, \mathrm{N}^k y]-\mathrm{N}^k[x, y]).\end{aligned}$$

It is clear that $\Phi_{\mathrm{N}}^0=0$ and that $\Phi_{\mathrm{N}}^1=2\,[\mathrm{N}, \mathrm{N}]_{\mu}$. Moreover,

$$\Phi_{\mathrm{N}}^k(x, y)=[\mathrm{N}, \mathrm{N}]_{\mu}(\mathrm{N}^{k-1} x, y)+[\mathrm{N}, \mathrm{N}]_{\mu}(x, \mathrm{N}^{k-1} y)+\mathrm{N}(\Phi_{\mathrm{N}}^{k-1}(x, y)).$$

Therefore, induction shows that, if $[\mathrm{N}, \mathrm{N}]_{\mu}=0$, then $\Phi_{\mathrm{N}}^k=0$ for all $k \geqq 0$. Summarizing this discussion, we obtain:

PROPOSITION 1.2. – *Let* μ *be a Lie algebra-structure on* E, *and let* N *be a Nijenhuis operator on* (E, μ). *Let* k *and* m *be nonnegative integers. Then*

(i) $\mathrm{N}^k\,.\,\mu$ *defines a Lie bracket;*

(ii) *the Nijenhuis torsion of* N^m *with respect to the Lie algebra-structure* $\mathrm{N}^k\,.\,\mu$ *vanishes;*

(iii) *the Lie brackets* $[\ ,\ ]_{\mathrm{N}^k\,.\,\mu}$ *and* $[\ ,\ ]_{\mathrm{N}^m\,.\,\mu}$ *are compatible;*

(iv) N^m *is a Lie algebra-morphism from* $(\mathrm{E}, \mathrm{N}^{k+m}\,.\,\mu)$ *to* $(\mathrm{E}, \mathrm{N}^k\,.\,\mu)$.

Proof. — Part (i) follows from lemma 1.2 and corollary 1.1 (i). In order to prove (ii), we remark that the following formula is valid for each integer $k \geqq 0$,

$$[N, N]_{N^{k+1}.\mu} = [[N, N]_{N^k.\mu}, N]$$

or

$$[N, N]_{N^{k+1}.\mu}(x, y) = [N, N]_{N^k.\mu}(Nx, y) + [N, N]_{N^k.\mu}(x, Ny) - N([N, N]_{N^k.\mu}(x, y)).$$

The proof is by computation, using the result of lemma 1.3. From this formula, it follows that if $[N, N]_\mu$ vanishes, then the torsion of N with respect to each iterated bracket $N^k.\mu$ vanishes. Moreover, by lemma 1.2, since the torsion of N with respect to the Lie bracket $N^k.\mu$ vanishes, for any integer $m \geqq 0$, that of N^m vanishes also.

Part (iii) follows from corollary 1.1 (ii), using the facts that for $k > m$, $N^k.\mu = N^{k-m}.(N^m.\mu)$ (which follows from lemma 1.3), and that N^{k-m} has vanishing Nijenhuis torsion with respect to $N^m.\mu$. Part (iv) follows from corollary 1.1 (iii), using $N^{k+m}.\mu = N^m.(N^k.\mu)$ and $[N^m, N^m]_{N^k.\mu} = 0$.

1.4. Abelian subalgebras

We shall now prove an additional property of the deformed Lie brackets which has applications in the theory of integrable systems.

PROPOSITION 1.3. — *Let* N *be a Nijenhuis operator on a Lie algebra* (E, μ), *and let k and m be nonnegative integers. Then*

(*a*) *The center* C_μ *of* (E, μ) *is an Abelian subalgebra of* (E, $N^m.\mu$).

(*b*) *The image* $N^k(C_\mu)$ *of* C_μ *under* N^k *is an Abelian subalgebra of* (E, $N^m.\mu$).

Proof. — Let x and y be in the center of (E, μ). Then

$$[x, y]_{N^m.\mu} = [N^m x, y]_\mu + [x, N^m y]_\mu - N^m[x, y]_\mu = 0,$$

which proves (*a*). Moreover, since the Nijenhuis torsion of N^k with respect to $N^m.\mu$ vanishes,

$$[N^k x, N^k y]_{N^m.\mu} = N^k[x, y]_{N^{m+k}.\mu} = 0,$$

which proves (*b*).

In [8], Caccese states a theorem due to Mishchenko and Fomenko in a form close to the preceding result.

2. DEFORMATION OF LIE BRACKETS AND HAMILTONIAN SYSTEMS

In this section, we explain the close relationship between *twilled Lie algebras* and Nijenhuis operators, and we show how the properties of Nijenhuis operators derived in the first section yield, as a corollary, involution theorems on the dual of a Lie algebra. Using Nijenhuis operators, a class of Hamiltonian systems of Toda type are obtained for each semisimple Lie algebra.

2.1. The Nijenhuis operator of a twilled Lie algebra

An operator with vanishing Nijenhuis torsion is easily constructed on a Lie algebra $\mathfrak{g}=(\mathrm{E}, \mu)$ which splits, as a vector space, into a direct sum of two Lie subalgebras, $\mathfrak{a}$ and $\mathfrak{b}$. The notion of a Lie algebra which is a direct sum of two Lie subalgebras was introduced and studied in [21], as a generalization of Drinfeld's Lie bialgebras, under the name "twilled extension", or, preferably, *twilled Lie algebra* (*"algèbre de Lie bicroisée"*, in French, *see* [2]). Shortly thereafter this notion was defined in [32], where it was called a bicrossproduct Lie algebra, and in [28], where it was called a double Lie algebra.

Let $\mathfrak{g}=\mathfrak{a}\oplus\mathfrak{b}$, and let π be the projection of $\mathfrak{g}$ onto $\mathfrak{a}$ parallel to $\mathfrak{b}$. For x in $\mathfrak{g}$, we shall write $x_\mathfrak{a}$ for the $\mathfrak{a}$-component, πx of x, and $x_\mathfrak{b}$ for the $\mathfrak{b}$-component, $x-\pi x$, of x. We shall also write $[\ ,\ ]$ for $[\ ,\ ]_\mu$.

Let N be any Nijenhuis operator on the Lie algebra $\mathfrak{a}$ such that, for x and y in $\mathfrak{g}$,

$$\mathrm{N}([x_\mathfrak{a}, y_\mathfrak{b}]_\mathfrak{a})=[\mathrm{N}x_\mathfrak{a}, y_\mathfrak{b}]_\mathfrak{a}. \tag{2.1}$$

If N is a Nijenhuis operator on $\mathfrak{a}$ which satisfies (2.1), denoting the canonical injection of $\mathfrak{a}$ into $\mathfrak{g}$ by i, then $N=i\circ \mathrm{N}\circ\pi$ is a Nijenhuis operator on $\mathfrak{g}$. In fact, for x and y in $\mathfrak{g}$,

$$\begin{aligned}
[N, N](x, y) &= [\mathrm{N}x_\mathfrak{a}, \mathrm{N}y_\mathfrak{a}]-\mathrm{N}[\mathrm{N}x_\mathfrak{a}, y_\mathfrak{a}+y_\mathfrak{b}]_\mathfrak{a} \\
&\quad -\mathrm{N}[x_\mathfrak{a}+x_\mathfrak{b}, \mathrm{N}y_\mathfrak{a}]_\mathfrak{a}+\mathrm{N}^2[x_\mathfrak{a}+x_\mathfrak{b}, y_\mathfrak{a}+y_\mathfrak{b}]_\mathfrak{a} \\
&= ([\mathrm{N}x_\mathfrak{a}, \mathrm{N}y_\mathfrak{a}]-\mathrm{N}[\mathrm{N}x_\mathfrak{a}, y_\mathfrak{a}]-\mathrm{N}[x_\mathfrak{a}, \mathrm{N}y_\mathfrak{a}]+\mathrm{N}^2[x_\mathfrak{a}, y_\mathfrak{a}]) \\
&\quad +\mathrm{N}(\mathrm{N}[x_\mathfrak{a}, y_\mathfrak{b}]_\mathfrak{a}+\mathrm{N}[x_\mathfrak{b}, y_\mathfrak{a}]_\mathfrak{a}-[\mathrm{N}x_\mathfrak{a}, y_\mathfrak{b}]_\mathfrak{a}-[x_\mathfrak{b}, \mathrm{N}y_\mathfrak{a}]_\mathfrak{a})+\mathrm{N}^2([x_\mathfrak{b}, y_\mathfrak{b}]_\mathfrak{a}).
\end{aligned}$$

The first term vanishes because N is a Nijenhuis operator on $\mathfrak{a}$, while the second term vanishes because of the additional requirement (2.1) on N, and the last term is zero because $\mathfrak{b}$ is a Lie subalgebra of $\mathfrak{g}$. Whence the Nijenhuis torsion of N vanishes.

The identity of $\mathfrak{a}$ is obviously a Nijenhuis operator on $\mathfrak{a}$ which satisfies condition (2.1). The associated Nijenhuis operator on $\mathfrak{g}$, $N=i\circ\pi$, is the projection of $\mathfrak{g}$ onto $\mathfrak{a}$. Similarly, the projection onto $\mathfrak{b}$ is a Nijenhuis operator on $\mathfrak{g}$.

Nontriviality for a twilled Lie algebra means that $\mathfrak{a}$ and $\mathfrak{b}$ are both different from $\{0\}$, and nontriviality for a Nijenhuis operator means that it is neither invertible nor equal to 0. When $\mathfrak{a}$ is neither $\{0\}$ nor $\mathfrak{g}$, then the projection of $\mathfrak{g}$ onto $\mathfrak{a}$ is a nontrivial Nijenhuis operator on $\mathfrak{g}$. Therefore

PROPOSITION 2.1. — *Any nontrivial twilled Lie algebra has a nontrivial Nijenhuis operator and, therefore, a hierarchy of deformed Lie brackets.*

Let us remark, however, that if the Nijenhuis operator is the projection of $\mathfrak{g}$ onto one of the subalgebras of a twilled Lie algebra, the corresponding "hierarchy" has only two elements, μ and $N.\mu$, since, in this case, $N^2=N$.

Conversely, let us show that any finite-dimensional Lie algebra admitting a Nijenhuis operator is a twilled Lie algebra. This twilled Lie algebra-structure is nontrivial if the Nijenhuis operator is nontrivial.

Let E be a finite-dimensional vector space, and let N be a linear mapping from E to E. Recall [17] that there exists a smallest integer $r\geqq 1$, called the *Riesz index* of N, such that the sequences of nested subspaces

$$\mathrm{Im}\ N\supset \mathrm{Im}\ N^2\supset \dots$$

and

$$\mathrm{Ker}\ N\subset \mathrm{Ker}\ N^2\subset \dots$$

both stabilize at rank r. Thus, by definition,

$$\mathrm{Im}\ N^r=\mathrm{Im}\ N^{r+1}=\dots,\quad \text{while}\ \mathrm{Im}\ N^{r-1}\neq \mathrm{Im}\ N^r,$$

and

$$\mathrm{Ker}\ N^r=\mathrm{Ker}\ N^{r+1}=\dots,\quad \text{while}\ \mathrm{Ker}\ N^{r-1}\neq \mathrm{Ker}\ N^r.$$

Moreover, E splits as the direct sum of the vector spaces $\mathrm{Im}\ N^r$ and $\mathrm{Ker}\ N^r$, the *Fitting components* of N. We shall set $\mathfrak{a}=\mathrm{Im}\ N^r$ and $\mathfrak{b}=\mathrm{Ker}\ N^r$.

We now assume that $\mathfrak{g}=(E, \mu)$ is a Lie algebra. We show that, if N is a Nijenhuis operator, then $\mathfrak{a}$ and $\mathfrak{b}$ are Lie subalgebras. In fact, by lemma 1.2, $[N, N]_\mu$ implies $[N^r, N^r]_\mu$. Thus, for x and y in $\mathfrak{g}$,

$$[N^r x, N^r y]-N^r[N^r x, y]-N^r[x, N^r y]+N^{2r}[x, y]=0.$$

Thus $[\mathfrak{a}, \mathfrak{a}]\subset\mathfrak{a}$ and, moreover, since $\mathrm{Ker}\ N^{2r}=\mathrm{Ker}\ N^r$, $[\mathfrak{b}, \mathfrak{b}]\subset\mathfrak{b}$.

By the definition of the Riesz index, $\mathfrak{a}=\mathrm{Im}\ N^r$ is invariant under N^r, and n, the restriction of N^r to $\mathfrak{a}$, is invertible. It follows from lemma 1.2 that n is a Nijenhuis operator on $\mathfrak{a}$. We show that, in addition, n satisfies equation (2.1). Since n is invertible, it suffices to prove that, for x and y in $\mathfrak{g}$,

$$\mathrm{n}\,([\mathrm{n}\,x_\mathfrak{a}, y_\mathfrak{b}]_\mathfrak{a}-\mathrm{n}\,[x_\mathfrak{a}, y_\mathfrak{b}]_\mathfrak{a})=0.$$

By definition, and since the Nijenhuis torsion of N^r vanishes,

$$\mathrm{N}([\mathrm{N}x_{\mathfrak{a}}, y_{\mathfrak{b}}]_{\mathfrak{a}} - \mathrm{N}[x_{\mathfrak{a}}, y_{\mathfrak{b}}]_{\mathfrak{a}}) = \mathrm{N}^r([\mathrm{N}^r x_{\mathfrak{a}}, y_{\mathfrak{b}}] - \mathrm{N}^r[x_{\mathfrak{a}}, y_{\mathfrak{b}}]) \\ = [\mathrm{N}^r x_{\mathfrak{a}}, \mathrm{N}^r y_{\mathfrak{b}}] - \mathrm{N}^r [x_{\mathfrak{a}}, \mathrm{N}^r y_{\mathfrak{b}}],$$

which expression is equal to 0 because N^r vanishes on $\mathfrak{b}$. Therefore N satisfies (2.1).

In conclusion, nontrivial Nijenhuis operators on a Lie algebra $\mathfrak{g}$ are in one-to-one correspondence with pairs $(\mathfrak{a}\oplus\mathfrak{b}, \mathrm{N})$, where $\mathfrak{a}\oplus\mathfrak{b}$ is a nontrivial twilled Lie algebra-structure on $\mathfrak{g}$, and where N is an invertible Nijenhuis operator on $\mathfrak{a}$. In the next subsection, we shall consider the idempotent Nijenhuis operators which correspond to the case where N is the identity of $\mathfrak{a}$.

2.2. The N-matrix approach to the Kostant-Symes theorem

In this subsection we present "an N-matrix approach" to the Kostant-Symes theorem, to be distinguished from, and compared with, Semenov-Tian-Shansky's "R-matrix approach" to the theory of integrable systems [41].

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with Lie bracket $[\ ,\ ]_\mu = [\ ,\ ]$. We consider the dual $\mathfrak{g}^*$ of $\mathfrak{g}$ equipped with the linear Poisson structure defined by μ ("Lie-Poisson structure" or "Kirillov-Kostant-Souriau structure"), and we denote the Poisson bracket on $C^\infty(\mathfrak{g}^*)$ by $\{\ ,\ \}_\mu$ or by $\{\ ,\ \}_{\mathfrak{g}^*}$. Then, for f and g in $C^\infty(\mathfrak{g}^*)$, and ξ in $\mathfrak{g}^*$,

$$\{f, g\}_{\mathfrak{g}^*}(\xi) = \langle \xi, [d_\xi f, d_\xi g] \rangle.$$

Casimir functions are functions on $\mathfrak{g}^*$ whose Poisson bracket with any function vanishes.

We shall prove that when N is an *idempotent* Nijenhuis operator on $(\mathfrak{g}, \mu)$, *i.e., when* $\mathfrak{g}$ *is a twilled Lie algebra and* N *is the projection of* $\mathfrak{g}$ *onto a factor*, then

$$\mathcal{N}:\quad f \in C^\infty(\mathfrak{g}^*) \to f \circ {}^t\mathrm{N} \in C^\infty(\mathfrak{g}^*)$$

is a Nijenhuis operator on $C^\infty(\mathfrak{g}^*)$. We first note the following facts:

LEMMA 2.1. — *For f and g in $C^\infty(\mathfrak{g}^*)$, and ξ in $\mathfrak{g}^*$,*

$$d_\xi(f \circ {}^t\mathrm{N}) = \mathrm{N}\, d_{{}^t\mathrm{N}\xi} f \tag{2.2}$$

and

$$(\{\mathcal{N}f, \mathcal{N}g\}_\mu - \mathcal{N}(\{\mathcal{N}f, g\}_\mu + \{f, \mathcal{N}g\}_\mu) + \mathcal{N}^2\{f, g\}_\mu)(\xi) \\ = \langle \xi, [\mathrm{N}, \mathrm{N}]_\mu (d_{{}^t\mathrm{N}\xi} f, d_{{}^t\mathrm{N}\xi} g) \rangle. \tag{2.3}$$

Proof. — The chain rule implies (i). To prove (ii) we write the definitions and use the preceding result. Since, by assumption, $N^2=N$, we obtain

$$(\{\mathcal{N} f, \mathcal{N} g\}_\mu - \mathcal{N}(\{\mathcal{N} f, g\}_\mu + \{f, \mathcal{N} g\}_\mu) + \mathcal{N}^2\{f, g\}_\mu)(\xi)$$
$$= \langle \xi, [N\, d_{{}^tN\xi} f, N\, d_{{}^tN\xi} g]_\mu \rangle$$
$$- \langle \xi, N\,[N\, d_{({}^tN)^2\xi} f, d_{{}^tN\xi} g]_\mu \rangle - \langle \xi, N\,[d_{{}^tN\xi} f, N\, d_{({}^tN)^2\xi} g]_\mu \rangle$$
$$+ \langle \xi, N^2\,[d_{({}^tN)^2\xi} f, d_{({}^tN)^2\xi} g]_\mu \rangle$$
$$= \langle \xi, [N, N]_\mu\, (d_{{}^tN\xi} f, d_{{}^tN\xi} g) \rangle.$$

PROPOSITION 2.2. — *If* N *is an idempotent Nijenhuis operator on* $(\mathfrak{g}, [\ ,\ ]_\mu)$, *then* $\mathcal{N}$ *is a Nijenhuis operator on* $(C^\infty(\mathfrak{g}^*), \{\ ,\ \}_\mu)$.

Proof. — The proposition follows from formula (2.3).

Since $\mathcal{N}$ is a Nijenhuis operator on $C^\infty(\mathfrak{g}^*)$, there is a deformed Poisson bracket of functions on $\mathfrak{g}^*$,

$$\{f, g\}_{\mathcal{N}} = \{\mathcal{N} f, g\}_\mu + \{f, \mathcal{N} g\}_\mu - \mathcal{N}\{f, g\}_\mu.$$

This bracket is compatible with the Lie-Poisson bracket $\{\ ,\ \}_\mu$, but the hierarchy reduces to just two such brackets since $\mathcal{N}^2 = \mathcal{N}$.

In the following proposition, we study the restriction of the Poisson bracket $\{\ ,\ \}_{\mathcal{N}}$ to the dual of the image of N.

We consider the splitting $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$, where $\mathfrak{a}$ is the image of N and $\mathfrak{b}$ its kernel. Then, $N = i \circ \pi$, where, as above, π is the projection of $\mathfrak{g}$ onto $\mathfrak{a}$, and i is the canonical injection of $\mathfrak{a}$ into $\mathfrak{g}$. The mapping ${}^t i$ is a morphism of Poisson manifolds from $\mathfrak{g}^*$ to $\mathfrak{a}^*$. In fact, since i is a Lie algebra-morphism from $\mathfrak{a}$ to $\mathfrak{g}$, for f and g in $C^\infty(\mathfrak{a}^*)$,

$$\{f, g\}_{\mathfrak{a}^*} \circ {}^t i = \{f \circ {}^t i, g \circ {}^t i\}_{\mathfrak{g}^*}.$$

Therefore, for any f and g in $C^\infty(\mathfrak{g}^*)$,

$$\{\mathcal{N} f, \mathcal{N} g\}_{\mathfrak{g}^*} = \{f \circ {}^t\pi \circ {}^t i, g \circ {}^t\pi \circ {}^t i\}_{\mathfrak{g}^*} = \{f \circ {}^t\pi, g \circ {}^t\pi\}_{\mathfrak{a}^*} \circ {}^t i.$$

Since

$$\{\mathcal{N} f, \mathcal{N} g\}_{\mathfrak{g}^*} = \mathcal{N}\{f, g\}_{\mathcal{N}},$$

and since $f \circ {}^t\pi$ is the restriction, $f|_{\mathfrak{a}^*}$, of f to $\mathfrak{a}^*$, we see that the Lie-Poisson bracket on $\mathfrak{a}^*$ and the bracket $\{\ ,\ \}_{\mathcal{N}}$ on $\mathfrak{g}^*$ are related by

$$\{f|_{\mathfrak{a}^*}, g|_{\mathfrak{a}^*}\}_{\mathfrak{a}^*} = \{f, g\}_{\mathcal{N}}|_{\mathfrak{a}^*}. \tag{2.4}$$

We have proved:

PROPOSITION 2.3. — *The linear mapping* ${}^t\pi$ *is a Poisson morphism from* $\mathfrak{a}^*$ *equipped with* $\{\ ,\ \}_{\mathfrak{a}^*}$ *to* $\mathfrak{g}^*$ *equipped with the Poisson structure* $\{\ ,\ \}_{\mathcal{N}}$.

Therefore $\mathfrak{a}^*$ with its canonical Poisson structure is a Poisson submanifold of $\mathfrak{g}^*$ equipped with $\{\ ,\ \}_{\mathcal{N}}$. From (2.4) and proposition 1.3 we obtain immediately

COROLLARY 2.1 (Kostant-Symes theorem [22]). — *If f and g are Casimir functions on* $\mathfrak{g}^*$, *then*

$$\{ f|_{\mathfrak{a}^*}, g|_{\mathfrak{a}^*} \}_{\mathfrak{a}^*} = 0.$$

In order to compare this "N-matrix approach" with the "R-matrix approach" of Semenov-Tian-Shansky, we first remark that the Lie algebra-structure $\{ \, , \, \}_{\mathcal{N}}$ on $\mathfrak{g}^*$ coincides with the Lie-Poisson structure defined by the deformed Lie bracket $\mathrm{N} . \mu$ when restricted to $\mathfrak{a}^*$. In fact, if $\xi \in \mathfrak{a}^*$,

$$\begin{aligned}\{ f, g \}_{\mathcal{N}} (\xi) &= \{ f \circ {}^t\mathrm{N}, g \}_{\mu}(\xi) + \{ f, g \circ {}^t\mathrm{N} \}_{\mu}(\xi) - \{ f, g \}_{\mu}({}^t\mathrm{N}\,\xi) \\ &= \langle \xi, [d_\xi (f \circ {}^t\mathrm{N}), d_\xi g]_\mu + [d_\xi f, d_\xi (g \circ {}^t\mathrm{N})]_\mu - \mathrm{N}\, [d_\xi f, d_\xi g]_\mu \rangle.\end{aligned}$$

Since ξ is in $\mathfrak{a}^*$, then

$$d_\xi (f \circ {}^t\mathrm{N}) = \mathrm{N}\, d_{{}^t\mathrm{N}\,\xi} f = \mathrm{N}\, d_\xi f,$$

and therefore

$$\begin{aligned}\{ f, g \}_{\mathcal{N}} (\xi) &= \langle \xi, [\mathrm{N}\, d_\xi f, d_\xi g]_\mu + [d_\xi f, \mathrm{N}\, d_\xi g]_\mu - \mathrm{N}\, [d_\xi f, d_\xi g]_\mu \rangle \\ &= \langle \xi, [d_\xi f, d_\xi g]_{\mathrm{N}.\mu} \rangle,\end{aligned}$$

proving that both brackets coincide on $\mathfrak{a}^*$.

Now, consider the R-bracket of Semenov-Tian-Shansky,

$$[x, y]_{\mathrm{R}} = [\mathrm{R}\, x, y] + [x, \mathrm{R}\, y],$$

where $\mathrm{R}\, x = x_{\mathfrak{a}} - x_{\mathfrak{b}}$.
Then,

$$[x, y]_{\mathrm{R}} = [x_{\mathfrak{a}}, y_{\mathfrak{a}}] - [x_{\mathfrak{b}}, y_{\mathfrak{b}}],$$

while

$$[x, y]_{\mathrm{N}.\mu} = [x_{\mathfrak{a}}, y] + [x, y_{\mathfrak{a}}] - [x, y]_{\mathfrak{a}} = [x_{\mathfrak{a}}, y_{\mathfrak{a}}] + [x_{\mathfrak{a}}, y_{\mathfrak{b}}]_{\mathfrak{b}} + [x_{\mathfrak{b}}, y_{\mathfrak{a}}]_{\mathfrak{b}}.$$

Thus the bracket $[\, , \,]_{\mathrm{N}.\mu}$ and the bracket $[\, , \,]_{\mathrm{R}}$ have the same projection onto $\mathfrak{a}$.

Thus, upon restriction to $\mathfrak{a}^*$, the bracket $\{ \, , \, \}_{\mathcal{N}}$ and the Lie-Poisson brackets associated with the deformed Lie bracket $[\, , \,]_{\mathrm{N}.\mu}$ and with the R-bracket $[\, , \,]_{\mathrm{R}}$ all coincide. In consequence, the Hamiltonian vector fields defined by these Poisson brackets on $\mathfrak{g}^*$ coincide on $\mathfrak{a}^*$.

2.3. Nijenhuis operators on semisimple Lie algebras and the Toda lattice

Let $\mathfrak{s}$ be a semisimple real or complex Lie algebra of rank l. If $\mathfrak{s}$ is real, we assume that it is split. We consider a Cartan subalgebra $\mathfrak{h}$ and a system of simple roots $\alpha_1, \ldots, \alpha_l$ with respect to $\mathfrak{h}$. Let

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$$

be the associated Borel subalgebra. Let $h_1, \ldots, h_l$ be a basis of $\mathfrak{h}$. Let x_j be an eigenvector of the simple root α_j, $j = 1, \ldots, l$. The generators

$h_1, \ldots, h_l, x_1, \ldots, x_l$ of $\mathfrak{g}$ satisfy the commutation relations

$$[h_i, h_j]=0, \qquad [h_i, x_j]=\mathrm{A}\,(i, j)\, x_j,$$

where $\mathrm{A}\,(i, j)=\alpha_j\,(h_i)$.

Let N be the projection of $\mathfrak{g}$ onto $\mathfrak{h}$, which, by the result of subsection 2.1, is a Nijenhuis operator on $\mathfrak{g}$. The deformed Lie bracket $[\ ,\ ]_{\mathrm{N}}$ on $\mathfrak{g}$ has the commutation relations

$$[h_i, h_j]_{\mathrm{N}}=0, \qquad [h_i, x_j]_{\mathrm{N}}=\mathrm{A}\,(i, j)\, x_j, \qquad [x_i, x_j]_{\mathrm{N}}=0.$$

Thus $\mathfrak{n}$ becomes an Abelian subalgebra of $(\mathfrak{g}, [\ ,\ ]_{\mathrm{N}})$. The vector subspace V of $\mathfrak{g}$ generated by $h_1, \ldots, h_l, x_1, \ldots, x_l$ is, in fact, a Lie subalgebra of $(\mathfrak{g}, [\ ,\ ]_{\mathrm{N}})$ whose Lie brackets are written above. Let V^* be the dual of the vector space V, with coordinates $b_1, \ldots, b_l, a_1, \ldots, a_l$ with respect to the dual basis $h_1^*, \ldots, h_l^*, x_1^*, \ldots x_l^*$. Their Poisson brackets in the Lie-Poisson structure $\{\ ,\ \}_{\mathrm{N}}$ are

$$\{b_i, b_j\}_{\mathrm{N}}=0, \qquad \{b_i, a_j\}_{\mathrm{N}}=\mathrm{A}\,(i, j)\, a_j, \qquad \{a_i, a_j\}_{\mathrm{N}}=0.$$

Let $\mathrm{H}=\mathrm{H}\,(a_1, \ldots, a_l, b_1, \ldots, b_l)$ be a function on V^*. The Hamiltonian system associated with H is

$$\left\{\begin{aligned} \dot{a}_i &= -a_i \sum_{j=1}^{l} \mathrm{A}\,(j, i)\, \frac{\partial \mathrm{H}}{\partial b_j} \\ \dot{b}_i &= \sum_{j=1}^{l} \mathrm{A}\,(i, j)\, a_j\, \frac{\partial \mathrm{H}}{\partial a_j}. \end{aligned}\right. \tag{2.5}$$

Whence the following result [13],

Proposition 2.4. — *Let* W *be a* $2l$*-dimensional vector space with coordinates* $a_1, \ldots, a_l, b_1, \ldots, b_l$. *When the entries of the matrix* $(\mathrm{A}\,(i, j))$ *are the components of a system of simple roots of a semisimple Lie algebra, the dynamical system* (2.5) *is Hamiltonian.*

In particular, for $\mathrm{H}=\sum_{i=1}^{l}\left(\frac{1}{2}(b_i)^2+a_i\right)$, we obtain the system

$$\left\{\begin{aligned} \dot{a}_i &= -a_i \sum_{j=1}^{l} \mathrm{A}\,(j, i)\, b_j \\ \dot{b}_i &= \sum_{j=1}^{l} \mathrm{A}\,(i, j)\, a_j \end{aligned}\right.$$

which had been considered in [1] and [42].

In [13], Flaschka shows that with a suitable choice of a system of simple roots of the Lie algebra $\mathfrak{s}=\mathfrak{sl}\,(l+1, \mathbb{C})$, the equations (2.5) are those of the *nonperiodic Toda lattice* in center of mass-coordinates.

We now show that the same method also yields the usual equations of the Toda lattice in the Flaschka coordinates $a_1, \ldots, a_l, b_1, \ldots, b_{l+1}$,

$$\begin{cases} \dot{a}_i = a_i\,(b_i - b_{i+1}) \\ \dot{b}_i = 2\,((a_{i-1})^2 - (a_i)^2) \end{cases}$$

Instead of a Borel subalgebra of the semisimple Lie algebra $\mathfrak{sl}\,(l+1, \mathbb{C})$, we consider the solvable Lie algebra $\mathfrak{b}$ of all upper triangular matrices of order $l+1$, generated by

$$h_1 = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \ldots, h_{l+1} = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & \ddots & & \\ & & & 0 & \\ & & & & 1 \end{pmatrix},$$

$$x_1 = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 0 & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \ldots, x_l = \begin{pmatrix} 0 & 0 & & & \\ & 0 & & & \\ & & \ddots & 0 & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

which satisfy the commutation relations

$$[h_i, h_j] = 0, \qquad [h_i, x_j] = \delta_{ij} x_j - \delta_{i,\,j+1}\, x_j.$$

Let N be the projection of $\mathfrak{b}$ onto the subalgebra of all diagonal matrices. For a Hamiltonian $\mathrm{H} = \mathrm{H}\,(a_1, \ldots, a_l, b_1, \ldots, b_{l+1})$ we obtain a Hamiltonian system

$$\begin{cases} \dot{a}_i = a_i \left(\dfrac{\partial \mathrm{H}}{\partial b_{i+1}} - \dfrac{\partial \mathrm{H}}{\partial b_i} \right) \\ \dot{b}_i = a_i\, \dfrac{\partial \mathrm{H}}{\partial a_i} - a_{i-1}\, \dfrac{\partial \mathrm{H}}{\partial a_{i-1}}. \end{cases}$$

For $\mathrm{H} = -(1/2) \sum_{i=1}^{l+1} (b_i)^2 - \sum_{i=1}^{l} (a_i)^2$, these are the usual equations of the nonperiodic Toda lattice.

3. DUALIZATION OF LIE BRACKETS BY MEANS OF A POISSON BIVECTOR

In this section we shall construct a Lie bracket on the space of differential 1-forms on a Poisson manifold, by a kind of dualization process, from the Lie bracket of vector fields, while in section 6, we shall show that this construction can be performed somewhat more generally on a *differential Lie algebra* equipped with a bivector with vanishing Schouten bracket.

3.1. Poisson manifolds

Let M be a Hausdorff, second-countable, smooth manifold. Here and below, "1-form" or simply "form" will stand for a "smooth differential 1-form", while "vector", "tensor", and "multivector" will stand for "smooth vector field", "smooth tensor field", and "smooth field of antisymmetric contravariant tensors", respectively. A multivector of rank k will be called a k-vector. Let $A = C^\infty M$ be the ring of smooth real-valued functions on M, let $E = \underline{TM}$ denote the A-module of smooth vector fields on M, and let $E^* = \underline{T^* M}$ be the dual A-module of smooth differential 1-forms. Let us first recall some standard definitions concerning the Poisson manifolds.

Let P be a bivector on M. Then P defines a biderivation, denoted by $\{\ ,\ \}_P$, of the ring of functions, $A = C^\infty M$. By definition, the Poisson bracket of f and g in $C^\infty M$ is the function

$$\{f, g\}_P = P(df, dg).$$

The *Schouten bracket* of P is the trivector [P, P] such that, for f_1, f_2 and f_3 in A,

$$[P, P](df_1, df_2, df_3) = 2 \oint \{\{f_1, f_2\}_P, f_3\}_P. \tag{3.1}$$

(Using the notation [,] for the Schouten bracket of multivectors is justified since it reduces to the ordinary Lie bracket in the case of two vector fields.) By definition, P is a *Poisson bivector* if

$$[P, P] = 0,$$

and (M, P) is a *Poisson manifold* if P is a Poisson bivector. It is clear from (3.1) that (M, P) is a Poisson manifold if and only if the antisymmetric bracket $\{\ ,\ \}_P$ satisfies Jacobi's identity. We shall denote by **P** the linear mapping from $E^* = \underline{T^* M}$ to $E = \underline{TM}$ defined by P, such that for α and β in E^*,

$$\langle \alpha, \mathbf{P}\beta \rangle = P(\alpha, \beta),$$

and by **[P, P]** the antisymmetric mapping from $E^* \times E^*$ to E defined by [P, P], such that for α, β and γ in E^*,

$$\langle \alpha, [\mathbf{P}, \mathbf{P}](\beta, \gamma) \rangle = [P, P](\alpha, \beta, \gamma).$$

3.2. The Lie bracket of 1-forms on a Poisson manifold

The bivector P also defines an antisymmetric bracket on the 1-forms on M in the following way. The bracket of 1-forms α and β is defined to be the 1-form $\{\alpha, \beta\}_{\mathrm{P}}$, where

$$\langle \{\alpha, \beta\}_{\mathrm{P}}, x \rangle = \langle \mathscr{L}_{\mathbf{P}\alpha}\,\beta, x \rangle - \langle \mathscr{L}_{\mathbf{P}\beta}\,\alpha, x \rangle + \mathscr{L}_x \langle \alpha, \mathbf{P}\beta \rangle, \quad (3.2)$$

for x in E. Here and below, $\mathscr{L}_x$ denotes the *Lie derivation* by $x \in \mathrm{E}$. It is clear that this expression is A-linear with respect to x, and that it therefore defines a 1-form $\{\alpha, \beta\}_{\mathrm{P}}$ for α and β in E*. Formula (3.2) can be rewritten as

$$\{\alpha, \beta\}_{\mathrm{P}} = \mathscr{L}_{\mathrm{P}\alpha}\,\beta - \mathscr{L}_{\mathrm{P}\beta}\,\alpha + d(\mathrm{P}(\alpha, \beta)). \quad (3.2')$$

As a matter of fact, definition (3.2) can be rewritten using only the Lie derivative of functions and the Lie bracket of vector fields, using the relation

$$\langle \mathscr{L}_y \alpha, x \rangle = \mathscr{L}_y \langle \alpha, x \rangle - \langle \alpha, [y, x] \rangle,$$

where x and y are vector fields and α is a 1-form. It is then easy to see that it actually generalizes to the case where E is a 'differential Lie algebra' (*see* section 6).

This bracket of 1-forms on a Poisson manifold is not so well-known as the Poisson bracket of functions. It was found, apparently independently, by Magri [30], [31] and by several other authors, Gel'fand and Dorfman [12] (it also plays an implicit role in [15]), Koszul in his study of Schouten brackets [23], Karasev [19] in the case of a Poisson-Lie group, and Coste, Dazord and Weinstein [9] in the context of their study of symplectic groupoids. It also figures in Weinstein's note [42] on the infinitesimal dressing transformations.

We now state two fundamental properties of this bracket which can be used to characterize it axiomatically. For α and β in E*, and for f and g in A,

$$\{\alpha, f\beta\}_{\mathrm{P}} = f\{\alpha, \beta\}_{\mathrm{P}} + (\mathscr{L}_{\mathrm{P}\alpha} f)\,\beta \quad (3.3)$$

and

$$\{df, dg\}_{\mathrm{P}} = -d(\{f, g\}_{\mathrm{P}}). \quad (3.4)$$

The bracket $\{\ ,\ \}_{\mathrm{P}}$ is the only bracket on 1-forms satisfying (3.3) and (3.4).

We consider the $\mathbb{R}$-bilinear mapping from $(\mathrm{A} \oplus \mathrm{E}^*) \times (\mathrm{A} \oplus \mathrm{E}^*)$ to $\mathrm{A} \oplus \mathrm{E}^*$ whose restriction to the functions is zero, whose restriction to the 1-forms is the bracket (3.2), and which maps a pair (α, f) to $\mathscr{L}_{\mathrm{P}\alpha} f$, and a pair (f, α) to $-\mathscr{L}_{\mathrm{P}\alpha} f$. Condition (3.3) is equivalent to the requirement that this bilinear mapping extend uniquely to a biderivation of the graded associative algebra $\Lambda(E^*)$, analogous to the Schouten bracket of multivectors obtained by an extension of the Lie bracket of vectors and the Lie derivation of functions. For this reason, the bracket $\{\ ,\ \}_P$ satisfying property (3.3) is sometimes called a *biderivation* of the A-module E^*. A formula for the corresponding biderivation on the algebra of all forms on M appears in the work of Koszul [23]. This *Koszul-Schouten bracket* will play an important role in section 6.

We shall now prove two fundamental identities that relate the bracket $\{\ ,\ \}_P$ to the Schouten bracket [P, P].

PROPOSITION 3.1. – *For any bivector* P, *and* 1*-forms* α *and* β *in* E^*,

$$\mathbf{P}\{\alpha,\beta\}_P-[\mathbf{P}\alpha,\mathbf{P}\beta]=(1/2)[\mathbf{P},\mathbf{P}](\alpha,\beta). \tag{3.5}$$

Proof. – We use the relation (*see* [31], [15])

$$[P,P](\alpha_1,\alpha_2,\alpha_3)=2\oint\langle \mathscr{L}_{\mathbf{P}\alpha_1}\alpha_3,\mathbf{P}\alpha_2\rangle. \tag{3.6}$$

The result follows from replacing $\{\ ,\ \}_P$ by its expression.

We shall now prove that the bracket $\{\ ,\ \}_P$ satisfies Jacobi's identity if and only if P is a Poisson bivector.

LEMMA 3.1. – *For* α_1, α_2 *and* α_3 *in* E^*,

$$\langle\oint\{\alpha_1,\{\alpha_2,\alpha_3\}_P\}_P,x\rangle=-(1/2)\oint\langle \mathscr{L}_{[\mathbf{P},\mathbf{P}](\alpha_1,\alpha_2)}\alpha_3,x\rangle + \mathscr{L}_x([P,P](\alpha_1,\alpha_2,\alpha_3)). \tag{3.7}$$

Proof. – We use the definition (3.1) and the relations (3.5) and (3.6). These identities imply

PROPOSITION 3.2. – (i) *The bracket* $\{\ ,\ \}_P$ *on* E^* *is a Lie bracket if and only if* P *is a Poisson bivector.*

(ii) *When* P *is a Poisson bivector,* **P** *is a Lie algebra-morphism from* (E^*, $\{\ ,\ \}_P$) *to* (E, [,]).

Proof. – It follows from lemma 3.1 that if $[P,P]=0$, then the antisymmetric bracket $\{\ ,\ \}_P$ satisfies Jacobi's identity. To prove the converse, we first remark that it follows from formulae (3.4) and (3.1) that for all f_1, f_2 and f_3 in A,

$$2\oint\{\{df_1,df_2\}_P,df_3\}_P=2d\left(\oint\{\{f_1,f_2\}_P,f_3\}_P\right) = d([P,P](df_1,df_2,df_3)). \tag{3.8}$$

If $\{\ ,\ \}_P$ is a Lie bracket, this expression must vanish identically and therefore the trivector $[P,P]$ must vanish. The proof of part (i) is therefore complete.

Part (ii) of the proposition follows immediately from proposition 3.1.

In section 6, we shall give an alternative proof of proposition 3.2, using the derivations of the graded associative algebra of the multivectors on M.

3.3. Example: the bracket of 1-forms on the dual of a Lie algebra

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with Lie bracket [,]. Then $\mathfrak{g}^*$ is a Poisson manifold. The Poisson bivector P of $\mathfrak{g}^*$ is such that

$$\mathbf{P}_\xi x = -\mathrm{ad}_x^* \xi,$$

for ξ in $\mathfrak{g}^*$ and x in $T_\xi^*(\mathfrak{g}^*)$ identified with $\mathfrak{g}$. For differential 1-forms α and β on $\mathfrak{g}^*$, considered as mappings from $\mathfrak{g}^*$ to $\mathfrak{g}$, we find

$$\{\alpha, \beta\}_{\mathrm{P}}(\xi) = (d_\xi \alpha)(\mathrm{ad}_{\beta_\xi}^* \xi) - (d_\xi \beta)(\mathrm{ad}_{\alpha_\xi}^* \xi) + [\alpha_\xi, \beta_\xi], \tag{3.9}$$

where $d_\xi \alpha$ is the differential at ξ of the mapping α. For α and β taken to be constant differential 1-forms on $\mathfrak{g}$ identified with elements x and y of $\mathfrak{g}$, this formula reduces to

$$\{x, y\}_{\mathrm{P}} = [x, y]. \tag{3.10}$$

Thus the Lie bracket of 1-forms on $\mathfrak{g}^*$ extends the Lie bracket of the given Lie algebra $\mathfrak{g}$.

4. DEFORMATION AND DUALIZATION: THE COMPATIBILITY CONDITION

In this section, as in the last, M is a Hausdorff, second-countable, smooth manifold, and we retain the other notations as well. We shall apply the results of section 1 to the deformation of the natural Lie algebra-structure of E by a *Nijenhuis tensor*, *i. e.*, a Nijenhuis operator on E which is also A-linear, and the results of section 3 concerning the dualization of the Lie bracket of vector fields by means of a Poisson bivector. Given both a Nijenhuis tensor and a Poisson bivector on a manifold, it is natural to require that the two processes, deformation and dualization, commute. This requirement – the *compatibility condition* – leads to the definition and first properties of the Poisson-Nijenhuis structures, which will be described in this section. Further properties of the hierarchy of Poisson structures defined by a Poisson-Nijenhuis structure will be described in the next section.

4.1. Deformed Lie bracket of vector fields and associated bracket on forms

We denote by μ the natural Lie algebra-structure on $E = \underline{TM}$, and by $[\ ,\ ]_\mu$, instead of by [,] as in the last section, the Lie bracket of vector fields, and, more generally, the Schouten bracket of multivectors which extends the Lie bracket of vector fields.

Let P be a Poisson bivector on M, *i. e.*, a bivector such that

$$[P, P]_\mu = 0. \tag{4.1}$$

We shall denote by $\nu = \nu(\mu, P)$ the Lie algebra-structure on E^* that was constructed, in section 3, from the naturel Lie algebra-structure μ on vector fields and from the Poisson bivector P. The corresponding Lie bracket on E^* will, however, be denoted by $\{\ ,\ \}_\nu$ or $\{\ ,\ \}_P^\mu$, instead of by $\{\ ,\ \}_P$, as in section 3. Recall (proposition 3.2) that the linear mapping **P** from E^* to E is a Lie algebra-morphism from (E^*, ν) to (E, μ).

Let N be a (1,1)-tensor with vanishing Nijenhuis torsion with respect to μ,

$$[N, N]_\mu = 0. \tag{4.2}$$

We denote by $\mu' = N.\mu$ the deformed Lie algebra-structure on E. Using the notations and results of section 1 applied to $E = \underline{TM}$, we consider the deformed Lie bracket of vector fields, denoted by $[\ ,\ ]_{N.\mu}$ or $[\ ,\ ]'$. We recall that N is a Lie algebra-morphism from $(E, N.\mu)$ to (E, μ). The deformed bracket has properties similar to those of the natural bracket, namely, it satisfies the identity, for x and y in E, and f in A,

$$[x, fy]' = f[x, y]' + (\mathscr{L}'_x f) y, \tag{4.3}$$

where, by definition,

$$\mathscr{L}'_x = \mathscr{L}_{Nx}.$$

It follows from this property that there exists a unique deformed derivation, denoted by $\mathscr{L}'_x$, of the algebra of tensors, that commutes with the contractions, such that

$$\mathscr{L}'_x f = \mathscr{L}_{Nx} f,$$
$$\mathscr{L}'_x y = [x, y]'.$$

In particular, for α in E^*,

$$\mathscr{L}'_x \alpha = \mathscr{L}_{Nx} \alpha - \mathscr{L}_x ({}^t N) \alpha. \tag{4.4}$$

Here the transpose ${}^tN$ of the linear map, N, is considered to be a (1,1)-tensor. This derivation law $\mathscr{L}'$ can also be denoted by $\mathscr{L}^{N.\mu}$, while the natural derivation law $\mathscr{L}$ will sometimes be referred to as $\mathscr{L}^\mu$, to stress that it is associated with the natural Lie bracket μ on E.

There is a deformed differential $d' = d_{N.\mu}$ on the space of all forms $\Lambda(E^*)$, such that

$$\mathscr{L}'_x = [i_x, d'],$$

where i_x denotes the interior product with x. In particular, for a function f in A,

$$d' f = {}^t N (df). \tag{4.5}$$

The map d' is a coboundary, *i. e.*, $(d')^2=0$.

It also follows from property (4.3) that there exists a unique biderivation, again denoted by $[\ ,\ ]_{N\,.\,\mu}$ or $[\ ,\ ]'$, of the graded associative algebra of multivectors $\Lambda\mathrm{E}$, whose restriction to the functions is zero, whose restriction to the vectors is $[\ ,\ ]'$, and which maps a pair (x, f) to $\mathscr{L}_{\mathrm{N}x} f$, and a pair (f, x) to $-\mathscr{L}_{\mathrm{N}x} f$. This biderivation is called the *Schouten bracket* associated with $\mathrm{N}\,.\,\mu$.

We now combine the processes of deformation and dualization and we consider the bracket $\{\ ,\ \}'_{\mathrm{P}}=\{\ ,\ \}^{\mathrm{N}\,.\,\mu}_{\mathrm{P}}$ on E^* which is defined by a formula similar to formula (3.2′) which defines $\{\ ,\ \}^{\mu}_{\mathrm{P}}$. We replace the Lie derivation law $\mathscr{L}$ by the deformed Lie derivation law $\mathscr{L}'$ and the differential d on functions by the deformed differential d'. Thus, by definition, for α and β in E^*,

$$\begin{aligned}\{\alpha,\beta\}'_{\mathrm{P}}&=\mathscr{L}'_{\mathbf{P}\alpha}\beta-\mathscr{L}'_{\mathbf{P}\beta}\alpha+d'\langle\alpha,\mathbf{P}\beta\rangle\\&=\mathscr{L}_{\mathbf{NP}\alpha}\beta-\mathscr{L}_{\mathbf{P}\alpha}({}^t\mathrm{N})\beta-\mathscr{L}_{\mathbf{NP}\beta}\alpha+\mathscr{L}_{\mathbf{P}\beta}({}^t\mathrm{N})\alpha+{}^t\mathrm{N}(d\langle\alpha,\mathbf{P}\beta\rangle),\end{aligned}\tag{4.6}$$

by formulas (4.4) and (4.5).

We shall now investigate the condition under which the bracket on E that we have just considered is a Lie bracket.

4.2. The compatibility condition: Poisson-Nijenhuis structures

We denote by NP the (2,0)-tensor associated with the linear mapping $\mathrm{N}\circ\mathbf{P}$ from E^* to E, defined by

$$\mathrm{NP}(\alpha,\beta)=\langle\alpha,\mathbf{NP}\beta\rangle=\langle{}^t\mathrm{N}\alpha,\mathbf{P}\beta\rangle=\mathrm{P}({}^t\mathrm{N}\alpha,\beta),$$

and by $\mathrm{P}\,{}^t\mathrm{N}$ the (2,0)-tensor defined by

$$\mathrm{P}\,{}^t\mathrm{N}(\alpha,\beta)=\langle\alpha,\mathbf{P}\,{}^t\mathrm{N}\beta\rangle=\mathrm{P}(\alpha,{}^t\mathrm{N}\beta).$$

We shall deal with the following diagram of linear maps:

$$\begin{tikzcd}(\mathrm{E}^*,\nu(\mu,\mathrm{P})) \arrow[r, "\mathbf{P}"] & (\mathrm{E},\mu)\\ (\mathrm{E}^*,\nu') \arrow[u, "{}^t\mathrm{N}"] \arrow[r, "\mathbf{P}"'] & (\mathrm{E},\mu'=\mathrm{N}\,.\,\mu) \arrow[u, "\mathrm{N}"']\end{tikzcd}$$

For ν', we can consider the following brackets on E^*:

(*a*) $\nu'=\nu(\mathrm{N}\,.\,\mu,\mathrm{P})$, *i. e.*, the bracket $\{\ ,\ \}'_{\mathrm{P}}=\{\ ,\ \}^{\mathrm{N}\,.\,\mu}_{\mathrm{P}}$ obtained by dualization from the deformed Lie bracket $[\ ,\ ]'=[\ ,\ ]_{\mathrm{N}\,.\,\mu}$ on E by means of the bivector P, which is given by formula (4.6),

(*b*) $\nu'=\nu\left(\mu,\dfrac{1}{2}(\mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N})\right)$, *i. e.*, the bracket $\{\ ,\ \}^{\mu}_{(1/2)(\mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N})}$ obtained by dualization of the natural Lie bracket $[\ ,\ ]=[\ ,\ ]_{\mu}$ on E by means of the bivector (1/2) (NP+P ^{t}N), which is defined by

$$\nu\left(\mu, \frac{1}{2}(\mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N})\right)(\alpha, \beta)=\{\alpha, \beta\}^{\mu}_{(1/2)(\mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N})}$$
$$=\frac{1}{2}(\mathscr{L}_{(\mathbf{NP}+\mathbf{P}\,{}^t\mathbf{N})\,\alpha}\beta-\mathscr{L}_{(\mathbf{NP}+\mathbf{P}\,{}^t\mathbf{N})\,\beta}\alpha+d\langle\alpha, (\mathbf{NP}+\mathbf{P}\,{}^t\mathbf{N})\,\beta\rangle), \quad (4.7)$$

(*c*) $\nu'={}^t\mathrm{N}.\nu(\mu, \mathrm{P})$, *i. e.*, the bracket $\{\ ,\ \}_{{}^t\mathrm{N}.\nu}$ obtained by deformation from the Lie bracket $\{\ ,\ \}_\nu=\{\ ,\ \}^\mu_\mathrm{P}$, where $\nu=\nu(\mu, \mathrm{P})$, by means of the (1,1)-tensor ^{t}N on E*, which is defined by

$$({}^t\mathrm{N}.\nu(\mu, \mathrm{P}))(\alpha, \beta)=\{\alpha, \beta\}_{{}^t\mathrm{N}.\nu}$$
$$=\{{}^t\mathrm{N}\,\alpha, \beta\}^\mu_\mathrm{P}+\{\alpha, {}^t\mathrm{N}\,\beta\}^\mu_\mathrm{P}-{}^t\mathrm{N}\{\alpha, \beta\}^\mu_\mathrm{P}. \quad (4.8)$$

These three brackets are not independent. Indeed,

LEMMA 4.1. — *Let* μ *be a Lie algebra-structure on* E. *Then, for any* (1,1)*-tensor* N *and for any bivector* P,

$$\nu(\mathrm{N}.\mu, \mathrm{P})+{}^t\mathrm{N}.\nu(\mu, \mathrm{P})=\nu(\mu, \mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N}).$$

Proof. — We obtain from the definition (4.8), for α and β in E*,

$$\begin{aligned}({}^t\mathrm{N}.\nu(\mu, \mathrm{P}))(\alpha, \beta)&=\{\alpha, \beta\}_{{}^t\mathrm{N}.\nu}=\{{}^t\mathrm{N}\,\alpha, \beta\}^\mu_\mathrm{P}+\{\alpha, {}^t\mathrm{N}\,\beta\}^\mu_\mathrm{P}-{}^t\mathrm{N}\{\alpha, \beta\}^\mu_\mathrm{P}\\ &=\mathscr{L}_{\mathbf{P}\,{}^t\mathrm{N}\alpha}\beta-\mathscr{L}_{\mathbf{P}\beta}({}^t\mathrm{N}\,\alpha)+d\langle {}^t\mathrm{N}\,\alpha, \mathbf{P}\,\beta\rangle\\ &\quad+\mathscr{L}_{\mathbf{P}\alpha}({}^t\mathrm{N}\,\beta)-\mathscr{L}_{\mathbf{P}\,{}^t\mathrm{N}\beta}\alpha+d\langle\alpha, \mathbf{P}{}^t\mathrm{N}\,\beta\rangle\\ &\qquad-{}^t\mathrm{N}(\mathscr{L}_{\mathbf{P}\alpha}\beta-\mathscr{L}_{\mathbf{P}\beta}\alpha+d\langle\alpha, \mathbf{P}\,\beta\rangle).\end{aligned}$$

By addition, using (4.6) and (4.7) we obtain the lemma.

It is not true in general that these brackets will satisfy Jacobi's identity because

(*a*) the Schouten bracket $[\mathrm{P}, \mathrm{P}]'=[\mathrm{P}, \mathrm{P}]_{\mathrm{N}.\mu}$ of the bivector P need not vanish,

(*b*) the Schouten bracket

$$[\mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N}, \mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N}]=[\mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N}, \mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N}]_\mu$$

of the bivector NP + P ^{t}N need not vanish,

(*c*) the Nijenhuis torsion $[{}^t\mathrm{N}, {}^t\mathrm{N}]_\nu$ of the (1,1)-tensor ^{t}N, with respect to the Lie algebra-structure $\nu=\nu(\mu, \mathrm{P})$, need not vanish.

Below, we show that a single *compatibility condition* on N and P implies that the preceding diagram commutes, that the three quantities mentioned above vanish, that the three brackets on E* considered above must coincide and satisfy Jacobi's identity, and that all the arrows in the diagram are Lie algebra-morphisms.

DEFINITION 4.1. — *A Nijenhuis tensor* N *and a Poisson bivector* P *on a manifold* M *are called compatible if* NP *is a bivector*, i. e.,

$$\mathrm{N}\circ\mathbf{P}=\mathbf{P}\circ{}^t\mathrm{N}, \quad (4.9)$$

and if

$$\nu(\mathrm{N}.\mu, \mathrm{P}) = {}^t\mathrm{N}.\nu(\mu, \mathrm{P}). \tag{4.10}$$

If the Nijenhuis tensor N *and the Poisson bivector* P *are compatible, the pair* (P, N) *is called a* Poisson-Nijenhuis structure *on* M, *and* (M, P, N) *is called a* Poisson-Nijenhuis manifold.

By (4.9), when N and P are compatible, $(1/2)(\mathrm{NP}+\mathrm{P}\,{}^t\mathrm{N})=\mathrm{NP}$. It follows from lemma 4.1 that, *if* N *and* P *are compatible, then the three brackets defined by* (4.6), (4.7) *and* (4.8) *coincide*, namely

$$\nu(\mathrm{N}.\mu, \mathrm{P}) = {}^t\mathrm{N}.\nu(\mu, \mathrm{P}) = \nu(\mu, \mathrm{NP}). \tag{4.11}$$

In order to express the compatibility condition, let us set

$$\mathrm{C}_\mu(\mathrm{P}, \mathrm{N})(\alpha, \beta) = \frac{1}{2}(\{\alpha, \beta\}_{{}^t\mathrm{N}.\nu(\mu, \mathrm{P})} - \{\alpha, \beta\}_{\mathrm{P}}^{\mathrm{N}.\mu}).$$

As a result of assumption (4.9), $\mathrm{C}_\mu(\mathrm{P}, \mathrm{N})$ is A-linear with respect to α and β. Thus $\mathrm{C}_\mu(\mathrm{P}, \mathrm{N})$ is a form-valued bivector on E which vanishes if and only if the processes of dualization and of deformation commute. From lemma 4.1, we obtain

$$\mathrm{C}_\mu(\mathrm{P}, \mathrm{N})(\alpha, \beta) = \{\alpha, \beta\}_{\mathrm{NP}}^{\mu} - \{\alpha, \beta\}_{\mathrm{P}}^{\mathrm{N}.\mu} = \{\alpha, \beta\}_{{}^t\mathrm{N}.\nu(\mu, \mathrm{P})} - \{\alpha, \beta\}_{\mathrm{NP}}^{\mu}.$$

Using the first equality and the definitions of the brackets, we find

$$\begin{aligned}\mathrm{C}_\mu(\mathrm{P}, \mathrm{N})(\alpha, \beta) = {} & \mathscr{L}_{\mathbf{P}\alpha}({}^t\mathrm{N}\,\beta) - \mathscr{L}_{\mathbf{P}\beta}({}^t\mathrm{N}\,\alpha) + {}^t\mathrm{N}\,\mathscr{L}_{\mathbf{P}\beta}\,\alpha - {}^t\mathrm{N}\,\mathscr{L}_{\mathbf{P}\alpha}\,\beta \\ & + d\langle \alpha, \mathbf{NP}\,\beta \rangle - {}^t\mathrm{N}\,d\langle \alpha, \mathbf{P}\,\beta \rangle.\end{aligned}$$

We see that this form-valued bivector $\mathrm{C}_\mu(\mathrm{P}, \mathrm{N})$ is none other than the tensor considered by Magri and Morosi in [30] and designated there by R(P, N). (We have avoided the use of the letter R because this letter is used in other contexts in the theory of the Yang-Baxter equation.) We note that this *differential concomitant* of the bivector P and the (1,1)-tensor N, which has the required tensoriel properties only when $\mathrm{NP} = \mathrm{P}\,{}^t\mathrm{N}$, was already considered by Schouten in 1953 ([40], formula (15)). In fact a staightforward calculation shows that the coordinate expression for $\mathrm{C}_\mu(\mathrm{P}, \mathrm{N})$ is, with obvious notations,

$$\mathrm{C}_m^{kj} = \mathrm{P}^{lj}\,\partial_l\,\mathrm{N}_m^k + \mathrm{P}^{kl}\,\partial_l\,\mathrm{N}_m^j - \mathrm{N}_m^l\,\partial_l\,\mathrm{P}^{kj} + \mathrm{N}_l^j\,\partial_m\,\mathrm{P}^{kl} - \mathrm{P}^{lj}\,\partial_m\,\mathrm{N}_l^k.$$

The tensor $\mathrm{C}_\mu(\mathrm{P}, \mathrm{N})$ also relates the torsion of N with respect to μ with the torsion of ${}^t\mathrm{N}$ with respect to $\nu(\mu, \mathrm{P})$.

Let P be a Poisson bivector and let N be a (1,1)-tensor. Then, for any α and β in E*, x in E,

$$\begin{aligned}& \langle [{}^t\mathrm{N}, {}^t\mathrm{N}]_{\nu(\mu, \mathrm{P})}(\alpha, \beta), x \rangle - \langle \alpha, [\mathrm{N}, \mathrm{N}]_\mu(x, \mathbf{P}\,\beta) \rangle \\ & \qquad = \langle \mathrm{C}_\mu(\mathrm{P}, \mathrm{N})({}^t\mathrm{N}\,\alpha, \beta), x \rangle - \langle \mathrm{C}_\mu(\mathrm{P}, \mathrm{N})(\alpha, \beta), \mathrm{N}\,x \rangle. \end{aligned} \tag{4.12}$$

To prove this formula, we first check the identity, for x in E and for α and β in E*,

$$\langle \{ ({}^t\mathrm{N})^2 \alpha, \beta \}_{\mathrm{P}}^{\mu} - {}^t\mathrm{N} \{ {}^t\mathrm{N} \alpha, \beta \}_{\mathrm{P}}^{\mu} - \{ {}^t\mathrm{N} \alpha, \beta \}_{\mathrm{NP}}^{\mu} + {}^t\mathrm{N} \{ \alpha, \beta \}_{\mathrm{NP}}^{\mu}, x \rangle + \langle \alpha, [\mathrm{N}, \mathrm{N}]_{\mu} (x, \mathbf{P} \beta) \rangle = 0,$$

which expresses the torsion of N with respect to μ, evaluated on x and $\mathrm{P}\beta$, in terms of brackets on forms. This identity can be rewritten as

$$\langle ({}^t\mathrm{N} \{ \alpha, \beta \}_{\mathrm{NP}}^{\mu} - \{ {}^t\mathrm{N} \alpha, {}^t\mathrm{N} \beta \}_{\mathrm{P}}^{\mu}) + \mathrm{C}_{\mu}(\mathrm{P}, \mathrm{N}) ({}^t\mathrm{N} \alpha, \beta), x \rangle + \langle \alpha, [\mathrm{N}, \mathrm{N}]_{\mu} (x, \mathbf{P} \beta) \rangle = 0,$$

from which it is clear that (4.12) follows. Formula (4.12) implies

LEMMA 4.2. — *If the Nijenhuis tensor* N *and the Poisson bivector* P *are compatible, then*

$$[{}^t\mathrm{N}, {}^t\mathrm{N}]_{\nu(\mu, \mathrm{P})} = 0.$$

It follows from this lemma and from corollary 1.1 that, when N is a Nijenhuis tensor that is compatible with P, the brackets ${}^t\mathrm{N} . \nu(\mu, \mathrm{P})$, $\nu(\mathrm{N} . \mu, \mathrm{P})$ and $\nu(\mu, \mathrm{NP})$ on E* coincide and satisfy Jacobi's identity. Moreover

PROPOSITION 4.1. — *Let* (P, N) *be a Poisson-Nijenhuis structure on* M. *Then,*

(i) *the bracket* ${}^t\mathrm{N} . \nu(\mu, \mathrm{P})$ *satisfies the Jacobi identity and is compatible with* $\nu(\mu, \mathrm{P})$, *and* ${}^t\mathrm{N}$ *is a Lie algebra-morphism from* $(\mathrm{E}^*, {}^t\mathrm{N} . \nu(\mu, \mathrm{P}))$ *to* $(\mathrm{E}^*, \nu(\mu, \mathrm{P}))$, i. e., *for all* α *and* β *in* E*,

$${}^t\mathrm{N} \{ \alpha, \beta \}_{{}^t\mathrm{N} . \nu(\mu, \mathrm{P})} = \{ {}^t\mathrm{N} \alpha, {}^t\mathrm{N} \beta \}_{\mathrm{P}}^{\mu}, \tag{4.13}$$

(ii) $\mathbf{P}$ *is a Lie algebra-morphism from* $(\mathrm{E}^*, {}^t\mathrm{N} . \nu(\mu, \mathrm{P}))$ *to* $(\mathrm{E}, \mathrm{N} . \mu)$,

$$\mathbf{P} \{ \alpha, \beta \}_{\mathrm{NP}}^{\mu} = [\mathbf{P} \alpha, \mathbf{P} \beta]_{\mathrm{N} . \mu}, \tag{4.14}$$

(iii) $\mathbf{NP}$ *is a Lie algebra-morphism from* $(\mathrm{E}^*, {}^t\mathrm{N} . \nu(\mu, \mathrm{P}))$ *to* (E, μ),

$$\mathbf{NP} \{ \alpha, \beta \}_{\mathbf{NP}}^{\mu} = [\mathbf{NP} \alpha, \mathbf{NP} \beta]_{\mu}, \tag{4.15}$$

(iv) $[\mathrm{P}, \mathrm{P}]_{\mathrm{N} . \mu} = 0$ *and* $[\mathrm{NP}, \mathrm{NP}]_{\mu} = 0$,

(v) $[\mathrm{P}, \mathrm{NP}]_{\mu} = 0$.

Proof. — In view of lemma 4.2, we obtain part (i) by applying corollary 1.1 to $\nu(\mu, \mathrm{P})$ and to ${}^t\mathrm{N}$. To prove (ii), we use definition (4.8) and assumption (4.9), and we apply proposition 3.1. We thus obtain

$$\begin{aligned} \mathbf{P} \{ \alpha, \beta \}_{\mathrm{NP}}^{\mu} &= \mathbf{P} ({}^t\mathrm{N} . \nu(\mu, \mathrm{P}) (\alpha, \beta)) \\ &= \mathbf{P} \{ {}^t\mathrm{N} \alpha, \beta \}_{\mathrm{P}}^{\mu} + \mathbf{P} \{ \alpha, {}^t\mathrm{N} \beta \}_{\mathrm{P}}^{\mu} - \mathbf{P} ({}^t\mathrm{N} \{ \alpha, \beta \}_{\mathrm{P}}^{\mu}) \\ &= [\mathbf{NP} \alpha, \mathbf{P} \beta]_{\mu} + [\mathbf{P} \alpha, \mathbf{NP} \beta]_{\mu} - \mathrm{N} [\mathbf{P} \alpha, \mathbf{P} \beta]_{\mu} = [\mathbf{P} \alpha, \mathbf{P} \beta]_{\mathrm{N} . \mu}. \end{aligned}$$

Part (iii) is a direct consequence of (i) and (ii). For part (iv) we use proposition 3.2 (i) and the results (ii) and (iii), above. Finally, by corollary 1.1, $\nu(\mu, P) + {}^t N.\nu(\mu.P)$ is a Lie algebra-structure on E*. By the compatibility condition, this bracket is $\nu(\mu, P + NP)$. Using proposition 3.2 (i) once more, we have proved that $P + NP$ is a Poisson bivector with respect to μ, whence (v).

Proposition 4.1 (iv) shows that, given a Poisson-Nijenhuis structure (P, N) on a manifold M, the bivector NP defines a *new Poisson structure* on M, and (v) proves that this Poisson structure is *compatible* with the given Poisson structure P.

In the next section, we shall study the iteration of the processes of deformation and dualization on a Poisson manifold.

5. HIERARCHIES OF BRACKETS OF A POISSON-NIJENHUIS STRUCTURE

In this section we consider a Poisson-Nijenhuis structure (P, N) on a Hausdorff, second-countable, smooth manifold M, as in section 4. We show that there exist hierarchies of Lie brackets on both E, the space of vector fields, and E*, the space of differential 1-forms, and that all the iterated deformations commute with all the dualizations, as indicated by the diagram at the end of this section.

5.1. The iterated Lie brackets on vector fields and on 1-forms

From the usual Lie algebra-structure μ on E, with Lie bracket $[\ ,\ ] = [\ ,\ ]_\mu$, and from the (1,1)-tensor N whose Nijenhuis torsion with respect to μ vanishes, we can construct the sequence of Lie algebra-structures,

$$\mu_0 = \mu, \qquad \mu_1 = \mu' = N.\mu, \ldots, \qquad \mu_k = N^k.\mu, \ldots,$$

with Lie brackets

$$[\ ,\ ]_0 = [\ ,\ ]_{\mu_0} = [\ ,\ ]_\mu, \qquad [\ ,\ ]_1 = [\ ,\ ]_{\mu_1} = [\ ,\ ]', \ldots,$$
$$[\ ,\ ]_k = [\ ,\ ]_{\mu_k}, \ldots$$

We have proved in proposition 1.2 that each of these iterated brackets is indeed a Lie algebra-bracket, and that they are compatible in pairs. Such a sequence of compatible Lie algebra-structures can be called a *hierarchy* of Lie algebra-structures.

Given μ and the Poisson bivector P, we now consider the Lie algebra-structure $\nu = \nu(\mu, P)$ on the space of 1-forms E*, with the Lie bracket $\{\ ,\ \}^\mu_P = \{\ ,\ \}_\nu$ that was defined and studied in sections 3 and 4. By proposition 4.1, we know that, when N and P are compatible, the Nijenhuis torsion of the (1,1)-tensor ${}^t N$ with respect to $\nu = \nu(\mu, P)$ vanishes.

Therefore, we can apply the results of section 1 to construct a sequence of Lie algebra-structures,

$$\nu_0=\nu, \qquad \nu_1={}^t\mathrm{N}\,.\,\nu=\nu', \ldots, \qquad \nu_k=({}^t\mathrm{N})^k\,.\,\nu, \ldots$$

with Lie brackets

$$\{\ ,\ \}_0=\{\ ,\ \}_{\nu_0}=\{\ ,\ \}_\nu, \qquad \{\ ,\ \}_1=\{\ ,\ \}_{\nu_1}, \ldots, \qquad \{\ ,\ \}_k=\{\ ,\ \}_{\nu_k}, \ldots$$

Again, these iterated brackets on the space of 1-forms are indeed Lie brackets, and they are compatible in pairs. Let us remark that, by lemma 1.3, for $k \geqq 0$, $i \geqq 0$,

$$\mathrm{N}^i\,.\,\mu_k=\mu_{k+i}, \tag{5.1}$$

$$({}^t\mathrm{N})^i\,.\,\nu_k=\nu_{k+i}. \tag{5.2}$$

Because of condition (4.9), each linear mapping $\mathrm{N}^k \circ \mathbf{P}$ is an antisymmetric map from E^* to E which therefore defines a bivector that we shall denote by $\mathrm{N}^k\mathrm{P}$. Thus, from the Poisson bivector P and the iterated Nijenhuis tensors N^k one can construct a sequence of bivectors,

$$\mathrm{P}_0=\mathrm{P}, \qquad \mathrm{P}_1=\mathrm{NP}, \ldots, \qquad \mathrm{P}_k=\mathrm{N}^k\mathrm{P}, \ldots$$

The following lemma shows that the properties which are valid for N, ${}^t\mathrm{N}$ and P, are also valid for the iterated tensors N^i, $({}^t\mathrm{N})^i$ and $\mathrm{N}^i\mathrm{P}$.

LEMMA 5.1. — *For each integer $k \geqq 0$, and for each integer $i \geqq 0$, x and y in* E, α *and* β *in* E^*,

$$\mathrm{N}^i[x, y]_{k+i}=[\mathrm{N}^i x, \mathrm{N}^i y]_k, \tag{5.3}$$

$$({}^t\mathrm{N})^i\{\alpha, \beta\}_{k+i}=\{({}^t\mathrm{N})^i\alpha, ({}^t\mathrm{N})^i\beta\}_k, \tag{5.4}$$

$$\mathbf{P}\{\alpha, \beta\}_k=[\mathbf{P}\alpha, \mathbf{P}\beta]_k, \tag{5.5}$$

and, more generally,

$$\mathrm{N}^i\mathbf{P}\{\alpha, \beta\}_{k+i}=[\mathrm{N}^i\mathbf{P}\alpha, \mathrm{N}^i\mathbf{P}\beta]_k. \tag{5.6}$$

Proof. — Formulae (5.3) and (5.4) express the morphism-properties of N^i and $({}^t\mathrm{N})^i$ which follow from proposition 1.2. To prove (5.5) we use the definition of $\{\ ,\ \}_k$, the fact that P is a Poisson bivector, and assumption (4.9). We obtain

$$\begin{aligned}\mathbf{P}\{\alpha, \beta\}_k&=\mathbf{P}(\{({}^t\mathrm{N})^k\alpha, \beta\}^\mu_\mathrm{P}+\{\alpha, ({}^t\mathrm{N})^k\beta\}^\mu_\mathrm{P}-({}^t\mathrm{N})^k\{\alpha, \beta\}^\mu_\mathrm{P})\\&=[\mathrm{N}^k\mathbf{P}\alpha, \mathbf{P}\beta]_\mu+[\mathbf{P}\alpha, \mathrm{N}^k\mathbf{P}\beta]_\mu-\mathrm{N}^k[\mathbf{P}\alpha, \mathbf{P}\beta]_\mu=[\mathbf{P}\alpha, \mathbf{P}\beta]_k.\end{aligned}$$

Now, formula (5.6) follows from (5.5) and (5.4).

5.2. Deformation and dualization of iterated Lie brackets

In section 4, we introduced a compatibility condition on N and P in order to ensure that the processes of deformation of μ under N and of dualization of μ under P commute. In this subsection we show that this compatibility condition actually implies that the same property is valid for any of the iterated Lie algebra-structures μ_j, $j \geqq 0$, for any of the iterated Nijenhuis tensors N^k, $k \geqq 1$, and any of the iterated bivectors $N^i P$, $i \geqq 0$. More precisely,

PROPOSITION 5.1. — *Let* (P, N) *be a Poisson-Nijenhuis structure on a manifold. Let k be a nonnegative integer. Then, for all nonnegative integers i and j such that* $0 \leqq i+j \leqq k$,

$$\nu_k = ({}^tN)^{k-i-j} . \nu(N^j . \mu, N^i P). \tag{5.7}$$

Proof. — A direct computation shows that, for each $i \geqq 1$, α and β in E^*,

$$(\{\alpha, \beta\}_{\nu(\mu,\, N^{i+1}P)} - \{\alpha, \beta\}_{{}^tN . \nu(\mu,\, N^i P)}) - ({}^tN\{\alpha, \beta\}_{\nu(\mu,\, N^i P)} - \{{}^tN\alpha, {}^tN\beta\}_{\nu(\mu,\, N^{i-1}P)}) = 0.$$

Since N and P are compatible, the relation

$$\nu(\mu, N^i P) = \nu_i \tag{5.8}$$

is valid for $i=0$ and $i=1$. Let us assume that this relation is valid for all integers less than or equal to $i \geqq 1$. Then the second term of the left-hand side of this identity vanishes by (5.4). Therefore

$$\nu(\mu, N^{i+1} P) = {}^tN . \nu(\mu, N^i P) = {}^tN . \nu_i = \nu_{i+1}. \tag{5.9}$$

The last equality uses relation (5.2). Thus relation (5.8) is proved by induction on i.

Lemma 4.1 implies that, for all $k \geqq 0$, $i \geqq 0$,

$$\nu(N^{k+1} . \mu, N^i P) + {}^tN . \nu(N^k . \mu, N^i P) = 2\nu(N^k . \mu, N^{i+1} P). \tag{5.10}$$

From (5.9) and (5.10) it follows that the relations

$$\nu(N^{k+1} . \mu, N^i P) = \nu(N^k . \mu, N^{i+1} P) = {}^tN . \nu(N^k . \mu, N^i P) \tag{5.11}$$

are valid for $k=0$ and any $i \geqq 0$. Let us assume that these relations are valid for any integer less than or equal to $k \geqq 0$ and any $i \geqq 0$. Then (5.10) shows that

$$\nu(N^{k+2} . \mu, N^i P) = \nu(N^{k+1} . \mu, N^{i+1} P) = {}^tN . \nu(N^{k+1} . \mu, N^i P),$$

proving by induction on k that relations (5.11) are valid for any $k \geqq 0$ and any $i \geqq 0$. These relations in turn show that

$$({}^tN)^j . \nu(N^k . \mu, N^i P) = \nu(N^{k+j} . \mu, N^i P) = \nu(N^k . \mu, N^{i+j} P),$$

for any $j \geqq 0$, thus proving the proposition.

5.3. The hierarchy of Poisson structures on a Poisson-Nijenhuis manifold

We shall now prove that, because N and P are compatible, the bivectors P, NP, . . ., $N^k P$, . . . are Poisson bivectors and compatible in pairs.

PROPOSITION 5.2. — *On a Poisson-Nijenhuis manifold* (M, P, N), *each bivector* $N^i P$ *is a Poisson bivector with respect to each Lie algebra-structure* $N^k.\mu = \mu_k$, *and for all nonnegative integers i, j and k,*

$$[N^i P, N^j P]_{N^k.\mu} = 0.$$

Proof. — Since, by relation (5.7), the brackets ν_{k+i} and $\nu(N^k.\mu, N^i P)$ coincide, it follows from formula (5.6) and proposition 3.1 that $N^i P$ is a Poisson bivector with respect to μ_k.

We shall prove that, moreover, $[N^i P, N^{i+m} P]_k = 0$, for $i \geqq 0$, $m \geqq 0$, $k \geqq 0$. From proposition 1.2 (iii) applied to $\nu(N^k.\mu, N^i P)$ and $({}^tN)^m.\nu(N^k.\mu, N^i P)$, and from formula (5.7) we know that $\nu(N^k.\mu, N^i P + N^{i+m} P)$ is a Lie algebra-structure on E*. Using proposition 3.2(i), we conclude that $N^i P + N^{i+m} P$ is a Poisson bivector with respect to μ_k.

(We have in fact used the generalizations of propositions 3.1 and 3.2 to the iterated Lie brackets μ_k. These are straightforward, and they also follow from the general results of section 6.)

In the particular case where $k=0$, the preceding proposition yields the fundamental property of the Poisson-Nijenhuis structures:

COROLLARY 5.1. — *On a Poisson-Nijenhuis manifold* (M, P, N), *there is a hierarchy of iterated Poisson structures,* i.e., *a sequence of Poisson structures* $N^i P$, $i \geqq 0$, *which are compatible in pairs.*

Combining the results of propositions 5.1 and 5.2, we see that the hypotheses $[N, N]_{\mu_0} = 0$, $[P, P]_{\mu_0} = 0$, $C_{\mu_0}(P, N) = 0$ imply that, for all nonnegative integers *i*, *j* and *k*,

$$[N^i, N^j]_{\mu_k} = 0, \qquad [({}^tN)^i, ({}^tN)^j]_{\nu_k} = 0,$$
$$[N^i.\mu, N^j.\mu] = 0, \qquad [N^i P, N^j P]_{\mu_k} = 0, \qquad C_{\mu_k}(N^i P, N^j) = 0.$$

In particular, proposition 5.2 shows that the following diagram is commutative and that all the arrows are Lie algebra-morphisms:

$$\begin{tikzcd}
(E^*, \nu_k) \arrow[r, "P"] & (E, \mu_k) \\
(E^*, \nu_{k+i}) \arrow[u, "({}^tN)^i"] \arrow[r, "P"'] \arrow[ur, "N^i P"] & (E, \mu_{k+i}) \arrow[u, "N^i"']
\end{tikzcd}$$

These results are in fact valid somewhat more generally on the *differential Lie algebras* to be studied in the next section.

6. DIFFERENTIAL LIE ALGEBRAS, GRADED DIFFERENTIAL ALGEBRAS AND SCHOUTEN ALGEBRAS

In this section, we shall define and study the differential Lie algebras, an appropriate algebraic framework for both the vector space of vector fields on a manifold and the vector space of differential 1-forms on a Poisson manifold. With a differential Lie algebra E we associate a graded differential algebra, which generalizes the algebra of forms $\Lambda(E^*)$ equipped with the de Rham cohomology operator, and a graded Lie algebra, which generalizes the algebra of multivectors ΛE equipped with the Schouten bracket. We shall show that the process of deformation described in section 1 can be carried out on a differential Lie algebra equipped with a Nijenhuis operator, while the dualization described in section 3 can be performed on a differential Lie algebra equipped with a bivector with vanishing Schouten bracket. We shall study the effect of deformations and dualizations on the cohomology operator of the associated graded differential algebra and on the Schouten bracket.

6.1. Differential Lie algebras

The linear space $E=\overline{TM}$ of smooth vector fields on a smooth manifold M is, in a natural way, both an A-module (where A denotes the associative and commutative $\mathbb{R}$-algebra, with unit, of the smooth, real-valued functions on M), and a Lie algebra over $\mathbb{R}$. These two structures are related by the identity

$$[x, fy]=f[x, y]+(\mathscr{L}_x f)y \tag{6.1}$$

for all x and y in E, and for all f in A. We have denoted by [,] the standard Lie algebra bracket on E and, by $\mathscr{L}_x f$, the Lie derivative of f by x. We remark that $x\in E\to\mathscr{L}_x$ is an $\mathbb{R}$-Lie algebra-morphism from (E, [,]) into the $\mathbb{R}$-Lie algebra of derivations of the ring A, and that it is A-linear. When one formalizes the above properties, one arrives at what Palais [37] called *Lie d-rings* over A. Some closely related variants have been studied by algebraists and geometers under various names: pseudo-Lie-algebras ([16], [38]), (K, R)- or (K, A)-Lie algebras [39], [5], Elie Cartan spaces and pre-spaces [10], [11], Lie modules [34], Lie-Cartan pairs [20]. (The list is not complete.) We shall consider a special class of Lie d-rings, which we shall call the *differential Lie algebras*. It will be convenient to first introduce the weaker notion of a *differential pre-Lie algebra*. Roughly speaking, the differential Lie algebras over an algebra are the algebraic counterparts of the Lie algebroids ([38], [9], [29]), just as the modules over a ring are the algebraic counterparts of the vector bundles.

DEFINITION 6.1. — *Let* K *be the field of real or complex numbers, and let* A *be an associative and commutative* K-*algebra with unit. Let* E *be a finitely generated projective* A-*module. Let* μ *be an antisymmetric,* K-*bilinear map, from* $E \times E$ *to* E. *We say that* (E, μ) *is a* differential pre-Lie algebra *over* A *if there exists an* A-*linear map* $\mathscr{L}^\mu$ *from* E *to the* K-*vector space of derivations of* A *such that*

$$\mu(x, fy) = f\,\mu(x, y) + (\mathscr{L}^\mu_x f)\,y, \tag{6.2}$$

for all x *and* y *in* E, *and for all* f *in* A.

Most of the results of this section require that the bidual of E can be identified with E, so, in order to simplify the exposition, we have added to the definition the requirement that E be a finitely generated projective A-module.

By adding the requirement that μ satisfy the Jacobi identity, we arrive at the following definition:

DEFINITION 6.2. — *We say that* (E, μ) *is a* differential Lie algebra *over* A *if* (E, μ) *is a differential pre-Lie algebra over* A *and if, in addition,* μ *is a* K-*Lie algebra-structure on* E, *and if* $\mathscr{L}^\mu$ *defines an* E-*module-structure on* A.

Thus when (E, μ) is a differential Lie algebra, $\mathscr{L}^\mu$ is a K-Lie algebra-morphism from (E, μ) to the K-Lie algebra of derivations of the ring A, *i.e.*,

$$\mathscr{L}^\mu_{[x,\, y]_\mu} = [\mathscr{L}^\mu_x, \mathscr{L}^\mu_y], \tag{6.3}$$

for all x and y in E.

Here are three examples of differential Lie algebras in differential geometry:

If M is a Hausdorff, second-countable, smooth manifold, the vector space $E = \underline{TM}$ of smooth vector fields on M is a finitely generated projective A-module, where A is the algebra of smooth functions on M, and therefore it is a differential Lie algebra with respect to the usual Lie bracket. If, moreover, M is a Poisson manifold, the associative algebra of smooth functions over M, with respect to the Poisson bracket, and the vector space of smooth differential 1-forms with respect to the bracket (3.2), are differential Lie algebras.

Let us assume that the A-module E contains an element which is not a torsion element, *i.e.*, there exists $y \in E$ such that $f \in A$ and $fy = 0$ imply $f = 0$. This assumption has the following consequences:

(*a*) The K-linear map $\mathscr{L}^\mu$ in the definition of differential pre-Lie algebras is uniquely defined.

(*b*) It is enough to assume that $\mathscr{L}^\mu$ takes values in the vector space of K-linear maps from A to A, since (6.2) implies

$$\begin{aligned}\mathscr{L}_z^\mu(fg)\,x &= \mu(z, fg\,x) - fg\,\mu(z, x)\\ &= f\,\mu(z, gx) + (\mathscr{L}_z^\mu f)\,gx - fg\,\mu(z, x)\\ &= (f\,(\mathscr{L}_z^\mu g) + (\mathscr{L}_z^\mu f)\,g)\,x,\end{aligned}$$

which proves that $\mathscr{L}^\mu$ takes values in the derivations of A.

(*c*) Property (6.3) of $\mathscr{L}^\mu$ in the definition of the differential Lie algebras becomes a consequence of the Jacobi identity for μ since, by relation (6.2) and by the antisymmetry of μ,

$$\begin{aligned}&[[x, y]_\mu, fz]_\mu + [[y, fz]_\mu, x]_\mu + [[fz, x]_\mu, y]_\mu\\ &\qquad - f\,([[x, y]_\mu, z]_\mu + [[y, z]_\mu, x]_\mu + [[z, x]_\mu, y]_\mu)\\ &\qquad\qquad = (\mathscr{L}^\mu_{[x, y]_\mu} f - \mathscr{L}_x^\mu \mathscr{L}_y^\mu f + \mathscr{L}_y^\mu \mathscr{L}_x^\mu f)\,z.\end{aligned}$$

These properties do not require the commutativity of A. They were proved by Herz [16] under the assumption that A is a skew field.

If the A-module E possesses two linearly independent elements, then in the definition of differential pre-Lie algebras, it is enough to assume that the map $\mathscr{L}^\mu$ is K-linear, since (6.2) implies that

$$\begin{aligned}\mu(fx, gy) &= fg\,\mu(x, y) - g\,(\mathscr{L}_y^\mu f)\,x + (\mathscr{L}^\mu_{fx} g)\,y\\ &= fg\,\mu(x, y) + f\,(\mathscr{L}_x^\mu g)\,y - (\mathscr{L}^\mu_{gy} f)\,x,\end{aligned}$$

whence $\mathscr{L}^\mu$ is necessarily A-linear.

In the particular case where A is a commutative field, this property was also proved by Herz.

We shall now consider the exterior algebras $\Lambda\,\mathrm{E}$ and $\Lambda(\mathrm{E}^*)$, and we shall show that a differential Lie algebra-structure on E gives rise to both a graded differential algebra-structure on $\Lambda(\mathrm{E}^*)$, and a graded Lie algebra-structure on $\Lambda\,\mathrm{E}$.

6.2. The graded differential algebra of a differential Lie algebra

We recall that a *graded differential algebra* (abbreviated as GDA) is a graded commutative algebra, with a derivation of degree 1 and square 0. When the derivation is not necessarily of square 0, we shall speak of a *pre-GDA*.

Let (E, μ) be a *differential pre-Lie algebra* over A. We denote by E^* the A-dual of E, $\mathrm{Hom}_\mathrm{A}(\mathrm{E}, \mathrm{A})$. Since E is finitely generated and projective, we can identify the A-module of q-linear (over A) antisymmetric maps from E to A with $\Lambda^q(\mathrm{E}^*)$, and we set $\Lambda(\mathrm{E}^*) = \bigoplus_{q\geq 0} \Lambda^q(\mathrm{E}^*)$. Then the operator, $d_\mu : \Lambda(E^*) \to \Lambda(E^*)$, defined for α in $\Lambda^q(E^*)$ by

$$(d_\mu \alpha)(x_1, \ldots, x_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \mathscr{L}^\mu_{x_i}(\alpha(x_1, \ldots, \hat{x}_i, \ldots, x_{q+1})) + \sum_{i<j} (-1)^{i+j} \alpha([x_i, x_j]_\mu, x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{q+1}), \tag{6.4}$$

where $x_1, \ldots, x_{q+1}$ are elements of E, and where $\hat{}$ indicates an omitted variable, is a *derivation of degree* 1 of the graded associative algebra $\Lambda(E^*)$ [37].

Thus, to each differential pre-Lie algebra-structure μ on E, there corresponds a derivation d_μ of the graded algebra $\Lambda(E^*)$ which is of degree 1 but not necessarily of square 0. Conversely, assume that $(\Lambda(E^*), d)$ is a graded algebra with a derivation of degree 1. Using the fact that the dual of E^* can be identified with E, we define

$$\mathscr{L}_x f = \langle df, x \rangle \tag{6.5}$$

and

$$\langle \alpha, \mu(x, y) \rangle = -(d\alpha)(x, y) + \mathscr{L}_x \langle \alpha, y \rangle - \mathscr{L}_y \langle \alpha, x \rangle, \tag{6.6}$$

where x and y are in E, $\alpha \in E^*$, and $f \in A$. Then (E, μ) is a differential pre-Lie algebra, and μ is the unique differential pre-Lie algebra-structure on E such that $d_\mu = d$, To summarize, when (E, μ) is a *differential pre-Lie algebra*, $(\Lambda(E^*), d_\mu)$ is a *pre-GDA*, and conversely.

When (E, μ) is a differential Lie algebra, the derivation d_μ is the restriction to the A-linear forms on E of the coboundary operator of the cohomology of the K-Lie algebra E with values in the E-module A, and therefore, when (E, μ) is a differential Lie algebra, the square of d_μ vanishes. The following proposition contains a direct proof of this fact and of its converse.

PROPOSITION 6.1. — *The differential pre-Lie algebra* (E, μ) *over* A *is a differential Lie algebra if and only if*

$$(d_\mu)^2 = 0.$$

Proof. — For $f \in A$,

$$((d_\mu)^2 f)(x, y) = (\mathscr{L}^\mu_x \mathscr{L}^\mu_y - \mathscr{L}^\mu_y \mathscr{L}^\mu_x - \mathscr{L}^\mu_{[x, y]_\mu}) f.$$

For $\alpha \in E^*$,

$$((d_\mu)^2 \alpha)(x, y, z) = (\mathscr{L}^\mu_x \mathscr{L}^\mu_y - \mathscr{L}^\mu_y \mathscr{L}^\mu_x - \mathscr{L}^\mu_{[x, y]_\mu})(\langle \alpha, z \rangle) + \langle \alpha, \oint [[x, y]_\mu, z]_\mu \rangle.$$

Thus, if (E, μ) is a differential Lie algebra, by (6.3) and by the Jacobi identity for μ, the derivation $(d_\mu)^2$ vanishes on A and on E^*, and therefore also on $\Lambda(E^*)$, since it follows from the assumptions made on E that A and E^* generate the algebra of A-multilinear antisymmetric maps from E to A. Conversely, if $(d_\mu)^2=0$, then $\mathscr{L}^\mu$ satisfies (6.3) and μ satisfies Jacobi's identity, so (E, μ) is a differential Lie algebra.

To summarize, when (E, μ) is a *differential Lie algebra*, (Λ(E*), d_μ) is a *graded differential algebra*, and conversely.

If ω is an A-linear E-valued q-form on E, then, by definition,

$$\vartheta_\omega^\mu=[i_\omega, d_\mu]=i_\omega \circ d_\mu-(-1)^{q-1} d_\mu \circ i_\omega,$$

where i_ω is defined as in subsection 1.2, and where [,] is the graded commutator. Then, for any ω, ϑ_ω^μ is a derivation of degree q of Λ(E*) which commutes with d_μ (in the graded sense).

When ω is an E-valued 0-form, *i.e.*, an element x of E, ϑ_x^μ is called the *Lie derivation* with respect to x. Then, ϑ_x^μ is a derivation of degree 0 of Λ(E*) which commutes with d_μ, and

$$\vartheta_x^\mu\langle \alpha, y\rangle-\langle \vartheta_x^\mu \alpha, y\rangle=\langle \alpha, [x, y]_\mu\rangle$$

and

$$\vartheta_{[x, y]_\mu}^\mu=[\vartheta_x^\mu, \vartheta_y^\mu].$$

The restriction of ϑ_x^μ to $\Lambda^0(\mathrm{E}^*)=\mathrm{A}$ is $\mathscr{L}_x^\mu$.

6.3. The Schouten algebra of a differential Lie algebra

The *Schouten bracket* (*see* [36], [23], and also [7], [33]) of multivectors on a manifold M is the unique $\mathbb{R}$-bilinear mapping [,] on ΛE with values in ΛE, where E=TM, which

(*a*) extends the Lie bracket on E,

(*b*) satisfies $[x, f]=\mathscr{L}_x f$, for all x in E, and for all f in A,

(*c*) is antisymmetric in the graded sense, *i.e.*,

$$[\mathrm{Q}', \mathrm{Q}]=-(-1)^{(q-1)(q'-1)}[\mathrm{Q}, \mathrm{Q}'],$$

(*d*) is a biderivation of the graded algebra ΛE, *i.e.*,

$$[\mathrm{Q}, \mathrm{Q}' \wedge \mathrm{Q}'']=[\mathrm{Q}, \mathrm{Q}'] \wedge \mathrm{Q}''+(-1)^{(q-1)q'}\mathrm{Q}' \wedge [\mathrm{Q}, \mathrm{Q}''],$$

for $\mathrm{Q}\in\Lambda^q\mathrm{E}$, $\mathrm{Q}'\in\Lambda^{q'}\mathrm{E}$, $\mathrm{Q}''\in\Lambda^{q''}\mathrm{E}$.

The Schouten bracket of multivectors satisfies the Jacobi identity in the graded sense,

$$\oint(-1)^{(q-1)(q''-1)}[\mathrm{Q}, [\mathrm{Q}', \mathrm{Q}'']]=0,$$

i.e., it is a graded Lie algebra-bracket on ΛE when the usual gradation is shifted by one. It should be stressed that this bracket differs from the bracket used by Lichnerowicz, denoted by $[\ ,\ ]^{\mathrm{L}}$, by a sign,

$$[\mathrm{Q}, \mathrm{Q}']=(-1)^{q+1}[\mathrm{Q}, \mathrm{Q}']^{\mathrm{L}}.$$

Similarly, given a differential pre-Lie algebra (E, μ) over A, there exists a unique K-bilinear mapping on the graded algebra ΛE with values in

ΛE, which possesses the above four properties. This bilinear mapping will be called the *Schouten bracket* of multivectors, and be denoted by $[\ ,\ ]_\mu$. We shall refer to $(\Lambda E, [\ ,\ ]_\mu)$ as the *Schouten graded pre-Lie algebra*, or simply as the *Schouten pre-algebra*, of (E, μ). When (E, μ) is a differential Lie algebra, the Schouten bracket $[\ ,\ ]_\mu$ satisfies the Jacobi identity (in the graded sense), and in this case we shall refer to $(\Lambda E, [\ ,\ ]_\mu)$ as the *Schouten graded Lie algebra*, or simply as the *Schouten algebra*, of (E, μ).

We shall call a differential pre-Lie algebra *nondegenerate* if the only element Q in ΛE that satisfies $[Q, Q']_\mu = 0$ for all $Q' \in \Lambda E$, is $Q = 0$. This property is clearly satisfied in the case where $E = \underline{TM}$ with the usual Lie bracket of vector fields.

6.4. Graded differential algebra and Schouten algebra associated with a deformed Lie bracket

In this subsection, we retain the notations of section 1 and we assume moreover that (E, μ) is a differential pre-Lie algebra over A. Let N be an A-linear mapping from E to E, and let

$$N.\mu = [\mu, N] = \mu'$$

be the deformed K-bilinear map on E. Then $(E, N.\mu)$ is a differential pre-Lie algebra with

$$\mathscr{L}_x^{N.\mu} = \mathscr{L}_{Nx}^\mu = \mathscr{L}'_x.$$

We shall first study the effect of the deformation on the associated pre-GDA $(\Lambda(E^*), d_\mu)$.

PROPOSITION 6.2. – *Let* (E, μ) *be a differential pre-Lie algebra and let* N *be an* A*-linear mapping from* E *to* E. *Then* $(E, N.\mu)$ *is a differential pre-Lie algebra. The derivation of degree* 1 *of the graded algebra,* $\Lambda(E^*)$, *associated with* $\mu' = N.\mu$, *and denoted by* d' *or* $d_{N.\mu}$, *satisfies*

$$d_{N.\mu} = [i_N, d_\mu] = \vartheta_N^\mu. \tag{6.7}$$

Proof. – It suffices to check that both derivations coincide on the 0-forms and on the 1-forms. In fact, for f in A and for x in E,

$$\langle [i_N, d_\mu] f, x\rangle = \langle i_N d_\mu f, x\rangle = d_\mu f(Nx) = \mathscr{L}_{Nx}^\mu f = \mathscr{L}_x^{N.\mu} f,$$

and for α in E^*, and for x and y in E,

$$\begin{aligned}
([i_N, d_\mu]\alpha)(x, y) &= (i_N d_\mu \alpha)(x, y) - d_\mu(i_N \alpha)(x, y)\\
&= d_\mu \alpha(Nx, y) + d_\mu \alpha(x, Ny) - (\mathscr{L}_x^\mu(i_N \alpha(y)) - \mathscr{L}_y^\mu(i_N \alpha(x)) - (i_N \alpha)([x, y]_\mu))\\
&= \mathscr{L}_{Nx}^\mu \langle \alpha, y\rangle - \mathscr{L}_y^\mu \langle \alpha, Nx\rangle - \langle \alpha, [Nx, y]_\mu\rangle\\
&\quad + \mathscr{L}_x^\mu \langle \alpha, Ny\rangle - \mathscr{L}_{Ny}^\mu \langle \alpha, x\rangle - \langle \alpha, [x, Ny]_\mu\rangle\\
&\quad - \mathscr{L}_x^\mu \langle \alpha, Ny\rangle + \mathscr{L}_y^\mu \langle \alpha, Nx\rangle + \langle \alpha, N[x, y]_\mu\rangle\\
&= \mathscr{L}_x^{N.\mu} \langle \alpha, y\rangle - \mathscr{L}_y^{N.\mu} \langle \alpha, x\rangle - \langle \alpha, [x, y]_{N.\mu}\rangle\\
&= (d_{N.\mu}\alpha)(x, y).
\end{aligned}$$

It is natural to introduce the Nijenhuis torsion of N in order to determine when the square of d_μ vanishes. It follows from the intrinsic definition of the Frölicher-Nijenhuis bracket that, for an E-valued form N on E,

$$(\vartheta_N^\mu)^2 = \vartheta_{[N,\,N]_\mu}^\mu. \tag{6.8}$$

(We remark that the bracket originally defined by Frölicher and Nijenhuis [14] is twice the one that we consider. Had we used their definition, the graded commutator $[\vartheta_N^\mu, \vartheta_N^\mu]$ would have replaced the square of ϑ_N^μ in the preceding formula.)

In view of the fact that the differential pre-Lie algebra (E, N . μ) is a differential Lie algebra if and only if $(d_{N\,.\,\mu})^2 = 0$, formulae (6.7) and (6.8) imply

COROLLARY 6.1. – *If* N *is a Nijenhuis operator on a differential Lie algebra* (E, μ), *then* (E, N . μ) *is a differential Lie algebra.*

This corollary is just corollary 1.1 (i). In the special case where $E = \underline{TM}$, it was proved by J. Lehmann-Lejeune [25]. *See* also the comprehensive work of C. M. de Barros ([10], [11]) who gave an algebraic framework for the theory.

We now determine the deformed Schouten bracket.

PROPOSITION 6.3. – *Let* N *be a Nijenhuis operator on a differential pre-Lie algebra* (E, μ). *The Schouten bracket* $[\ ,\ ]_{N\,.\,\mu}$ *defined on* ΛE *by the differential pre-Lie algebra structure* N . μ *satisfies*

$$[Q, Q']_{N\,.\,\mu} = [i_{^tN} Q, Q']_\mu + [Q, i_{^tN} Q']_\mu - i_{^tN} [Q, Q']_\mu.$$

Proof. – In this formula, the elements of ΛE are considered to be forms on E*, and the transpose ${}^tN$ of N as an E*-valued 1-form on E*. The formula is valid when Q and Q′ are of degree 0 or 1, and therefore it is valid in general.

6.5. Dualization of the Lie bracket by means of a Poisson bivector on a differential Lie algebra. The associated Schouten algebra and graded differential algebra

In section 3 we studied the problem of dualizing the Lie bracket μ on $E = \underline{TM}$ by means of an A-linear mapping, **P**, from E* to E corresponding to a Poisson bivector P. We can proceed in exactly the same manner in the case of a differential Lie algebra (E, μ): formula (6.9) below generalizes formula (3.2).

We remark that in the process of dualization, the vectors and the 1-forms are exchanged. Starting from a differential Lie algebra-structure on the space of vector fields on a manifold, after dualization by means of a Poisson bivector, we obtain the differential Lie algebra-structure on the space of 1-forms that has already been considered in section 3. We shall see that the associated graded differential algebra is that of the multivectors with the Lichnerowicz cohomology operator [27], while the Schouten algebra is that of the forms on the manifold with the graded Lie algebra-structure defined by Koszul in [23].

Let (E, μ) be a differential pre-Lie algebra over A. Let P a bivector on E, *i.e.*, an element of $\Lambda^2 E$, and let $\mathbf{P}$ be the linear mapping from E^* to E defined by

$$\langle \alpha, \mathbf{P}\beta \rangle = P(\alpha, \beta),$$

for α in E^*, β in E^*. For elements α and β in E^*, and for an element x in E, we set

$$\begin{aligned}\langle \{\alpha, \beta\}_P^\mu, x\rangle = \langle \alpha, [\mathbf{P}\beta, x]_\mu\rangle - \langle \beta, [\mathbf{P}\alpha, x]_\mu\rangle \\ + \mathscr{L}^\mu_{\mathbf{P}\alpha}\langle \beta, x\rangle - \mathscr{L}^\mu_{\mathbf{P}\beta}\langle \alpha, x\rangle + \mathscr{L}^\mu_x\langle \alpha, \mathbf{P}\beta\rangle. \qquad (6.9)\end{aligned}$$

Then E^*, with the bilinear mapping $\nu : E^* \times E^* \to E^*$ defined by $\nu(\alpha, \beta) = \{\alpha, \beta\}_P^\mu$, is a differential pre-Lie algebra, with

$$\mathscr{L}^\nu_\alpha = \mathscr{L}^\mu_{\mathbf{P}\alpha}. \qquad (6.10)$$

We shall set $\nu = \nu(\mu, P)$, and we shall sometimes denote the bracket of 1-forms $\{\alpha, \beta\}_P^\mu$, defined in (6.9), by $[\alpha, \beta]_\nu$.

By the construction of subsection 6.2, the differential pre-Lie algebra-structure $\nu(\mu, P)$ on E^* defines a derivation d_ν of degree 1, on $\Lambda(E^{**})$ which we identify with ΛE. The following proposition relates this derivation d_ν of ΛE with the Schouten bracket defined by the given pre-Lie algebra-structure μ on E.

Proposition 6.4. — *Let* (E, μ) *be a differential pre-Lie algebra, and let* P *be a bivector on* E. *The derivation of degree* 1 *of the graded algebra* ΛE *corresponding to the bracket* $\nu = \{\ ,\ \}_P^\mu$ *on* E^* *is*

$$d_\nu = [P, .]_\mu,$$

where $[\ ,\ ]_\mu$ *denotes the Schouten bracket on* ΛE *defined by* μ.

Proof. — By the definition of d_ν,

$$\begin{aligned}d_\nu Q(\alpha_1, \ldots, \alpha_{q+1}) = \sum_{i=1}^{q+1} (-1)^{i+1} \mathscr{L}^\mu_{\mathbf{P}\alpha_i}(Q(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_{q+1})) \\ + \sum_{i<j} (-1)^{i+j} Q(\{\alpha_i, \alpha_j\}_P^\mu, \alpha_1, \ldots, \hat{\alpha}_i, \ldots, \hat{\alpha}_j, \ldots, \alpha_{q+1}).\end{aligned}$$

Let us check that both derivations, d_ν and $[P, .]_\mu$, coincide on elements of $A = \Lambda^0 E$ and $E = \Lambda^1 E$. In fact, let f be an element of A. Then, for α in E^*,

$$d_\nu f(\alpha) = \mathscr{L}^\mu_{\mathbf{P}\alpha} f = \langle d_\mu f, \mathbf{P}\alpha\rangle = -\langle \alpha, \mathbf{P} d_\mu f\rangle,$$

while

$$[\mathrm{P}, f]_\mu = -\mathbf{P}\, d_\mu f.$$

Now let x be an element of E. Then, for α and β in E^*,

$$\begin{aligned}(d_\nu x)(\alpha, \beta) &= \mathscr{L}^\mu_{\mathbf{P}\alpha}\langle \beta, x\rangle - \mathscr{L}^\mu_{\mathbf{P}\beta}\langle \alpha, x\rangle - \langle \{\alpha, \beta\}^\mu_{\mathrm{P}}, x\rangle \\ &= -\langle \alpha, [\mathbf{P}\beta, x]_\mu\rangle + \langle \beta, [\mathbf{P}\alpha, x]_\mu\rangle - \mathscr{L}^\mu_x\langle \alpha, \mathbf{P}\beta\rangle,\end{aligned}$$

by the definitions of d_ν and $\{\ ,\ \}^\mu_{\mathrm{P}}$, while

$$\begin{aligned}[\mathrm{P}, x]_\mu(\alpha, \beta) &= -(\mathscr{L}^\mu_x \mathrm{P})(\alpha, \beta) \\ &= -(\mathscr{L}^\mu_x(\mathrm{P}(\alpha, \beta)) - \langle \alpha, \mathbf{P}(\mathscr{L}^\mu_x \beta)\rangle + \langle \beta, \mathbf{P}(\mathscr{L}^\mu_x \alpha)\rangle) \\ &= -(\mathscr{L}^\mu_x\langle \alpha, \mathbf{P}\beta\rangle - \langle \alpha, \mathscr{L}^\mu_x \mathbf{P}\beta\rangle + \langle \beta, \mathscr{L}^\mu_x \mathbf{P}\alpha\rangle).\end{aligned}$$

Thus, when $\mathrm{E} = \underline{\mathrm{TM}}$, and P is a Poisson bivector, up to a sign, d_ν is nothing other than the cohomology operator on $\Lambda\mathrm{E}$ introduced by Lichnerowicz [27].

Conversely, under the assumption that E is projective and finitely generated, given a derivation, D, of degree 1 on $\Lambda\mathrm{E}$, we know that there exists a unique differential pre-Lie algebra-structure, ν, on E^* such that $\mathrm{D} = d_\nu$. Given (E, μ) and P as above, let us consider $\mathrm{D} = [\mathrm{P}, .]_\mu$ which is, in fact, a derivation of degree 1 on $\Lambda\mathrm{E}$. Then the bilinear mapping $\{\ ,\ \}^\mu_{\mathrm{P}}$ on E^* defined by (6.9) is the *unique* differential pre-Lie algebra-structure on E^* such that the corresponding derivation on $\Lambda\mathrm{E}$ is $[\mathrm{P}, .]_\mu$.

It follows from the preceding characterization of $\nu = \{\ ,\ \}^\mu_{\mathrm{P}}$, and from proposition 6.1, that $\{\ ,\ \}^\mu_{\mathrm{P}}$ is a Lie bracket on E^* if and only if $[\mathrm{P}, [\mathrm{P}, .]_\mu]_\mu = 0$. Assume that (E, μ) is a differential Lie algebra. By the Jacobi identity for the Schouten bracket $[\ ,\ ]_\mu$ on $\Lambda\mathrm{E}$, this condition is equivalent to $[[\mathrm{P}, \mathrm{P}]_\mu, .]_\mu = 0$. Therefore, if P is a Poisson bivector, then $\{\ ,\ \}^\mu_{\mathrm{P}}$ is a Lie bracket. If, moreover, the differential Lie algebra (E, μ) is *nondegenerate*, then the converse is true. We have thus obtained an alternative proof of proposition 3.2 (i) and, more generally,

PROPOSITION 6.5. — *Let* (E, μ) *be a differential Lie algebra, and let* P *be a bivector on* E. *A sufficient condition for the formula* (6.9) *to define a Lie bracket* $\{\ ,\ \}^\mu_{\mathrm{P}}$ *on* E^* *is that* P *be a Poisson bivector. When* (E, μ) *is a nondegenerate differential Lie algebra this condition is also necessary.*

This proof of the Jacobi identity for $\{\ ,\ \}^\mu_{\mathrm{P}}$ seems to be new, but the fact that d_ν is nothing but the Lichnerowicz cohomology operator had been noted independently, in the case of a Poisson manifold, by Bhaskara and Viswanath ([3], [4]). An algebraic version of the theory, in which no algebraic assumptions are made on the A-module E, and several applications can be found in a recent preprint by Huebschmann [18].

We now compute $d_\nu \mathrm{P}$ according to the definition, and we apply proposition 6.4. This computation yields the following explicit expression,

$$[\mathrm{P}, \mathrm{P}]_\mu(\alpha_1, \alpha_2, \alpha_3) = 2\oint\langle \mathscr{L}^\mu_{\mathbf{P}\alpha_1}\alpha_3, \mathbf{P}\alpha_2\rangle,$$

where, by definition,

$$\langle \mathscr{L}_x^\mu \alpha, y \rangle = \mathscr{L}_x^\mu \langle \alpha, y \rangle - \langle \alpha, [x, y]_\mu \rangle.$$

Thus we have checked that the definitions of the Schouten bracket of P adopted here and in section 3 [*see* formula (3.6)] agree. Moreover, we find, as in formula (3.5),

$$(1/2)\,[\mathbf{P}, \mathbf{P}]_\mu (\alpha, \beta) = \mathbf{P}\{\alpha, \beta\}_\mathrm{P}^\mu - [\mathbf{P}\alpha, \mathbf{P}\beta]_\mu. \tag{6.11}$$

We now assume that P is a Poisson bivector and that P has an inverse Ω (the *symplectic case*). We set $\mathbf{P}^{-1} = \mathbf{\Omega}$. Then, from $\mathbf{P}[\alpha, \beta]_\nu = [\mathbf{P}\alpha, \mathbf{P}\beta]_\mu$, we obtain

$$\mathbf{\Omega}[x, y]_\mu = [\mathbf{\Omega} x, \mathbf{\Omega} y]_\nu,$$

for all x and y in E. Dualizing the Lie bracket $\nu = \nu(\mu, \mathrm{P})$ by means of the 2-form Ω, we obtain a bracket $\mu(\nu, \Omega)$ on E that satisfies the same relation and therefore coincides with the given bracket. Applying proposition 6.4, we obtain the formula

$$d_\mu = [\Omega, \,.\,]_\nu \tag{6.12}$$

that relates the de Rham cohomology operator d_μ with the Koszul-Schouten bracket $[\ ,\ ]_\nu$ on $\Lambda(\mathrm{E}^*)$. In particular, given a nondegenerate differential Lie algebra (E, μ), an invertible 2-form Ω on E satisfies $[\Omega, \Omega]_\nu = 0$, where $\nu = \nu(\mu, \Omega^{-1})$, if and only if Ω is d_μ-closed.

By the construction of section 6.3, the differential pre-Lie algebra-structure $\nu(\mu, \mathrm{P})$ on E* defines a $\Lambda(\mathrm{E}^*)$-valued K-bilinear mapping on $\Lambda(\mathrm{E}^*)$, the Schouten bracket, denoted by $[\ ,\ ]_\nu$. It is clear from the definitions that if $\mathrm{E} = \underline{\mathrm{TM}}$, where (M, P) is a Poisson manifold, this bracket on $\Lambda(\mathrm{E}^*)$ is the bracket introduced by Koszul in [23], p. 266, because both brackets coincide on forms of degree 0 or 1. (In [23], the Poisson bivector is denoted by w and the corresponding Schouten bracket on $\Lambda(\mathrm{E}^*)$ is denoted by $[\ ,\ ]_w$.)

Let us set

$$\partial_\nu = [i_\mathrm{P}, d_\mu],$$

where i_P is the interior product of forms with the bivector P. When P is a Poisson bivector, the K-linear operator ∂_ν on $\Lambda(\mathrm{E}^*)$ of degree -1, is the Poisson homology operator, denoted by Δ in [23] and by δ in [6]. Koszul's formula relates the dualized Schouten bracket $[\ ,\ ]_\nu$ on $\Lambda(\mathrm{E}^*)$ with the differential d_ν and the exterior product

$$[\alpha, \beta]_\nu = (-1)^{a+1} (\partial_\nu \alpha \wedge \beta + (-1)^a \alpha \wedge \partial_\nu \beta - \partial_\nu (\alpha \wedge \beta)),$$

for α in $\Lambda^a(\mathrm{E}^*)$, and β in $\Lambda(\mathrm{E}^*)$.

The following diagram summarizes the relations between the various structures, μ, d_μ and $[\ ,\ ]_\mu$, their deformations, $\mathrm{N}.\mu$, $d_{\mathrm{N}.\mu}$ and $[\ ,\ ]_{\mathrm{N}.\mu}$, studied in section 6.4, and their dualizations, $\nu=\nu(\mu, P)$, d_ν and $[\ ,\ ]_\nu$, studied in this section.

$$
\begin{array}{ccccc}
d_\nu=[P,\,.\,]_\mu & \leftarrow & \left\{\begin{array}{l}\nu=\nu(\mu,P)\\ \mathscr{L}^\nu_\alpha=\mathscr{L}^\mu_{P\alpha}\end{array}\right. & \rightarrow & [\alpha,\beta]_\nu=(-1)^{a+1}(\partial_\nu\alpha\wedge\beta+(-1)^a\alpha\wedge\partial_\nu\beta-\partial_\nu(\alpha\wedge\beta)) \\
\uparrow & & \uparrow & & \uparrow \\
d_\mu & \leftarrow & \mu,\mathscr{L}^\mu & \rightarrow & [\ ,\ ]_\mu \\
\downarrow & & \downarrow & & \downarrow \\
d_{N.\mu}=[i_N,d_\mu] & \leftarrow & \left\{\begin{array}{l}N.\mu=[\mu,N]\\ \mathscr{L}^{N.\mu}_x=\mathscr{L}^\mu_{Nx}\end{array}\right. & \rightarrow & [Q,Q']_{N.\mu}=[i_N Q,Q']_\mu+[Q,i_N Q']_\mu-i_N[Q,Q']_\mu
\end{array}
$$

6.6. Morphisms of differential Lie algebras

We now show that to each morphism between differential Lie algebras there corresponds a morphism between their graded differential algebras (with the arrow reversed) and a morphism between their Schouten algebras, thus bringing out the functoriality of the constructions of subsections 6.2 and 6.3. Let (E, μ) and (F, ν) be differential pre-Lie algebras over the K-algebra A. We say that a mapping, ρ, from (F, ν) to (E, μ) is a *morphism of differential pre-Lie algebras* if it is A-linear and if, for all y and y' in F, and for all f in A,

$$\rho[y, y']_\nu=[\rho y, \rho y']_\mu \tag{6.13}$$

and

$$\mathscr{L}^\nu_y f=\mathscr{L}^\mu_{\rho y} f. \tag{6.14}$$

(If the image of ρ contains an element which is not a torsion element, the second condition is a consequence of the first.) The same conditions define a *morphism of differential Lie algebras.*

For example, a direct consequence of formula (6.11) is the following

LEMMA 6.1. — *A bivector* P *defines a morphism* **P** *of differential pre-Lie algebras from* $(E^*, \nu(\mu, P))$ *to* (E, μ) *if and only if the Schouten bracket of* P *with respect to* μ *vanishes.*

Let $\Lambda^q({}^t\rho)$ denote the q-th exterior power of the transpose of ρ. (In particular, for $q=0$, one obtains the identity of A.) Then

PROPOSITION 6.6. — *An* A-*linear mapping* ρ *from* F *to* E *is a morphism of differential pre-Lie algebras from* (F, ν) *to* (E, μ) *if and only if* $\Lambda({}^t\rho)$ *is a morphism of degree* 0 *of pre-GDA's from* $(\Lambda(E^*), d_\mu)$ *to* $(\Lambda(F^*), d_\nu)$, i. e., *the following diagram is commutative for all* $q \geqq 0$,

$$\begin{array}{ccc} \Lambda^q(E^*) & \xrightarrow{\Lambda^q({}^t\rho)} & \Lambda^q(F^*) \\ {}^{d_\mu}\downarrow & & \downarrow{}^{d_\nu} \\ \Lambda^{q+1}(E^*) & \xrightarrow{\Lambda^{q+1}({}^t\rho)} & \Lambda^{q+1}(F^*) \end{array}$$

Proof. — In fact, for $q=0$, the commutativity of the diagram is equivalent to (6.14). For $q=1$, using (6.14), we obtain for α in E^*, y and y' in E,

$$\begin{aligned} \Lambda^2({}^t\rho)(d_\mu\alpha)(y, y') - d_\nu(({}^t\rho)\alpha)(y, y') &= \mathcal{L}^\mu_{\rho y}\langle \alpha, \rho y'\rangle - \mathcal{L}^\mu_{\rho y'}\langle \alpha, \rho y\rangle - \langle \alpha, [\rho y, \rho y']_\mu\rangle \\ &\quad -(\mathcal{L}^\nu_y\langle \alpha, \rho y'\rangle - \mathcal{L}^\nu_{y'}\langle \alpha, \rho y\rangle - \langle \alpha, \rho[y, y']_\nu\rangle) \\ &= \langle \alpha, \rho[y, y']_\nu - [\rho y, \rho y']_\mu\rangle. \end{aligned}$$

Therefore the diagram commutes for $q=1$ if and only if (6.13) is satisfied. The diagram commutes for all q's if and only if it commutes for $q=0$ and $q=1$ since A and E^* generate $\Lambda(E^*)$, and since d_μ and d_ν are derivations.

In the case where $F=E^*$, with $\nu=\nu(\mu, P)$, where P is a bivector on E, the preceding diagram becomes

$$\begin{array}{ccc} \Lambda^q(E^*) & \xrightarrow{(-1)^q\Lambda^q \mathbf{P}} & \Lambda^q E \\ {}^{d_\mu}\downarrow & & \downarrow{}^{d_\nu} \\ \Lambda^{q+1}(E^*) & \xrightarrow{(-1)^{q+1}\Lambda^{q+1} \mathbf{P}} & \Lambda^{q+1} E \end{array}$$

As a corollary of proposition 6.6 and lemma 6.1, we obtain

COROLLARY 6.2. — *Let* (E, μ) *be a differential Lie algebra. When* P *is a Poisson bivector on* (E, μ), *the extension of* $-\mathbf{P}$ *to the exterior algebra* $\Lambda(E^*)$, *whose restriction to* $\Lambda^q(E^*)$ *is* $(-1)^q\Lambda^q\mathbf{P}$, *intertwines the de Rham cohomology operator* d_μ *and the cohomology operator* $d_\nu=[\mathbf{P}, .]_\mu$,

$$(\Lambda^{q+1}\mathbf{P})(d_\mu\alpha) + d_\nu((\Lambda^q\mathbf{P})\alpha) = 0, \tag{6.15}$$

for α *in* $\Lambda^q(E^*)$.

This corollary is to be compared with the result of Koszul in [23], p. 266, and with that of Krasilshchik [24], p. 102. In [23] the restriction of γ_w to the 1-forms should be $i(\alpha)w$ and not $-i(\alpha)w$. The result in [24], taking into account the signs in the definition of the operator ∂_h and of the Schouten bracket, coincides with ours.

We now consider the morphism of Schouten pre-algebras associated with a morphism of differential pre-Lie algebras. To each morphism of differential pre-Lie algebras ρ from (F, ν) to (E, μ), there coresponds a morphism $\Lambda\rho$ of the Schouten pre-algebras

$$\Lambda\rho : (\Lambda F, [\ ,\ ]_\nu) \to (\Lambda E, [\ ,\ ]_\mu),$$

i. e., the following formula is valid

$$(\Lambda^{q+q'-1}\rho)[Q, Q']_\nu = [(\Lambda^q\rho)(Q), (\Lambda^{q'}\rho)(Q')]_\mu, \tag{6.16}$$

for Q in $\Lambda^q F$ and Q' in $\Lambda^{q'} F$. In fact, by (6.13) and (6.14), this formula is valid when the degrees of Q and Q' are 0 or 1.

In particular, if P is a Poisson bivector on a differential Lie algebra (E, μ), and if $\nu = \nu(\mu, P)$, then by lemma 6.1, $\mathbf{P} : (E^*, \nu) \to (E, \mu)$ is a morphism of differential Lie algebras and therefore $\Lambda\mathbf{P}$ is a morphism of graded Lie algebras from the Schouten algebra $(\Lambda(E^*), [\ ,\ ]_\nu)$ of (E^*, ν) to the Schouten algebra $(\Lambda E, [\ ,\ ]_\mu)$ of (E, μ). We have thus given a new proof of a result of Koszul (formula 3.3 of [23]).

6.7. Conclusion: Poisson-Nijenhuis structures on differential Lie algebras

In this section, we have set out a general theory of differential Lie algebras, which constitutes a unified framework for the study of the Lie brackets on both TM, the space of vector fields, and T*M, the space of differential 1-forms on a manifold. We have seen how the deformation of a differential Lie algebra (E, μ) by means of a Nijenhuis operator N gives rise to a deformed derivation, $d_{N.\mu} = [i_N, d_\mu]$, of $\Lambda(E^*)$, and its dualization by means of a Poisson bivector P gives rise to a dualized derivation, $d_\nu = [P, .]_\mu$, of ΛE, and we have also introduced and studied the associated Schouten algebras. The definition of the *Poisson-Nijenhuis structures* that was given in section 4 can be carried over unchanged to the case of differential Lie algebras. The proofs, in sections 4 and 5, of the properties of a Poisson-Nijenhuis structure (P, N) on a manifold and of the hierarchy of iterated Poisson structures P, NP, ..., $N^i P$, ... which it defines, are algebraic, and they apply *verbatim* to the case of a Poisson-Nijenhuis structure on a differential Lie algebra. Other proofs of these same properties can alternatively be obtained by studying, instead of the brackets $\nu(N.\mu, P)$, ${}^tN.\nu(\mu, P)$, $\nu\left(\mu, \frac{1}{2}(NP + P\,{}^tN)\right)$ or $\nu(\mu, NP + P\,{}^tN)$, their associated derivations. For example, let ${}^tN$ be considered as an E^*-valued 1-form on E^*, so $i_{{}^tN} P = NP + P\,{}^tN$. Then lemma 4.1 can be reformulated as

$$d_{\nu(N.\mu, P)} + d_{{}^tN.\nu(\mu, P)} = d_{\nu(\mu, i_{{}^tN}P)}.$$

To obtain a proof of this formula as a corollary of the general results of section 6, we first use proposition 6.4 to write $d_{\nu(N.\mu, P)} = [P, .]_{N.\mu}$, and we apply proposition 6.3 to this deformed Schouten bracket.

Then we use proposition 6.2 to write $d_{t_{\mathrm{N}} \,.\, \mathrm{v}(\mu, \mathrm{P})} = [i_{t_{\mathrm{N}}}, d_{\mathrm{v}(\mu, \mathrm{P})}]$ and we use proposition 6.4 to write $d_{\mathrm{v}(\mu, \mathrm{P})} = [\mathrm{P}, \,.\,]_\mu$. When we add both expressions, we obtain the result. Similarly, the compatibility condition, that was introduced in definition 4.1, can be interpreted in terms of derivations on $\Lambda \mathrm{E}$.

Examples of Poisson-Nijenhuis structures on Lie groups and on the dual of a Lie algebra can be found in [31 *a*]. There are open problems concerning the classification under equivalence relations of the Poisson-Nijenhuis structures on manifolds.

The properties of the Poisson-Nijenhuis structures that were established in this article constitute an appropriate framework for the study of integrable Hamiltonian systems because a special case of these structures, the linear Poisson-Nijenhuis structures, can be constructed on the duals of associative Lie algebras by means of solutions of the modified Yang-Baxter equation [21 *a*]. Such linear Poisson-Nijenhuis structures on the dual of a Lie algebra give rise to quadratic Poisson structures and most of the familiar integrable systems can be regarded as reductions of systems which are bi-hamiltonian with respect to such a pair of a linear and a quadratic Poisson structure.

Note added in proof.

We would like to thank S. Sternberg for calling to our attention the article of M. Gerstenhaber, The Cohomology structure of an associative ring, *Ann. Math.*, **78**, 1963, pp. 267-288, which states the axioms of what we have called a *Schouten algebra* (*see* section 6.3, *supra*), and for showing us his unpublished manuscript with B. Kostant, "Anti-Poisson algebras and current algebras", which contains a definition of the differential Lie algebras, which they call (L, M)-systems, explains their relationship with the Schouten algebras, which they call $\mathbb{Z}$-graded Poisson algebras, and provides applications to the commutation relations in various algebras of currents, results obtained and announced *circa* 1970.

A contravariant definition of the symplectic manifolds had already been advocated by R. Jost in 1964. The intrinsic definition of the Schouten bracket of multivectors was introduced by W. M. Tulczyjew, The graded Lie algebra of multivector fields and the generalized Lie derivative of forms, *Bull. Acad. Pol. Sci.*, T. **22**, 1974, pp. 937-942, which is anterior to the sources that we cited, as a preliminary to his "Poisson brackets and canonical manifolds", *ibid.*, pp. 931-935, where the Schouten bracket of the Poisson bivector is explicitly considered. Tulczyjew asserts in his introduction that "the duality between forms and multivector fields is incomplete without some differential operation in the algebra of multivectors dual in a sense to the exterior differential of forms" and he states that "such [a] differential operation" is furnished by the Schouten differential concomitants. Proposition 6.4 and the other results of section 6.5 *supra* define that "differential operation" and prove his claim.

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