Quantified Multimodal Logics in Simple Type Theory

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Abstract. We present an embedding of quantified multimodal logics into simple type theory and prove its soundness and completeness. A correspondence between $QK\pi$ models for quantified multimodal logics and Henkin models is established and exploited.

Our embedding supports the application of off-the-shelf higher-order theorem provers for reasoning within and about quantified multimodal logics. Moreover, it provides a starting point for further logic embeddings and their combinations in simple type theory.

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1. Motivation

There are two approaches to automate reasoning in modal logics. The *direct* approach develops specific calculi and tools for the task; the *translational* approach transforms modal logic formulas into first-order logic and applies standard first-order tools.[1]

In previous work [10, 7, 11] we have extended the translational approach, presenting a sound and complete embedding of propositional multimodal logics into simple type theory (higher-order logic). In this paper we extend this work to quantified multimodal logics.

Multimodal logics with quantification for propositional variables have been studied by others before, including Kripke [26], Bull [14], Fine [16, 17], Kaplan [24], and Kremer [25]. Also first-order modal logics [20, 22] have

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[1] An overview on tools and provers for both the direct and the translational approach is available at `http://www.cs.man.ac.uk/~schmidt/tools/`.

been studied in numerous publications. We are interested here in multimodal logics with quantification over both propositional and first-order variables, a combination investigated, for example, by Fitting [18]. In contrast to Fitting we here pursue the translational approach. Moreover, Fitting has only studied the particular case of quantified monomodal logic S5, while we are interested in multimodal K structures.

Our approach has several advantages:

- The syntax and semantics of simple type theory are well understood [21, 2, 1, 9]. Studying (quantified) multimodal logics as fragments of simple type theory can thus help to better understand semantical issues.
- For simple type theory, various automated proof tools are available, including Isabelle/HOL [29], HOL [30], LEO-II [12], and TPS [5]. Employing the transformation presented in this paper, these systems become immediately applicable to quantified multimodal logics or fragments of them.
- The embedding studied in this article provides a fruitful basis for further logic embeddings and logic combinations in simple type theory [8]. Moreover, even meta properties of embedded logics and combinations of logics can be formalized and automatically analyzed with the above provers. In fact we conjecture that our approach will perform comparably better on the meta level.
- The systematic study of embeddings of multimodal logics in simple type theory can identify fragments of simple type theory that have interesting computational properties. This can foster improvements to proof tactics in interactive proof assistants.

Our paper is organized as follows. In Section 2 we briefly review simple type theory and adapt Fitting's [18] notion of quantified multimodal logics. In Section 3 we extend our previous work [10, 7, 11] and present an embedding of quantified multimodal logic in simple type theory. This embedding is shown sound and complete in Section 4.

2. Preliminaries

2.1. Simple Type Theory

Classical higher-order logic or *simple type theory* STT [3, 15] is built on top of the simply typed λ-calculus. The set T of simple types is usually freely generated from a set of basic types $\{o, \iota\}$ (where o is the type of Booleans and ι is the type of individuals) using the function type constructor $\rightarrow$. Instead of $\{o, \iota\}$ we here consider a set of base types $\{o, \iota, \mu\}$, providing an additional base type μ (the type of possible worlds).

The simple type theory language STT is defined by (α, $\beta \in \mathrm{T}$):

$$\begin{aligned} s,t \quad ::= \quad & p_\alpha \mid X_\alpha \mid (\lambda X_\alpha \text{.} s_\beta)_{\alpha\rightarrow\beta} \mid (s_{\alpha\rightarrow\beta}\, t_\alpha)_\beta \mid (\neg_{o\rightarrow o}\, s_o)_o \mid \\ & (s_o \vee_{o\rightarrow o\rightarrow o} t_o)_o \mid (s_\alpha =_{\alpha\rightarrow\alpha\rightarrow o} t_\alpha)_o \mid (\Pi_{(\alpha\rightarrow o)\rightarrow o}\, s_{\alpha\rightarrow o})_o \end{aligned}$$

p_α denotes typed constants and X_α typed variables (distinct from p_α). Complex typed terms are constructed via abstraction and application. Our logical connectives of choice are $\neg_{o\to o}$, $\vee_{o\to o\to o}$, $=_{\alpha\to\alpha\to o}$ and $\Pi_{(\alpha\to o)\to o}$ (for each type α). From these connectives, other logical connectives can be defined in the usual way. We often use binder notation $\forall X_\alpha . s$ for $\Pi_{(\alpha\to o)\to o}(\lambda X_\alpha . s_o)$. We denote *substitution* of a term A_α for a variable X_α in a term B_β by $[A/X]B$. Since we consider α-conversion implicitly, we assume the bound variables of B avoid variable capture. Two common relations on terms are given by β-reduction and η-reduction. A β-redex has the form $(\lambda X . s)t$ and β-reduces to $[t/X]s$. An η-redex has the form $(\lambda X . sX)$ where variable X is not free in s; it η-reduces to s. We write $s =_\beta t$ to mean s can be converted to t by a series of β-reductions and expansions. Similarly, $s =_{\beta\eta} t$ means s can be converted to t using both β and η. For each $s \in L$ there is a unique *β-normal form* and a unique *$\beta\eta$-normal form.*

The semantics of STT is well understood and thoroughly documented in the literature [21, 1, 2, 9]; our summary below is adapted from Andrews [4].

A *frame* is a collection $\{D_\alpha\}_{\alpha\in\mathrm{T}}$ of nonempty sets D_α, such that $D_o = \{T, F\}$ (for truth and falsehood). The $D_{\alpha\to\beta}$ are collections of functions mapping D_α into D_β. The members of D_ι are called *individuals*. An *interpretation* is a tuple $\langle\{D_\alpha\}_{\alpha\in\mathrm{T}}, I\rangle$ where function I maps each typed constant c_α to an appropriate element of D_α, which is called the *denotation* of c_α (the logical symbols $\neg_{o\to o}$, $\vee_{o\to o\to o}$, $\Pi_{(\alpha\to o)\to o}$, and $=_{\alpha\to\alpha\to o}$ are always given the standard denotations). A *variable assignment* ϕ maps variables X_α to elements in D_α. An interpretation $\langle\{D_\alpha\}_{\alpha\in\mathrm{T}}, I\rangle$ is a *Henkin model* (equivalently, a *general model*) if and only if there is a binary function V such that $\mathrm{V}_\phi\, s_\alpha \in D_\alpha$ for each variable assignment ϕ and term $s_\alpha \in L$, and the following conditions are satisfied for all ϕ and all $s, t \in L$: (a) $\mathrm{V}_\phi X_\alpha = \phi X_\alpha$, (b) $\mathrm{V}_\phi\, p_\alpha = I p_\alpha$, (c) $\mathrm{V}_\phi(s_{\alpha\to\beta}\, t_\alpha) = (\mathrm{V}_\phi\, s_{\alpha\to\beta})(\mathrm{V}_\phi t_\alpha)$, and (d) $\mathrm{V}_\phi(\lambda X_\alpha . s_\beta)$ is that function from D_α into D_β whose value for each argument $z \in D_\alpha$ is $\mathrm{V}_{[z/X_\alpha]\phi} s_\beta$, where $[z/X_\alpha]\phi$ is that variable assignment such that $([z/X_\alpha]\phi)X_\alpha = z$ and $([z/X_\alpha]\phi)Y_\beta = \phi Y_\beta$ if $Y_\beta \neq X_\alpha$. (Since $I\neg$, $I\vee$, $I\Pi$, and $I{=}$ always denote the standard truth functions, we have $\mathrm{V}_\phi\,(\neg s) = T$ if and only if $\mathrm{V}_\phi\, s = F$, $\mathrm{V}_\phi\,(s \vee t) = T$ if and only if $\mathrm{V}_\phi\, s = T$ or $\mathrm{V}_\phi\, t = T$, $\mathrm{V}_\phi\,(\forall X_\alpha . s_o) = \mathrm{V}_\phi\,(\Pi^\alpha(\lambda X_\alpha . s_o)) = T$ if and only if for all $z \in D_\alpha$ we have $\mathrm{V}_{[z/X_\alpha]\phi}\, s_o = T$, and $\mathrm{V}_\phi\,(s = t) = T$ if and only if $\mathrm{V}_\phi\, s = \mathrm{V}_\phi\, t$. Moreover, we have $\mathrm{V}_\phi\, s = \mathrm{V}_\phi\, t$ whenever $s =_{\beta\eta} t$.) It is easy to verify that Henkin models obey the rule that everything denotes, that is, each term t_α always has a denotation $\mathrm{V}_\phi\, t_\alpha \in D_\alpha$. If an interpretation $\langle\{D_\alpha\}_{\alpha\in\mathrm{T}}, I\rangle$ is a Henkin model, then the function V_ϕ is uniquely determined.

We say that formula $A \in L$ is *valid in a model* $\langle\{D_\alpha\}_{\alpha\in\mathrm{T}}, I\rangle$ if and only if $\mathrm{V}_\phi A = T$ for every variable assignment ϕ. A model for a set of formulas H is a model in which each formula of H is valid. A formula A is Henkin-valid if and only if A is valid in every Henkin model. We write $\models^{\mathrm{STT}} A$ if A is Henkin-valid.

2.2. Quantified Multimodal Logic

First-order quantification can be constant domain or varying domain. Below we only consider the constant domain case: every possible world has the same domain. We adapt the presentation of syntax and semantics of quantified modal logic from Fitting [18]. In contrast to Fitting we are not interested in S5 structures but in the more general case of K.

Let IV be a set of first-order (individual) variables, PV a set of propositional variables, and SYM a set of predicate symbols of any arity. Like Fitting, we keep our definitions simple by not having function or constant symbols; our language has no terms other than variables. While Fitting [18] studies quantified monomodal logic, we are interested in quantified multimodal logic. Hence, we introduce multiple $\Box_r$ operators for symbols r from an index set S. The grammar for our quantified multimodal logic QML is thus

$$s, t \quad ::= \quad P \mid k(X^1, \ldots, X^n) \mid \neg\, s \mid s \vee t \mid \forall X_\blacksquare s \mid \forall P_\blacksquare s \mid \Box_r\, s$$

where $P \in \mathrm{PV}$, $k \in \mathrm{SYM}$, and $X, X^i \in \mathrm{IV}$.

Further connectives, quantifiers, and modal operators can be defined as usual. We also obey the usual definitions of free variable occurrences and substitutions.

Fitting introduces three different notions of semantics: $\mathrm{QS5}\pi^-$, $\mathrm{QS5}\pi$, and $\mathrm{QS5}\pi^+$. We study related notions $\mathrm{QK}\pi^-$, $\mathrm{QK}\pi$, and $\mathrm{QK}\pi^+$ for a modal context K, and we support multiple modalities.

A *$QK\pi^-$ model* is a structure $M = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ such that $(W, (R_r)_{r\in S})$ is a multimodal frame (that is, W is the set of possible worlds and the R_r are accessibility relations between worlds in W), D is a non-empty set (the first-order domain), P is a non-empty collection of subsets of W (the propositional domain), and the I_w are interpretation functions mapping each n-place relation symbol $k \in \mathrm{SYM}$ to some n-place relation on D in world w.

A *variable assignment* $g = (g^{iv}, g^{pv})$ is a pair of maps $g^{iv} : \mathrm{IV} \longrightarrow D$ and $g^{pv} : \mathrm{PV} \longrightarrow P$, where g^{iv} maps each individual variable in IV to a an object in D and g^{pv} maps each propositional variable in PV to a set of worlds in P.

Validity of a formula s for a model $M = (W, (R_r)_{r\in S}, D, P, I_w)$, a world $w \in W$, and a variable assignment $g = (g^{iv}, g^{pv})$ is denoted as $M, g, w \models s$ and defined as follows, where $[a/Z]g$ denotes the assignment identical to g except that $([a/Z]g)(Z) = a$:

$M, g, w \models k(X^1, \ldots, X^n)$	if and only if	$\langle g^{iv}(X^1), \ldots, g^{iv}(X^n)\rangle \in I_w(k)$
$M, g, w \models P$	if and only if	$w \in g^{pv}(P)$
$M, g, w \models \neg\, s$	if and only if	$M, g, w \not\models s$
$M, g, w \models s \vee t$	if and only if	$M, g, w \models s$ or $M, g, w \models t$
$M, g, w \models \forall X_\blacksquare\, s$	if and only if	$M, ([d/X]g^{iv}, g^{pv}), w \models s$ for all $d \in D$

$M, g, w \models \forall Q_\bullet\, s$	if and only if	$M, (g^{iv}, [p/Q]g^{pv}), w \models s$ for all $p \in P$
$M, g, w \models \Box_r\, s$	if and only if	$M, g, v \models s$ for all $v \in W$ with $\langle w, v\rangle \in R_r$

A QKπ^- model $M = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ is a *QKπ model* if for every variable assignment g and every formula $s \in$ QML, the set of worlds $\{w \in W \mid M, g, w \models s\}$ is a member of P.

A QKπ model $M = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ is a *QKπ^+ model* if every world $w \in W$ is member of an atom in P. The *atoms* of P are minimal non-empty elements of P: no proper subsets of an atom are also elements of P.

A QML formula s is *valid in model M for world w* if $M, g, w \models s$ for all variable assignments g. A formula s is *valid in model M* if $M, g, w \models s$ for all g and w. Formula s is *QKπ-valid* if s is valid in all QKπ models, when we write $\models^{\mathrm{QK}\pi} s$; we define QK$\pi^-$-valid and QK$\pi^+$-valid analogously.

In the remainder we mainly focus on QKπ models. These models naturally correspond to Henkin models, as we shall see in Section 4.

3. Embedding Quantified Multimodal Logic in STT

The idea of the encoding is simple. We choose type ι to denote the (non-empty) set of individuals and we reserve a second base type μ to denote the (non-empty) set of possible worlds. The type o denotes the set of truth values. Certain formulas of type $\mu \to o$ then correspond to multimodal logic expressions. The multimodal connectives $\dot\neg$, $\dot\vee$, and $\dot\Box$, become λ-terms of types $(\mu \to o) \to (\mu \to o)$, $(\mu \to o) \to (\mu \to o) \to (\mu \to o)$, and $(\mu \to \mu \to o) \to (\mu \to o) \to (\mu \to o)$ respectively.

Quantification is handled as usual in higher-order logic by modeling $\forall X_\bullet s$ as $\Pi(\lambda X_\bullet s)$ for a suitably chosen connective Π, as we remarked in Section 2. Here we are interested in defining two particular modal Π-connectives: $\mathbf{\Pi}^\iota$, for quantification over individual variables, and $\mathbf{\Pi}^{\mu\to o}$, for quantification over modal propositional variables that depend on worlds, of types $(\iota \to (\mu \to o)) \to (\mu \to o)$ and $((\mu \to o) \to (\mu \to o)) \to (\mu \to o)$, respectively.

In previous work [10] we have discussed first-order and higher-order modal logic, including a means of explicitly excluding terms of certain types. The idea was that no proper subterm of $t_{\mu\to o}$ should introduce a dependency on worlds. Here we skip this restriction. This leads to a simpler definition of a quantified multimodal language QMLSTT below, and it does not affect our soundness and completeness results.

Definition 3.1 (Modal operators). The modal operators $\dot\neg$, $\dot\vee$, $\dot\Box$, $\mathbf{\Pi}^\iota$, and $\mathbf{\Pi}^{\mu\to o}$ are defined as follows:

$$\dot\neg_{(\mu\to o)\to(\mu\to o)} = \lambda\phi_{\mu\to o\bullet}\lambda W_{\mu\bullet}\neg(\phi\, W)$$
$$\dot\vee_{(\mu\to o)\to(\mu\to o)\to(\mu\to o)} = \lambda\phi_{\mu\to o\bullet}\lambda\psi_{\mu\to o\bullet}\lambda W_{\mu\bullet}\,\phi\, W \vee \psi\, W$$

$$\begin{aligned}
\Box_{(\mu\to\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda R_{\mu\to\mu\to o}.\lambda\phi_{\mu\to o}.\lambda W_{\mu}.\forall V_{\mu}.\neg(R\,W\,V) \vee \phi\,V \\
\mathbf{\Pi}^{\iota}_{(\iota\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{\iota\to(\mu\to o)}.\lambda W_{\mu}.\forall X_{\iota}.\phi\,X\,W \\
\mathbf{\Pi}^{\mu\to o}_{((\mu\to o)\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{(\mu\to o)\to(\mu\to o)}.\lambda W_{\mu}.\forall P_{\mu\to o}.\phi\,P\,W
\end{aligned}$$

Note that our encoding actually only employs the second-order fragment of simple type theory enhanced with lambda-notation.

Further operators can be introduced, for example,

$$\begin{aligned}
\top_{(\mu\to o)\to(\mu\to o)} &= \forall P_{\mu\to o}.P \vee \neg P \\
\bot_{(\mu\to o)\to(\mu\to o)} &= \neg\top \\
\wedge_{(\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}.\lambda\psi_{\mu\to o}.\neg(\neg\phi \vee \neg\psi) \\
\supset_{(\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}.\lambda\psi_{\mu\to o}.\neg\phi \vee \psi \\
\Diamond_{(\mu\to\mu\to o)\to(\mu\to o)\to(\mu\to o)} &= \lambda R_{\mu\to\mu\to o}.\lambda\phi_{\mu\to o}.\neg(\Box\,R\,(\neg\phi)) \\
\mathbf{\Sigma}^{\iota}_{(\iota\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{\iota\to(\mu\to o)}.\neg(\mathbf{\Pi}^{\iota}(\lambda X_{\iota}.\neg(\phi\,X))) \\
\mathbf{\Sigma}^{\mu\to o}_{((\mu\to o)\to(\mu\to o))\to(\mu\to o)} &= \lambda\phi_{(\mu\to o)\to(\mu\to o)}.\neg(\mathbf{\Pi}^{\mu\to o}(\lambda P_{\mu\to o}.\neg(\phi\,P)))
\end{aligned}$$

We could also introduce further modal operators, such as the difference modality D, the global modality E, nominals with !, or the @ operator (cf. the recent work of Kaminski and Smolka [23] in the propositional hybrid logic context):

$$\begin{aligned}
D_{(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}.\lambda W_{\mu}.\exists V_{\mu}.W \neq V \wedge \phi V \\
E_{(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}.\phi \vee D\,\phi \\
!_{(\mu\to o)\to(\mu\to o)} &= \lambda\phi_{\mu\to o}.E\,(\phi \wedge \neg(D\,\phi)) \\
@_{\mu\to(\mu\to o)\to(\mu\to o)} &= \lambda W_{\mu}.\lambda\phi_{\mu\to o}.\phi\,W
\end{aligned}$$

For defining QMLSTT-propositions we fix a set IVSTT of individual variables of type ι, a set PVSTT of propositional variables of type $\mu\to o$, and a set SYMSTT of n-ary (curried) predicate constants of types $\underbrace{\iota\to\ldots\to\iota}_{n}\to(\mu\to o)$. The latter types will be abbreviated as $\iota^n\to(\mu\to o)$ in the remainder. Moreover, we fix a set SSTT of accessibility relation constants of type $\mu\to\mu\to o$.

Definition 3.2 (QMLSTT-propositions). QMLSTT-propositions are defined as the smallest set of simply typed λ-terms for which the following hold:

- Each variable $P_{\mu\to o}\in$ PVSTT is an atomic QMLSTT-proposition, and if $X^j_{\iota}\in$ IVSTT (for $j=1,\ldots,n$) and $k_{\iota^n\to(\mu\to o)}\in$ SYMSTT, then the term $(k\,X^1\ldots X^n)_{\mu\to o}$ is an atomic QMLSTT-proposition.
- If ϕ and ψ are QMLSTT-propositions, then so are $\neg\phi$ and $\phi\vee\psi$.
- If $r_{\mu\to\mu\to o}\in$ SSTT is an accessibility relation constant and if ϕ is an QMLSTT-proposition, then $\Box\,r\,\phi$ is a QMLSTT-proposition.
- If $X_{\iota}\in$ IVSTT is an individual variable and ϕ is a QMLSTT-proposition then $\mathbf{\Pi}^{\iota}(\lambda X_{\iota}.\phi)$ is a QMLSTT-proposition.
- If $P_{\mu\to o}\in$ PVSTT is a propositional variable and ϕ is a QMLSTT-proposition then $\mathbf{\Pi}^{\mu\to o}(\lambda P_{\mu\to o}.\phi)$ is a QMLSTT-proposition.

We write $\Box_r \phi$, $\forall X_\iota . \phi$, and $\forall P_{\mu\to o} . \phi$ for $\Box\, r\, \phi$, $\mathbf{\Pi}^\iota(\lambda X_\iota . \phi)$, and $\mathbf{\Pi}^{\mu\to o}(\lambda P_{\mu\to o} . \phi)$, respectively.

Because the defining equations in Definition 3.1 are themselves formulas in simple type theory, we can express proof problems in a higher-order theorem prover elegantly in the syntax of quantified multimodal logic. Using rewriting or definition expanding, we can reduce these representations to corresponding statements containing only the basic connectives $\neg$, $\vee$, $=$, Π^ι, and $\Pi^{\mu\to o}$ of simple type theory.

Example. The following QMLSTT proof problem expresses that in all accessible worlds there exists truth:

$$\Box_r\, \exists P_{\mu\to o} .\, P$$

The term rewrites into the following $\beta\eta$-normal term of type $\mu \to o$

$$\lambda W_\mu . \forall Y_\mu . \neg(r\, W\, Y) \vee (\neg \forall P_{\mu\to o} . \neg(P\, Y))$$

Next, we define validity of QMLSTT propositions $\phi_{\mu\to o}$ in the obvious way: a QML-proposition $\phi_{\mu\to o}$ is valid if and only if for all possible worlds w_μ we have $w_\mu \in \phi_{\mu\to o}$, that is, if and only if $\phi_{\mu\to o}\, w_\mu$ holds.

Definition 3.3 (Validity). Validity is modeled as an abbreviation for the following simply typed λ-term:

$$\text{valid} = \lambda \phi_{\mu\to o} . \forall W_\mu . \phi\, W$$

Alternatively, we could define validity simply as $\Pi_{(\mu\to o)\to o}$.

Example. We analyze whether the proposition $\Box_r\, \exists P_{\mu\to o} .\, P$ is valid or not. For this, we formalize the following proof problem

$$\text{valid } (\Box_r\, \exists P_{\mu\to o} .\, P)$$

Expanding this term leads to

$$\forall W_\mu . \forall Y_\mu . \neg(r\, W\, Y) \vee (\neg \forall X_{\mu\to o} . \neg(X\, Y))$$

It is easy to check that this term is valid in Henkin semantics: put $X = \lambda Y_\mu . \top$.

An obvious question is whether the notion of quantified multimodal logics we obtain via this embedding indeed exhibits the desired properties. In the next section, we prove soundness and completeness for a mapping of QML-propositions to QMLSTT-propositions.

4. Soundness and Completeness of the Embedding

In our soundness proof, we exploit the following mapping of QKπ models into Henkin models. We assume that the QML logic L under consideration is constructed as outlined in Section 2 from a set of individual variables IV, a set of propositional variables PV, and a set of predicate symbols SYM. Let $\Box_{r^1}, \ldots, \Box_{r^n}$ for $r^i \in S$ be the box operators of L.

Definition 4.1 (QMLSTT logic $L^{\mathbf{STT}}$ for QML logic L). Given an QML logic L, define a mapping $\dot{\_}$ as follows:

$$\dot{X} = X_\iota \text{ for every } X \in \text{IV}$$
$$\dot{P} = P_{\mu\to o} \text{ for every } P \in \text{PV}$$
$$\dot{k} = k_{\iota^n \to (\mu\to o)} \text{ for every n-ary } k \in \text{SYM}$$
$$\dot{r} = r_{\mu\to\mu\to o} \text{ for every } r \in S$$

The QMLSTT logic L^{STT} is obtained from L by applying Def. 3.2 with IVSTT $= \{\dot{X} \mid X \in \text{IV}\}$, PVSTT $= \{\dot{P} \mid P \in \text{PV}\}$, SYMSTT $= \{\dot{k} \mid k \in \text{SYM}\}$, and SSTT $= \{\dot{r} \mid r \in S\}$. Our construction obviously induces a one-to-one correspondence $\dot{\_}$ between languages L and L^{STT}.

Moreover, let $g = (g^{iv} : \text{IV} \longrightarrow D, g^{pv} : \text{PV} \longrightarrow P)$ be a variable assignment for L. We define the corresponding variable assignment

$$\dot{g} = (\dot{g}^{iv} : \text{IVSTT} \longrightarrow D = D_\iota,\ \dot{g}^{pv} : \text{PVSTT} \longrightarrow P = D_{\mu\to o})$$

for L^{STT} so that $\dot{g}(X_\iota) = \dot{g}(\dot{X}) = g(X)$ and $\dot{g}(P_{\mu\to o}) = \dot{g}(\dot{P}) = g(P)$ for all $X_\iota \in$ IVSTT and $P_{\mu\to o} \in$ PVSTT.

Finally, a variable assignment $\dot{g}$ is lifted to an assignment for variables Z_α of arbitrary type by choosing $\dot{g}(Z_\alpha) = d \in D_\alpha$ arbitrarily, if $\alpha \neq \iota, \mu \to o$.

We assume below that L, L^{STT}, g and $\dot{g}$ are defined as above.

Definition 4.2 (Henkin model H^Q for QKπ model Q). Given a QKπ model $Q = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ for L, a Henkin model $H^Q = \langle \{D_\alpha\}_{\alpha\in\text{T}}, I\rangle$ for L^{STT} is constructed as follows. We choose

- the set D_μ as the set of possible worlds W,
- the set D_ι as the set of individuals D (cf. definition of $\dot{g}^{iv}$),
- the set $D_{\mu\to o}$ as the set of sets of possible worlds P (cf. definition of $\dot{g}^{pv}$),[2]
- the set $D_{\mu\to\mu\to o}$ as the set of relations $(R_r)_{r\in S}$,
- and all other sets $D_{\alpha\to\beta}$ as (not necessarily full) sets of functions from D_α to D_β; for all sets $D_{\alpha\to\beta}$ the rule that everything denotes must be obeyed, in particular, we require that the sets $D_{\iota^n\to(\mu\to o)}$ contain the elements $Ik_{\iota^n\to(\mu\to o)}$ as characterized below.

The interpretation I is as follows:

- Let $k_{\iota^n\to(\mu\to o)} = \dot{k}$ for $k \in$ SYM and let $X^i_\iota = \dot{X}^i$ for $X^i \in$ IV. We choose $Ik_{\iota^n\to(\mu\to o)} \in D_{\iota^n\to(\mu\to o)}$ such that

  $$(I\, k)(\dot{g}(X^1_\iota), \ldots, \dot{g}(X^n_\iota), w) = T$$

  for all worlds $w \in D_\mu$ such that $Q, g, w \models k(X^1, \ldots, X^n)$, that is, if $\langle g(X^1), \ldots, g(X^n)\rangle \in I_w(k)$. Otherwise $(I\, k)(\dot{g}(X^1_\iota), \ldots, \dot{g}(X^n_\iota), w) = F$.

[2] To keep things simple, we identify sets with their characteristic functions.

- Let $r_{\mu\to\mu\to o} = \dot{r}$ for $r \in S$. We choose $Ir_{\mu\to\mu\to o} \in D_{\mu\to\mu\to o}$ such that $(Ir_{\mu\to\mu\to o})(w,w') = T$ if $\langle w,w'\rangle \in R_r$ in Q and $(Ir_{\mu\to\mu\to o})(w,w') = F$ otherwise.

  It is not hard to verify that $H^Q = \langle\{D_\alpha\}_{\alpha\in \mathrm{T}}, I\rangle$ is a Henkin model.

Lemma 4.3. *Let $Q = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ be a QKπ model and let $H^Q = \langle\{D_\alpha\}_{\alpha\in T}, I\rangle$ be a Henkin model for Q. Furthermore, let $s_{\mu\to o} = \dot{s}$ for $s \in L$. Then for all worlds $w \in W$ and variable assignments g we have $Q, g, w \models s$ in Q if and only if $\mathrm{V}_{[w/W_\mu]\dot{g}}(s_{\mu\to o} W_\mu) = T$ in H^Q.*

Proof. The proof is by induction on the structure of $s \in L$.

Let $s = P$ for $P \in \mathrm{PV}$. By construction of Henkin model H^Q and by definition of $\dot{g}$, we have for $P_{\mu\to o} = \dot{P}$ that $\mathrm{V}_{[w/W_\mu]\dot{g}}(P_{\mu\to o}\, W_\mu) = \dot{g}(P_{\mu\to o})(w) = T$ if and only if $Q, g, w \models P$, that is, $w \in g(P)$.

Let $s = k(X^1, \ldots, X^n)$ for $k \in \mathrm{SYM}$ and $X^i \in \mathrm{IV}$. By construction of Henkin model H^Q and by definition of $\dot{g}$, we have for $\dot{k}(\dot{X}^1, \ldots, \dot{X}^n) = (k_{\iota^n\to(\mu\to o)}\, X^1_\iota \ldots X^n_\iota)$ that

$$\mathrm{V}_{[w/W_\mu]\dot{g}}((k_{\iota^n\to(\mu\to o)} X^1_\iota \ldots X^n_\iota)\, W_\mu) = (I\,k)(\dot{g}(X^1_\iota), \ldots, \dot{g}(X^n_\iota), w) = T$$

if and only if $Q, g, w \models k(X^1, \ldots, X^n)$, that is, $\langle g(X^1), \ldots, g(X^n)\rangle \in I_w(k)$.

Let $s = \neg t$ for $t \in L$. We have $Q, g, w \models \neg s$ if and only $Q, g, w \not\models s$, which is equivalent by induction to $\mathrm{V}_{[w/W_\mu]\dot{g}}(t_{\mu\to o}\, W_\mu) = F$ and hence to $\mathrm{V}_{[w/W_\mu]\dot{g}}\, \neg(t_{\mu\to o}\, W_\mu) =_{\beta\eta} \mathrm{V}_{[w/W_\mu]\dot{g}}((\neg\, t_{\mu\to o})\, W_\mu) = T$.

Let $s = (t \vee l)$ for $t, l \in L$. We have $Q, g, w \models (t \vee l)$ if and only if $Q, g, w \models t$ or $Q, g, w \models l$. The latter condition is equivalent by induction to $\mathrm{V}_{[w/W_\mu]\dot{g}}(t_{\mu\to o}\; W_\mu) = T$ or $\mathrm{V}_{[w/W_\mu]\dot{g}}(l_{\mu\to o}\; W_\mu) = T$ and therefore to $\mathrm{V}_{[w/W_\mu]\dot{g}}(t_{\mu\to o}\; W_\mu) \vee (l_{\mu\to o}\, W_\mu) =_{\beta\eta} \mathrm{V}_{[w/W_\mu]\dot{g}}(t_{\mu\to o}\; \vee\, l_{\mu\to o}\; W_\mu) = T$.

Let $s = \Box_r\, t$ for $t \in L$. We have $Q, g, w \models \Box_r\, t$ if and only if for all u with $\langle w, u\rangle \in R_r$ we have $Q, g, u \models t$. The latter condition is equivalent by induction to this one: for all u with $\langle w, u\rangle \in R_r$ we have $\mathrm{V}_{[u/V_\mu]\dot{g}}(t_{\mu\to o}\, V_\mu) = T$. That is equivalent to

$$\mathrm{V}_{[u/V_\mu][w/W_\mu]\dot{g}}(\neg(r_{\mu\to\mu\to o}\, W_\mu\, V_\mu) \vee (t_{\mu\to o}\, V_\mu)) = T$$

and thus to

$$\mathrm{V}_{[w/W_\mu]\dot{g}}(\forall Y_\mu\mathbf{.}(\neg(r_{\mu\to\mu\to o}\, W_\mu\, Y_\mu) \vee (t_{\mu\to o}\, Y_\mu))) =_{\beta\eta} \mathrm{V}_{[w/W_\mu]\dot{g}}(\Box_r\, t\, W_\mu) = T.$$

Let $s = \forall X\mathbf{.}\, t$ for $t \in L$ and $X \in \mathrm{IV}$. We have $Q, g, w \models \forall X\mathbf{.}\, t$ if and only if $Q, [d/X]g, w \models t$ for all $d \in D$. The latter condition is equivalent by induction to $\mathrm{V}_{[d/X_\iota][w/W_\mu]\dot{g}}(t_{\mu\to o}\, W_\mu) = T$ for all $d \in D_\iota$. That condition is equivalent to

$$\mathrm{V}_{[w/W_\mu]\dot{g}}(\Pi^\iota_{(\iota\to o)\to o}(\lambda X_\iota\mathbf{.} t_{\mu\to o}\, W_\mu)) =_{\beta\eta}$$
$$\mathrm{V}_{[w/W_\mu]\dot{g}}((\lambda V_\mu\mathbf{.}\, (\Pi^\iota_{(\iota\to o)\to o}\, (\lambda X_\iota\mathbf{.} t_{\mu\to o}\, V_\mu)))\, W_\mu) = T$$

and so by definition of $\mathbf{\Pi}^\iota$ to $\mathrm{V}_{[w/W_\mu]\dot{g}}((\mathbf{\Pi}^\iota_{(\iota\to(\mu\to o))\to(\mu\to o)}\, (\lambda X_\iota\mathbf{.} t_{\mu\to o}))\, W_\mu) = \mathrm{V}_{[w/W_\mu]\dot{g}}((\forall X_\iota\mathbf{.}\, t_{\mu\to o})\, W_\mu) = T$.

The case for $s = \forall P\mathbf{.}\, t$ where $t \in L$ and $P \in \mathrm{PV}$ is analogous to $s = \forall X\mathbf{.}\, t$. ☐

We exploit this result to prove the soundness of our embedding.

Theorem 4.4 (Soundness for QKπ semantics). *Let $s \in L$ be a QML proposition and let $s_{\mu\to o} = \dot{s}$ be the corresponding QMLSTT proposition. If $\models^{STT}$ $(valid\, s_{\mu\to o})$ then $\models^{QK\pi} s$.*

Proof. By contraposition, assume $\not\models^{\mathrm{QK}\pi} s$: that is, there is a QK$\pi$ model $Q = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$, a variable assignment g and a world $w \in W$, such that $Q, g, w \not\models s$. By Lemma 4.3, we have $\mathrm{V}_{[w/W_\mu]\dot{g}}\,(s_{\mu\to o}\, W_\mu) = F$ in a Henkin model H^Q for Q. Thus, $\mathrm{V}_{\dot{g}}\,(\forall W_{\mu\centerdot}\,(s_{\mu\to o}\, W)) =_{\beta\eta} \mathrm{V}_{\dot{g}}\,(\mathrm{valid}\, s_{\mu\to o}) = F$. Hence, $\not\models^{\mathrm{STT}} (\mathrm{valid}\, s_{\mu\to o})$. □

In order to prove completeness, we reverse our mapping from Henkin models to QKπ models.

Definition 4.5 (QML logic $L^{\mathbf{QML}}$ for QMLSTT logic L). The mapping $\bar{\cdot}$ is defined as the reverse map of $\dot{\cdot}$ from Def. 4.1.

The QML logic L^{QML} is obtained from QMLSTT logic L by choosing $\mathrm{IV} = \{\bar{X}_\iota \mid X_\iota \in \mathrm{IVSTT}\}$, $\mathrm{PV} = \{\bar{P}_{\mu\to o} \mid P_{\mu\to o} \in \mathrm{PVSTT}\}$, $\mathrm{SYM} = \{\bar{k}_{\iota^n\to(\mu\to o} \mid k_{\iota^n\to(\mu\to o)} \in \mathrm{SYMSTT}\}$, and $S = \{\bar{r}_{\mu\to\mu\to o} \mid r_{\mu\to\mu\to o} \in \mathrm{SSTT}\}$.

Moreover, let $g : \mathrm{IVSTT}\cup\mathrm{PVSTT} \longrightarrow D\cup P$ be a variable assignment for L. The corresponding variable assignment $\bar{g} : \mathrm{IV}\cup\mathrm{PV} \longrightarrow D \cup P$ for L^{QML} is defined as follows: $\bar{g}(X) = \bar{g}(\bar{X}_\iota) = g(X_\iota)$ and $\bar{g}(P) = \bar{g}(\bar{P}_{\mu\to o}) = g(P_{\mu\to o})$ for all $X \in \mathrm{IV}$ and $P \in \mathrm{PV}$.

We assume below that L, L^{QML}, g and $\bar{g}$ are defined as above.

Definition 4.6 (QKπ^- model Q^H for Henkin model H). Given a Henkin model $H = \langle\{D_\alpha\}_{\alpha\in\mathrm{T}}, I\rangle$ for QMLSTT logic L, we construct a QML model $Q^H = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ for L^{QML} by choosing $W = D_\mu$, $D = D_\iota$, and $P = D_{\mu\to o}$. Moreover, let $k = \bar{k}_{\iota^n\to(\mu\to o)}$ and let $X^i = \bar{X}^i_\iota$. We choose $I_w(k)$ such that $\langle\bar{g}(X^1), \ldots, \bar{g}(X^n)\rangle \in I_w(k)$ if and only if

$$(I\,k)(g(X^1_\iota), \ldots, g(X^n_\iota), w) = T.$$

Finally, let $r = \bar{r}_{\mu\to\mu\to o}$. We choose R_r such that $\langle w, w'\rangle \in R_r$ if and only if $(I r_{\mu\to\mu\to o})(w, w') = T$.

It is not hard to verify that $Q^H = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ meets the definition of QKπ^- models. Below we will see that it also meets the definition of QKπ models.

Lemma 4.7. *Let $Q^H = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ be a QKπ^- model for a given Henkin model $H = \langle\{D_\alpha\}_{\alpha\in T}, I\rangle$. Furthermore, let $s = \bar{s}_{\mu\to o}$. For all worlds $w \in W$ and variable assignments g we have $\mathrm{V}_{[w/W_\mu]g}\,(s_{\mu\to o} W_\mu) = T$ in H if and only if $Q^H, \bar{g}, w \models s$ in Q^H.*

Proof. The proof is by induction on the structure of $s_{\mu\to o} \in L$ and it is similar to the proof of Lemma 4.3. □

With the help of Lemma 4.7, we now show that the QKπ^- models we construct in Def. 4.6 are in fact always QKπ models. Thus, Henkin models never relate to QKπ^- models that do not already fulfill the QKπ criterion.

Lemma 4.8. *Let $Q^H = (W, (R_r)_{r\in S}, D, P, (I_w)_{w\in W})$ be a QKπ^- model for a given Henkin model $H = \langle\{D_\alpha\}_{\alpha\in T}, I\rangle$. Then Q^H is also a QKπ model.*

Proof. We need to show that for every variable assignment $\bar{g}$ and formula $s = \bar{s}_{\mu\to o}$ the set $\{w \in W \mid Q^h, \bar{g}, w \models s\}$ is a member of P in Q^H. This is a consequence of the rule that everything denotes in the Henkin model H. To see this, consider $\mathrm{V}_g s_{\mu\to o} = \mathrm{V}_g(\lambda V_\mu\text{.} s_{\mu\to o} V)$ for variable V_μ not occurring free in $s_{\mu\to o}$. By definition of Henkin models this denotes that function from $D_\mu = W$ to truth values $D_o = \{T, F\}$ whose value for each argument $w \in D_\mu$ is $\mathrm{V}_{[w/V_\mu]g}(s\, V)$, that is, $s_{\mu\to o}$ denotes the characteristic function $\lambda w \in W\text{.} \mathrm{V}_{[w/V_\mu]g}\,(s_{\mu\to o} V_\mu) = T$ which we identify with the set $\{w \in W \mid \mathrm{V}_{[w/V_\mu]g}\,(s_{\mu\to o} V_\mu) = T\}$. Hence, we have $\{w \in W \mid \mathrm{V}_{[w/V_\mu]g}\,(s_{\mu\to o} V_\mu) = T\} \in D_{\mu\to o}$. By the choice of $P = D_{\mu\to o}$ in the construction of Q^H we know $\{w \in W \mid \mathrm{V}_{[w/V_\mu]g}\,(s_{\mu\to o} V_\mu) = T\} \in P$. By Lemma 4.7 we get $\{w \in W \mid Q^h, \bar{g}, w \models s\} \in P$. □

Theorem 4.9 (Completeness for QKπ models). *Let $s_{\mu\to o}$ be a QMLSTT proposition and let $s = \bar{s}_{\mu\to o}$ be the corresponding QML proposition. If $\models^{QK\pi} s$ then $\models^{STT}$ (valid $s_{\mu\to o}$).*

Proof. By contraposition, assume $\not\models^{\mathrm{STT}}$ (valid $s_{\mu\to o}$): there is a Henkin model $H = \langle\{D_\alpha\}_{\alpha\in \mathrm{T}}, I\rangle$ and a variables assignment g such that V_g (valid $s_{\mu\to o}) = F$. Hence, for some world $w \in D_\mu$ we have $\mathrm{V}_{[w/W_\mu]g}\,(s_{\mu\to o} W_\mu) = F$. By Lemma 4.7 we then get $Q^H, \bar{g}, w \not\models^{\mathrm{QK}\pi^-} s$ for $s = \bar{s}_{\mu\to o}$ in QKπ^- model Q^H for H. By Lemma 4.8 we know that Q^H is actually a QKπ model. Hence, $\not\models^{\mathrm{QK}\pi} s$. □

Our soundness and completeness results obviously also apply to fragments of QML logics.

Corollary 4.10. *The reduction of our embedding to propositional quantified multimodal logics (which only allow quantification over propositional variables) is sound and complete.*

Corollary 4.11. *The reduction of our embedding to first-order multimodal logics (which only allow quantification over individual variables) is sound and complete.*

Corollary 4.12. *The reduction of our embedding to propositional multimodal logics (no quantification) is sound and complete.*

5. Conclusion

We have presented a straightforward embedding of quantified multimodal logics in simple type theory and we have shown that this embedding is sound and complete for QKπ semantics. This entails further soundness and completeness results of our embedding for fragments of quantified multimodal logics. We have formally explored the natural correspondence between QKπ models and Henkin models.

Non-quantified and quantified (normal) multimodal logics can thus be uniformly seen as natural fragments of simple type theory and their semantics (except some weak notions such as QKπ^- models) can be studied from the perspective of the well understood semantics of simple type theory. Vice versa, via our embedding we can characterize some computationally interesting fragments of simple type theory, which in turn may lead to some powerful proof tactics for higher-order proof assistants.

In experiments we applied the embedding presented in this paper for reasoning *within* and *about* combinations of multimodal logics [8]. For example, a formulation of the well known wise men puzzle in quantified multimodal logic can be solved with our theorem prover LEO-II in a few milliseconds. We obtain similar performance results for the verification of meta-properties such as the equivalence of different axiomatizations of modal logic S5. Interestingly, even higher-order model finders such as Nitpick [13] can be fruitfully applied, for example, to verify the consistency of our logic embeddings and their combinations within our framework.

Future work includes further extensions of our embedding to full higher-order modal logics [19, 27]. A first suggestion in direction of higher-order modal logics has already been made [10]. This proposal does not yet address intensionality aspects. However, combining it with non-extensional notions of models for simple type theory [9, 28] appears a promising direction.

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