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SPLIT QUATERNIONS AND ROTATIONS IN SEMI EUCLIDEAN SPACE $\mathbb{E}_2^4$

LEVENT KULA AND YUSUF YAYLI

ABSTRACT. We review the algebraic structure of $\mathbb{H}'$ and show that $\mathbb{H}'$ has a scalar product that allows as to identify it with semi Euclidean $\mathbb{E}_2^4$. We show that a pair q and p of unit split quaternions in $\mathbb{H}'$ determines a rotation $R_{qp} : \mathbb{H}' \to \mathbb{H}'$. Moreover, we prove that R_{qp} is a product of rotations in a pair of orthogonal planes in $\mathbb{E}_2^4$. To do that we call upon one tool from the theory of second ordinary differential equations.

1. Introduction

Quaternion algebra, which is customarily denoted $\mathbb{H}$ in his honor, enunciated by Hamilton, has played a significant role recently in several areas of the physical science; namely, in differential geometry, in analysis and synthesis of mechanism and machines, simulation of particle motion in molecular physics and quaternionic formulation of equation of motion in theory of relativity. Agrawal [1] gave some algebraic properties of Hamilton operators. Also, quaternions have been expressed in terms of 4×4 matrices by means of these operators.

Inoguchi [2] reformulated the Gauss-Codazzi equations in form familiar to the theory of integrable system in Minkowski 3-space $\mathbb{E}_1^3$. The main tool of this reformulation was split quaternion numbers (also called Gödel quaternions in the literature).

Kula and Yayli [3] defined dual split quaternions and gave some algebraic properties of dual split quaternions. Moreover they gave the screw motion, in $\mathbb{R}_1^3$, using the properties of the Hamilton operators defined in that paper.

Tain [6] examined solutions of the quaternionic matrix equations and has established a group of universal factorization equalities for quaternions.

Weiner and Wilkens [7] showed that any rotation in $\mathbb{E}^4$ is a product of rotation in a pair of orthogonal two-dimensional subspaces.

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In this paper, we show that there exist rotations $R_1, R_2 : \mathbb{H}' \to \mathbb{H}'$ and a pair of orthogonal planes W_1 and W_2 in $\mathbb{H}'$, such that the restrictions $R_1 |_{W_1}$ and $R_2 |_{W_2}$ are identities on their respective planes and $R_{qp} = R_1 \circ R_2 = R_2 \circ R_1$. If W_1 is a timelike plane of index 1, then W_2 is also. W_1 is a timelike plane of index 2 if and only if W_2 is a spacelike plane. In this case, $\mathbb{H}' = W_1 \oplus W_2$ and R_{qp} rotates vectors in the W_1 through a determined angle θ_1 (or hyperbolic angle φ_1) and vectors in the W_2 through a determined angle θ_2 (or hyperbolic angle φ_1).

2. Split quaternions and semi Euclidean space $\mathbb{E}_2^4$

A split quaternion q is an expression of the form

$$q = a_0 + a_1 i + a_2 j + a_3 k,$$

where a_0, a_1, a_2 and a_3 are real numbers, and i, j, k are split quaternionic units which satisfy the non-commutative multiplication rules

$$\begin{aligned} i^2 &= -1, \ j^2 = k^2 = 1, \\ ij &= -ji = k, \ jk = -kj = -i, \end{aligned}$$

and

$$ki = -ik = j.$$

Let us denote the algebra of split quaternions by $\mathbb{H}'$ and its natural basis by $\{1, i, j, k\}$. An element of $\mathbb{H}'$ is called a split quaternion [2].
If $q = a_0 + a_1 i + a_2 j + a_3 k$ and $p = b_0 + b_1 i + b_2 j + b_3 k$ be the two split quaternions and let $r = qp$, then r is given by

$$r = S_q S_p + g(V_q, V_p) + S_q V_p + S_p V_q + V_q \wedge V_p,$$

where

$$\begin{aligned} &S_q = a_0, \ S_p = b_0, \ g(V_q, V_p) = -a_1 b_1 + a_2 b_2 + a_3 b_3, \\ &V_q = a_1 i + a_2 j + a_3 k, \ V_p = b_1 i + b_2 j + b_3 k, \\ &V_q \wedge V_p = (a_3 b_2 - a_2 b_3) i + (a_3 b_1 - a_1 b_3) j + (a_1 b_2 - a_2 b_1) k. \end{aligned}$$

Definition 2.1. $\mathbb{E}^n$ with the metric tensor

$$\langle u, v \rangle = -\sum_{i=1}^{v} u_i v_i + \sum_{j=v+1}^{n} u_j v_j, \quad u, v \in \mathbb{R}^n, \quad 0 \leqslant \nu \leqslant n$$

is called semi Euclidean space and is denoted by $\mathbb{E}_\nu^n$ when ν is called the index of the metric. The resulting semi Euclidean space $\mathbb{E}_\nu^n$ reduced to $\mathbb{E}^n$ if $\nu = 0$. For $n \geqslant 2$, $\mathbb{E}_1^n$ is called Minkowski n-space; if $n = 4$ it is the simplest example of a relativistic space time [4].

Definition 2.2. Let $\mathbb{E}_\nu^n$ be a semi Euclidean space furnished with a metric tensor $\langle \ , \ \rangle$. A vector $w \in \mathbb{E}_\nu^n$ is called

spacelike if $\langle w, w \rangle > 0$ or $w = 0$, null if $\langle w, w \rangle = 0$ and $w \neq 0$,
timelike if $\langle w, w \rangle < 0$.

The norm $\|w\|$ of a vector $w \in \mathbb{E}_\nu^n$ is $|\langle w, w\rangle|^{\frac{1}{2}}$, two vectors w_1 and w_2 in $\mathbb{E}_\nu^n$ are said to be orthogonal, if $\langle w_1, w_2\rangle = 0$ [4].

Theorem 2.1. *Let $\mathbb{E}_1^3$ be a Minkowski 3-space furnished with a metric tensor $g(u,v) = -u_1v_1 + u_2v_2 + u_3v_3$, $u, v \in \mathbb{E}^3$. Then we have the following:*

(i) *Every orthonormal set of three vectors is a basis for $\mathbb{E}_1^3$.*
(ii) *Every orthonormal basis has two spacelike vectors and one timelike vector.*
(iii) *For every orthonormal pair $\{u, v\}$ of vectors,*

$$\{u,\ v,\ u \wedge v = (u_3v_2 - u_2v_3,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)\}$$

is an orthonormal basis.
(iv) *For every unit spacelike or unit timelike vector v, there is an orthonormal basis containing v* [5].

Definition 2.3. Let V be a 2-dimensional vector subspace of $\mathbb{E}_2^4$. Then V is said to be

(i) timelike plane of index 1 if and only if V have timelike and spacelike vectors.
(ii) timelike plane of index 2 if and only if every nonzero vector in V is timelike.
(iii) spacelike plane if and only if every vector in V is spacelike.

Hereafter we identify $\mathbb{H}'$ with the semi Euclidean space $\mathbb{E}_2^4$, where

$$\mathbb{E}_2^4 = \left\{q = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4 : \langle q, q\rangle = -a_0^2 - a_1^2 + a_2^2 + a_3^2\right\}.$$

3. Exponential maps for a unit split quaternion

Let $q = a_0 + a_1 i + a_2 j + a_3 k$ is a unit split quaternion, $N_q = a_0^2 + a_1^2 - a_2^2 - a_3^2 = 1$. Then there is a real number θ and a pure split quaternion

$$u = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{a_1^2 - a_2^2 - a_3^2}}$$

such that $q = \cos\theta + u\sin\theta$ if $g(\vec{V_q}, \vec{V_q}) = -a_1^2 + a_2^2 + a_3^2 < 0$. Since $u^2 = -1$, the power series expansion of e^t leads to

$$e^{u\theta} = \sum_{n=0}^{\infty} \frac{(u\theta)^n}{n!} = \cos\theta + u\sin\theta,$$

providing equivalent representation for a unit split quaternion

$$q = a_0 + a_1 i + a_2 j + a_3 k = \cos\theta + u\sin\theta = e^{u\theta}.$$

Since each component of $e^{u\theta}$ is a differentiable function of θ, it is not difficult to verify that

$$\frac{d}{d\theta} e^{u\theta} = -\sin\theta + u\cos\theta = ue^{u\theta} = e^{u\theta}u.$$

If $g(\overrightarrow{V_q}, \overrightarrow{V_q}) = -a_1^2 + a_2^2 + a_3^2 > 0$, then there is a real number φ and a pure split quaternion

$$v = \frac{a_1 i + a_2 j + a_3 k}{\sqrt{-a_1^2 + a_2^2 + a_3^2}}$$

such that $q = \cosh\varphi + v \sinh\varphi$. Because $v^2 = 1$, we get

$$e^{v\varphi} = \sum_{n=0}^{\infty} \frac{(v\varphi)^n}{n!} = \cosh\varphi + v\sinh\varphi.$$

In this case, the other representation for the unit split quaternion

$$q = \cosh\varphi + v\sinh\varphi = e^{v\varphi}.$$

The differential of $e^{v\varphi}$ is

$$\frac{d}{d\varphi} e^{v\varphi} = \sinh\varphi + v\cosh\varphi = ve^{v\varphi} = e^{v\varphi}v.$$

4. Rotations in $\mathbb{E}_1^3$

We introduce the $\mathbb{R}$-linear transformations representing left and right multiplication in $\mathbb{H}'$. Let q be a split quaternion. Then $L_q : \mathbb{H}' \to \mathbb{H}'$ and $R_q : \mathbb{H}' \to \mathbb{H}'$ are defined as follows:

$$L_q(x) = qx, \qquad R_q(x) = xq, \quad x \in \mathbb{H}'.$$

If q is a unit split quaternion, then both L_q and R_q are semi orthogonal transformations of $\mathbb{H}'$. Therefore, for unit split quaternions q and p, the mapping $R_{qp} : \mathbb{H}' \to \mathbb{H}'$ defined by

$$R_{qp} = L_q \circ R_p = R_p \circ L_q$$

is also a semi orthogonal transformation of $\mathbb{H}'$.

If q is a split quaternion, then transformations L and R are, respectively, defined as

$$\Phi(q) = \begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & a_3 & -a_2 \\ a_2 & a_3 & a_0 & -a_1 \\ a_3 & -a_2 & a_1 & a_0 \end{bmatrix} \tag{4.1}$$

and

$$\Psi(q) = \begin{bmatrix} a_0 & -a_1 & a_2 & a_3 \\ a_1 & a_0 & -a_3 & a_2 \\ a_2 & -a_3 & a_0 & a_1 \\ a_3 & a_2 & -a_1 & a_0 \end{bmatrix}. \tag{4.2}$$

Lemma 4.1. *If q and p are split quaternions and λ is a real number and L and R are operators as defined in equations* (4.1) *and* (4.2), *respectively, then the following identities hold:*

(i) $q = p \Leftrightarrow \Phi(q) = \Phi(p) \Leftrightarrow \Psi(q) = \Psi(p)$.

(ii) $\Phi(q+p) = \Phi(q) + \Phi(p),\ \Psi(q+p) = \Psi(q) + \Psi(p)$.

(iii) $\Phi(\lambda q) = \lambda\Phi(q),\ \Psi(\lambda q) = \lambda\Psi(q)$.

(iv) $qp = \Phi(q)p,\ \ qp = \Psi(p)q,\ \ \Phi(q)\Psi(p) = \Psi(p)\Phi(q)$.

(v) $\Phi(qp) = \Phi(q)\Phi(p),\ \Psi(qp) = \Psi(p)\Psi(q)$.

(vi) $\Phi(\overline{q}) = \varepsilon(\Phi(q))^T\varepsilon,\ \Psi(\overline{q}) = \varepsilon(\Psi(q))^T\varepsilon,\ \varepsilon = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix}$.

(vii) $\Phi(q^{-1}) = \Phi^{-1}(q),\ \Psi(q^{-1}) = \Psi^{-1}(q),\ N_q \neq 0$.

(viii) $\det[\Phi(q)] = (N_q)^2 = \langle q,q\rangle^2,\ \det[\Psi(q)] = (N_q)^2 = \langle q,q\rangle^2$.

(ix) $\Phi(q) = C(\Psi(q))^T C,\ C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

$C^{-1} = C^T = C,\ C^2 = I_4$ [3].

Proposition 4.1. *If $q = e^{u\theta}$ and $u^2 = -1$, then $R_{q\overline{q}} : \mathbb{E}_1^3 \to \mathbb{E}_1^3$ defined by $R_{q\overline{q}}(x) = qx\overline{q}$ is a rotation in the plane orthogonal to timelike vector $\overrightarrow{u}$ through an angle 2θ.*

Proof. Since $R_{q\overline{q}}$ is a semi orthogonal transformation of $\mathbb{H}'$ and also preserve the pure split quaternion u, $R_{q\overline{q}}$ fixes the one-dimensional subspace K spanned by timelike vector $\overrightarrow{u}$, hence fixes its orthogonal complement $K^{\perp}$ in the pure split quaternions as well. Let $v = \overrightarrow{v}$ be a spacelike split vector such that $v^2 = 1$ in $K^{\perp}$. From Theorem 2.1, $\alpha = uv = \overrightarrow{u} \wedge \overrightarrow{v}$ is a spacelike split vector. Notice that $uv = -vu$, since u and v are orthogonal. This implies that $ve^{-u\theta} = e^{u\theta}v$. Accordingly,

$$R_{q\overline{q}}(v) = e^{u\theta}ve^{-u\theta} = e^{2u\theta}v = \cos(2\theta)\overrightarrow{v} + \sin(2\theta)\overrightarrow{\alpha}.$$

Thus $R_{q\overline{q}}(u)$ yield a finite rotation about the timelike vector $\overrightarrow{u}$ with the angle 2θ in Minkowski 3-space. □

Proposition 4.2. *If $q = e^{v\theta}$ and $v^2 = 1$, then $R_{q\overline{q}} : \mathbb{E}_1^3 \to \mathbb{E}_1^3$ defined by $R_{q\overline{q}}(x) = qx\overline{q}$ is a rotation in the plane orthogonal to spacelike vector $\overrightarrow{v}$ through a hyperbolic angle 2φ.*

Proof. Here, $R_{q\overline{q}}$ fixes the one- dimensional subspace N spanned by spacelike vector $\overrightarrow{v}$. Let $w = \overrightarrow{w}$ be a timelike (spacelike) split vector in its orthogonal complement $N^{\perp}$ such that $w^2 = -1$ $(w^2 = 1)$. Thus from Theorem 2.1, $\gamma = vw = \overrightarrow{v} \wedge \overrightarrow{w}$ is a spacelike (timelike) split vector. Since w and v are orthogonal, $vw = -wv$. Hence, $we^{-v\theta} = e^{v\theta}w$. Consequently,

$$R_{q\overline{q}}(w) = e^{v\theta}we^{-v\theta} = e^{2v\theta}w = \cosh(2\varphi)\overrightarrow{w} + \sinh(2\varphi)\overrightarrow{\gamma}.$$

Thus $R_{q\overline{q}}(w)$ yield a finite rotation about the timelike vector $\overrightarrow{v}$ with the hyperbolic angle 2φ in Minkowski 3-space. □

5. How to solve linear equation over $\mathbb{H}'$

In this section, we examine the general solution the following linear equation

$$qx - xp = 0$$

over $\mathbb{H}'$.

Definition 5.1. Two split quaternions q and p are said to be similar if there exists a split quaternion a, $N_a \neq 0$, such that $a^{-1}qa = p$; this is written as $q \sim p$. Obviously, the similar quaternions have the same norm. $\sim$ is an equivalence relation on the split quaternions.

Proposition 5.1. *Let $q = a_0 + a_1 i + a_2 j + a_3 k$ be a split quaternion with $g(V_q, V_q) < 0$. Then there exist another quaternion a such that $a^{-1}qa = \xi + \eta i$ is a complex number with $\eta > 0$.*

Proof. Consider the equation of split quaternions

$$qx = x\left(a_0 + \sqrt{(a_1)^2 - (a_2)^2 - (a_3)^2}\; i\right). \tag{5.1}$$

It is easy to verify that

$$x = \left(a_1 + \sqrt{(a_1)^2 - (a_2)^2 - (a_3)^2}\right) - a_3 j + a_2 k$$

is a solution to equation (5.1), if $(a_2)^2 + (a_3)^2 \neq 0$. For the case q is a complex number, $j^{-1}qj = \overline{q}$. □

Proposition 5.2. *Let $q = a_0 + a_1 i + a_2 j + a_3 k$ be a split quaternion with $g(V_q, V_q) > 0$. Then there exist another quaternion c, $N_c \neq 0$, such that $c^{-1}qc = \xi + \zeta j$ is a hyperbolic number with $\zeta > 0$.*

Proof. Consider the equation of split quaternions

$$qy = y\left(a_0 + \sqrt{-(a_1)^2 + (a_2)^2 + (a_3)^2}\; j\right). \tag{5.2}$$

Then $y = a_1 - a_3 j + \left(a_2 - \sqrt{-(a_1)^2 + (a_2)^2 + (a_3)^2}\right) k$ is a solution to equation (5.2), if $a_2 \leq 0$. If $a_2 > 0$ then

$$y = \left(a_2 + \sqrt{-(a_1)^2 + (a_2)^2 + (a_3)^2}\right) + a_3 i + a_1 k$$

is a solution to equation (5.2). For the case q is a hyperbolic number, $i^{-1}qi = \overline{q}$. □

Now we consider the following linear equation

$$qx - xp = 0, \quad q, p, x \in \mathbb{H}' \tag{5.3}$$

over $\mathbb{H}'$. According to Lemma 4.1, the equation (5.3) is equivalent to

$$[\Phi(q) - \Psi(p)]\, x = 0,$$

which is a simple system of linear equations over $\mathbb{H}'$. In order to symbolically solve it, we need to examine some operation properties on the matrix $\Phi(q) - \Psi(p)$.

Proposition 5.3. *Let $q = a_0 + a_1 i + a_2 j + a_3 k$, $p = b_0 + b_1 i + b_2 j + b_3 k \in \mathbb{H}'$ be given, and denote $\Theta(q,p) = \Phi(q) - \Psi(p)$. Then*

(i) *If q and p are two split quaternion with $g(V_q,V_q) < 0$, $g(V_p,V_p) < 0$ (or $g(V_q,V_q) < 0$, $g(V_p,V_p) < 0$). Then the determinant of $\Theta(q,p)$ is*

$$\begin{aligned} &|\Theta(q,p)| \\ =\ & \left[s^2 + \left(\sqrt{-g(V_q,V_q)} - \sqrt{-g(V_p,V_p)}\right)^2\right] \\ & \times \left[s^2 + \left(\sqrt{-g(V_q,V_q)} + \sqrt{-g(V_p,V_p)}\right)^2\right] \\ =\ & s^4 - 2s^2\left[(g(V_q,V_q)) + (g(V_p,V_p))\right] + \left[(g(V_q,V_q)) - (g(V_p,V_p))\right]^2 \end{aligned}$$

where $s = a_0 - b_0$.

(ii) *If $a_0 \neq b_0$, or $g(V_q,V_q) \neq g(V_p,V_p)$, then $\Theta(q,p)$ is regular and its inverse can be expressed*

$$\begin{aligned} \Theta^{-1}(q,p) &= \Phi^{-1}\left(q^2 - 2b_0 q + N_p\right)(\Phi(q) - \Psi(\overline{p})) \\ &= \Phi^{-1}\left(2(a_0 - b_0)q + N_p - N_q\right)(\Phi(q) - \Psi(\overline{p})) \end{aligned}$$

and

$$\begin{aligned} \Theta^{-1}(q,p) &= \Psi^{-1}\left(p^2 - 2a_0 p + N_q\right)(\Phi(\overline{q}) - \Psi(p)) \\ &= \Psi^{-1}\left(2(b_0 - a_0)p + N_q - N_p\right)(\Phi(\overline{q}) - \Psi(p)). \end{aligned}$$

(iii) *If $a_0 = b_0$, and $g(V_q,V_q) = g(V_p,V_p)$, then $\Theta(q,p)$ is singular and has a generalized inverse as follows*

$$\Theta^{-}(q,p) = \frac{1}{4(V_q)^2}\Theta(q,p) = \frac{1}{4(V_q)^2}\left(\Phi(V_q) - \Psi(V_p)\right).$$

Proof. Let q and p be two split quaternion with $g(V_q,V_q) < 0$, $g(V_p,V_p) < 0$. Then q and p are similar to complex numbers z_1 and z_2, respectively. From Proposition 5.1, there are a, $b \in \mathbb{H}'$ such that

$$q = a z_1 a^{-1}, \quad p = b z_2 b^{-1}, \quad N_a \neq 0, \quad N_b \neq 0$$

Now applying Lemma 4.1 (v) to both them we obtain

$$\Phi(q) = \Phi(a)\Phi(z_1)\Phi\left(a^{-1}\right), \quad \Psi(p) = \Psi\left(b^{-1}\right)\Psi(z_2)\Psi(b).$$

Therefore the determinant of $\Theta(q,p)$ is

$$\begin{aligned} |\Theta(q,p)| &= \left|\Phi(a)\Phi(z_1)\Phi\left(a^{-1}\right) - \Psi\left(b^{-1}\right)\Psi(z_2)\Psi(b)\right| \\ &= |\Phi(a)|\left|\Phi(z_1) - \Phi\left(a^{-1}\right)\Psi\left(b^{-1}\right)\Psi(z_2)\Psi(b)\Phi(a)\right|\left|\Phi\left(a^{-1}\right)\right| \\ &= \left|\Phi(z_1) - \Psi\left(b^{-1}\right)\Psi(z_2)\Psi(b)\right| \\ &= \left|\Psi\left(b^{-1}\right)\right|\left|\Psi(b)\Phi(z_1)\Psi\left(b^{-1}\right) - \Psi(z_2)\right||\Psi(b)| \\ &= |\Phi(z_1) - \Psi(z_2)|. \end{aligned}$$

From this equation, the determinant can be calculated easily.

If q and p are two split quaternion with $g(V_q, V_q) > 0$, $g(V_p, V_p) > 0$. Then q and p are similar to hyperbolic numbers w_1 and w_2, respectively. From Proposition 5.2, there are c, $d \in \mathbb{H}'$ such that

$$q = cw_1c^{-1}, \quad p = dw_2d^{-1} \quad N_c \neq 0, \quad N_d \neq 0.$$

Similarly, we have

$$\begin{aligned} |\Theta(q,p)| &= \left|\Phi(c)\Phi(w_1)\Phi\left(c^{-1}\right) - \Psi\left(d^{-1}\right)\Psi(w_2)\Psi(d)\right| \\ &= |\Phi(w_1) - \Psi(w_2)|. \end{aligned}$$

Thus the determinant can be calculated easily.
The results in (ii). come from the following two equalities

$$[\Phi(q) - \Psi(\overline{p})]\Theta(q,p) = \Phi^{-1}\left(q^2 - 2b_0q + N_p\right),$$

$$[\Phi(\overline{q}) - \Psi(p)]\Theta(q,p) = \Psi^{-1}\left(p^2 - 2a_0p + N_q\right).$$

For $a_0 = b_0$ and $g(V_q, V_q) = g(V_p, V_p)$,

$$\Theta^3(q,p) = 4(V_q)^2\Theta(q,p),$$

that is,

$$\Theta(q,p)\left[\frac{1}{4(V_q)^2}\Theta(q,p)\right]\Theta(q,p) = \Theta(q,p).$$

So we have (iii). □

Theorem 5.4. *Let $q \in \mathbb{H}'$ and $q \notin \mathbb{R}$, Then the general solution of the equation*

$$qx = xq \tag{5.4}$$

is

$$x = a + \frac{1}{(V_q)^2}(V_q)a(V_q),$$

where $a \in \mathbb{H}'$ is arbitrary, or equivalently,

$$x = \lambda_1 + \lambda_2 q, \tag{5.5}$$

where λ_1, $\lambda_2 \in \mathbb{R}$ are arbitrary.

Proof. According to Lemma 4.1, the equation (5.4) is equivalent to $\Theta(q,q)x = 0$, and the general solution of this equation can be expressed as

$$x = 2\left[I_4 - \Theta^-(q,q)\Theta(q,q)\right]a,$$

where a is an arbitrary split quaternion. From Proposition 5.3(iii), we have

$$\begin{aligned} x &= 2\left[I_4 - \frac{1}{4(V_q)^2}\Theta^2(q,q)\right]a \\ &= 2\left[I_4 - \frac{1}{4(V_q)^2}\left(2(V_q)^2 I_4 - 2\Phi(V_q)\Psi(V_q)\right)\right]a \\ &= \left[I_4 + \frac{1}{(V_q)^2}\Phi(V_q)\Psi(V_q)\right]a \end{aligned} \tag{5.6}$$

Next let $a = (V_q)\, b$ in (5.6), where $b \in \mathbb{H}'$ is arbitrary. Then equation (5.6) becomes

$$\begin{aligned} x &= (V_q)\, b + b\, (V_q) \\ &= \lambda_1 + \lambda_2 V_q, \end{aligned}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, which is equivalent to (5.5). □

Theorem 5.5. *Let q, $p \in \mathbb{H}'$ be given. If*

$$a_0 = b_0, \quad g\,(V_q, V_q) = g\,(V_p, V_p) \neq 0 \tag{5.7}$$

then the general solution of linear equation

$$qx = xp \tag{5.8}$$

is

$$x = a + \frac{1}{(V_q)^2}\,(V_q)\, a\, (V_p), \tag{5.9}$$

where $a \in \mathbb{H}'$ is arbitrary; in particular, if $q \neq \overline{p}$, $\overline{p} = b_0 - b_1 i - b_2 j - b_3 k$, i.e., $V_q + V_p \neq 0$, then the general solution of (5.8) *can be written as*

$$x = \lambda_1\,(V_q + V_p) + \lambda_2 \left[(V_q)\,(V_p) + (V_q)^2\right] \tag{5.10}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$.

Proof. According Lemma 4.1, the equation (5.8) is equivalent to

$$[\Phi\,(q) - \Psi\,(p)]\, x = \Theta\,(q,p)\, x = 0,$$

and this equation has a nonzero solution if and only if $|\Theta\,(q,p)| = 0$, which is equivalent, by Proposition 5.3(i), to (5.7). In that case, the general solution of this equation can be expressed as

$$x = 2\left[I_4 - \Theta^-\,(q,p)\,\Theta\,(q,p)\right] a,$$

where a is an arbitrary real vector. Now substituting $\Theta^-\,(q,q)$ in Proposition 5.3(iii) in it, we get

$$\begin{aligned} x &= 2\left[I_4 - \frac{1}{4\,(V_q)^2}\Theta^2\,(q,p)\right] a \\ &= 2\left[I_4 - \frac{1}{4\,(V_q)^2}\left(2\,(V_q)^2 I_4 - 2\Phi\,(V_q)\,\Psi\,(V_p)\right)\right] a \\ &= \left[I_4 + \frac{1}{(V_q)^2}\Phi\,(V_q)\,\Psi\,(V_p)\right] a. \end{aligned}$$

Returning it to quaternion form by Lemma 4.1, we have (5.9). If $q \neq \overline{p}$ in (5.9), then we set $a = V_q$ and $a = (V_q)^2$ in (5.9), respectively, and (5.9) becomes

$$x_1 = V_q + V_p, \quad x_2 = (V_q)^2 + V_q V_p.$$

Thus (5.10) is also a solution to (5.8) under (5.7). The independence of x_1 and x_2 can be seen from two simple facts that $S_{x_1} = 0$ and $S_{x_2} \neq 0$. Therefore

(5.10) is exactly the general solution to (5.8), since the rank of $\Theta(q,p)$ is two under (5.7). □

6. Semi orthogonal transformations of $\mathbb{E}_2^4$

In this section, we seek two-dimensional invariant subspace for the mapping R_{qp}.

Proposition 6.1. *Let $\widetilde{x} : \mathbb{R} \to \mathbb{R}^4$ satisfy a 2th-order linear homogeneous differential equation. Then the image of $\widetilde{x}$ lies in a 2-dimensional subspace of $\mathbb{R}^4$.*

Proof. Let $x_0 = \widetilde{x}(0)$ and $x_0' = \frac{d\widetilde{x}}{dt}(0)$ be the initial position and initial velocity for the given curve $\widetilde{x}$. Additionally, suppose that $\widetilde{x}$ satisfies the second-order linear homogenous differential equation

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + \beta x = 0,$$

where α and β are differentiable real valued functions of t. From standard ODE theory, we know that when two solution of this differential equation have the same initial position and the same initial velocity, the two solution are identical.

Now suppose that f_i $(i = 0, 1)$ are real valued functions that satisfy the differential equation $f'' + \alpha f' + \beta f = 0$ and, in addition, $f_0(0) = 1$, $f_0'(0) = 0$, $f_1(0) = 0$, $f_1'(0) = 1$. Then $x(t) = f_0(t)x_0 + f_1(t)x_0'$ satisfies the same ODE as $\widetilde{x}$ and has the same initial position and initial velocity. Thus the curve $\widetilde{x} = f_0(t)x_0 + f_1(t)x_0'$, so we see that the image of $\widetilde{x}$ lies in the subspace of $\mathbb{R}^4$ spanned by x_0 and x_0' [7]. □

Proposition 6.2. *Suppose that $\widetilde{x} : \mathbb{R} \to \mathbb{H}'$ satisfies the differential equation*

$$\frac{d^2\widetilde{x}}{dt^2} + s^2\widetilde{x} = 0,$$

where $s > 0$ is a constant, and that initial position vector $\widetilde{x}(0)$ and the initial velocity vector $\frac{d\widetilde{x}}{dt}(0)$ satisfy the conditions

$$\|\widetilde{x}(0)\| = s^{-1}\left\|\frac{d\widetilde{x}}{dt}(0)\right\|, \quad \left\langle \widetilde{x}(0), \frac{d\widetilde{x}}{dt}(0)\right\rangle = 0.$$

Then $\widetilde{x}(1) = R(\widetilde{x}(0))$, where R is a rotation in the plane of the image of $\widetilde{x}$ through an angle s in the direction that turns $\widetilde{x}(0)$ toward $s^{-1}\frac{d\widetilde{x}}{dt}(0)$.

Proof. Following the construction in the proof of Proposition 6.1, we choose $f_0(t) = \cos(st)$ and $f_1(t) = s^{-1}\sin(st)$. Then

$$\widetilde{x}(t) = \widetilde{x}(0)\cos(st) + s^{-1}\frac{d\widetilde{x}}{dt}(0)\sin(st),$$

which show that

$$\widetilde{x}(1) = \widetilde{x}(0)\cos(s) + s^{-1}\frac{d\widetilde{x}}{dt}(0)\sin(s).$$

That the rotation R exists follows from the assumptions that $\widetilde{x}(0)$ and $s^{-1}\frac{d\widetilde{x}}{dt}(0)$ are orthogonal vectors and have the same length. □

Proposition 6.3. *Suppose that $\widetilde{y}:\mathbb{R}\to\mathbb{H}'$ satisfies the differential equation*

$$\frac{d^2\widetilde{y}}{dt^2} - r^2\widetilde{y} = 0,$$

where $r>0$ is a constant, and that initial position vector $\widetilde{y}(0)$ and the initial velocity vector $\frac{d\widetilde{y}}{dt}(0)$ satisfy the conditions

$$\|\widetilde{y}(0)\| = r^{-1}\left\|\frac{d\widetilde{y}}{dt}(0)\right\|, \quad \left\langle \widetilde{y}(0), \frac{d\widetilde{y}}{dt}(0)\right\rangle = 0.$$

Then $\widetilde{y}(1) = R(\widetilde{y}(0))$, where R is a rotation in the plane of the image of $\widetilde{y}$ through an angle r in the direction that turns $\widetilde{y}(0)$ toward $r^{-1}\frac{d\widetilde{y}}{dt}(0)$.

Proof. Similarly, we choose $f_0(t) = \cosh(rt)$ and $f_1(t) = r^{-1}\sinh(rt)$. Then

$$\widetilde{y}(t) = \widetilde{y}(0)\cosh(rt) + r^{-1}\frac{d\widetilde{y}}{dt}(0)\sinh(rt),$$

which show that

$$\widetilde{y}(1) = \widetilde{y}(0)\cosh(r) + r^{-1}\frac{d\widetilde{y}}{dt}(0)\sinh(r).$$

That the rotation R exists follows from the assumptions that $\widetilde{y}(0)$ and $r^{-1}\frac{d\widetilde{y}}{dt}(0)$ are orthogonal vectors and have the same length. □

6.1. Representation of a rotation in $\mathbb{E}_2^4$ by means of timelike split vectors

First, notice that R_{qp}^t does make sense; in fact, for any real t, let R_{qp}^t be defined by

$$R_{qp}^t(x) = e^{u_1\theta_1 t}xe^{u_2\theta_2 t}, \ (u_1)^2 = -1, \ (u_2)^2 = -1. \tag{6.1}$$

To each quaternion x we associate a curve $\widetilde{x}:\mathbb{R}\to\mathbb{H}'$ defined by $\widetilde{x}(t) = R_{qp}^t(x)$. We will compute two derivative of $\widetilde{x}$. As the first derivative of $\widetilde{x}$ we obtain

$$\frac{d\widetilde{x}}{dt}(t) = u_1\theta_1 e^{u_1\theta_1 t}xe^{u_2\theta_2 t} + e^{u_1\theta_1 t}xe^{u_2\theta_2 t}u_2\theta_2 = u_1\theta_1\widetilde{x}(t) + \widetilde{x}(t)u_2\theta_2. \tag{6.2}$$

Differentiating the left hand and right hand sides of (6.2), at the same time using (6.2) to eliminate first order derivatives of $\widetilde{x}(t)$, we have

$$\frac{d^2\widetilde{x}}{dt^2}(t) = -\left[(\theta_1)^2 + (\theta_2)^2\right]\widetilde{x}(t) + 2\theta_1\theta_2 u_1\widetilde{x}(t)u_2. \tag{6.3}$$

Now if it happened that

$$u_1\widetilde{x}(t)u_2 = \lambda(t)\widetilde{x}(t), \tag{6.4}$$

where λ is a real valued function, then $\widetilde{x}$ would satisfy a linear homogeneous second order ordinary differential equation with real coefficients. By Proposition 6.1, the image of $\widetilde{x}$ would lie in a two dimensional subspace and necessarily the span of $x=\widetilde{x}(0)$ and $\widetilde{x}(1)$ would be an invariant subspace. Note however that $\|u_1\widetilde{x}(t)u_2\| = \|\widetilde{x}(t)\|$, because u_1 and u_2 are unit split vectors. Thus if equation (6.4) were to hold, then λ would have to be either the constant function 1 or the constant function -1. In fact, we can simplify the condition of equation (6.4) with $\lambda = \pm 1$.
On the other hand

$$u_1\widetilde{x}(t)u_2 = \pm x.$$

Proposition 6.4. *For x in $\mathbb{H}'$, $u_1xu_2 = \pm x$ if and only if $u_1\widetilde{x}(t)u_2 = \pm x$ holds for all t.*

Proof. Assume that $u_1\widetilde{x}(t)u_2 = \pm x$ for all t. Since $x=\widetilde{x}(0)$, $u_1xu_2 = \pm x$. If $u_1xu_2 = \pm x$, then

$$\pm R^t_{qp}(x) = R^t_{qp}(u_1xu_2) = e^{u_1\theta_1 t}u_1xu_2e^{u_2\theta_2 t} = u_1R^t_{qp}(x)u_2.$$

This completes the proof. □

First we look for those x in $\mathbb{H}'$ that satisfy one the linear equations $u_1x \pm xu_2 = 0$. To do this we introduce a basis for $\mathbb{H}'$. A natural choose is the set consisting of 1, u_1, u_2, and u_1u_2. Of course, this is not a basis if $u_2 = \pm u_1$. We first consider the case where $u_2 = \pm u_1$ and look for solutions to $u_1x \pm xu_1 = 0$. From Theorem 5.4, 1 and u_1 are solutions to $u_1x - xu_1 = 0$. Moreover, the solutions to $u_1x + xu_1 = 0$ are the pure quaternions x that are orthogonal to u_1. Thus the solution spaces to the two equations $u_1x \pm xu_1 = 0$ give a decomposition of $\mathbb{H}'$ into the sum of two 2-dimensional orthogonal subspaces. In this case $u_1 \neq \pm u_2$, from Theorem 5.5, $(u_1 + u_2)$ and $(u_1u_2 - 1)$ are solutions to $u_1x - xu_2 = 0$. Using (6.2), we see that for $x = u_1 + u_2$

$$\frac{d\widetilde{x}}{dt}(0) = u_1\theta_1(u_1 + u_2) + (u_1 + u_2)u_2\theta_2 = (\theta_1 + \theta_2)(u_1u_2 - 1).$$

Therefore $\widetilde{x}(0) = u_1 + u_2$ is orthogonal to $\widetilde{x}'(0)$ and thus to $u_1u_2 - 1$. Furthermore, we can show that $u_2 - u_1$ and $u_1u_2 + 1$ are orthogonal solution of $u_1x + xu_2 = 0$. Finally, it is easy to check that each pair of vectors is orthogonal to the other pair. Thus the vectors $u_1 + u_2$, $u_1u_2 - 1$, $u_2 - u_1$ and $u_1u_2 + 1$ constitute an orthogonal basis for $\mathbb{H}'$.

Assume that $u_1 \neq -u_2$, let $x = u_1 + u_2$, and recall that this x satisfy $u_1 x - x u_2 = 0$ (or equivalently $u_1 x u_2 = -x$). For this x, (6.3) becomes

$$\frac{d^2\widetilde{x}}{dt^2}(t) + (\theta_1 + \theta_2)^2 \widetilde{x}(t) = 0.$$

Hence $|\theta_1 + \theta_2|^{-1} \left\| \frac{d\widetilde{x}}{dt}(0) \right\| = \|\widetilde{x}(0)\|$ and $\left\langle \widetilde{x}(0), \frac{d\widetilde{x}}{dt}(0) \right\rangle = 0$. The same kind of results hold in the remaining cases when $u_1 \neq u_2$.

We consolidate what we have learned into our theorem:

Theorem 6.5. *Let $q = e^{u_1\theta_1}$ and $p = e^{u_2\theta_2}$, where $(u_1)^2 = -1$, $(u_2)^2 = -1$. The semi orthogonal transformation R_{qp}^t of $\mathbb{H}'$ is a product of two rotations in orthogonal planes. If $u_1 \neq \pm u_2$, then R_{qp}^t rotates the plane spanned by $u_1 + u_2$ and $u_1 u_2 - 1$ through the angle $|\theta_1 + \theta_2|$ and the plane spanned by $u_2 - u_1$ and $u_1 u_2 + 1$ through the angle $|\theta_1 - \theta_2|$. If $u_1 = \pm u_2$, then the invariant planes are the span of $1, u_1$ and its orthogonal complement.*

Proposition 6.6. *Let $(u_1)^2 = -1$, $(u_2)^2 = -1$, and $u_1 \neq \pm u_2$.*

(i) *If $\langle u_1, u_2 \rangle > 1$, then the subspace spanned by $\{u_1 + u_2, u_1 u_2 - 1\}$ is spacelike plane and the subspace spanned by $\{u_2 - u_1, u_1 u_2 + 1\}$ is timelike plane of index 2.*

(ii) *If $\langle u_1, u_2 \rangle < -1$, then the subspace spanned by $\{u_1 + u_2, u_1 u_2 - 1\}$ is timelike plane of index 2 and the subspace spanned by $\{u_2 - u_1, u_1 u_2 + 1\}$ is spacelike plane.*

6.2. Representation of a rotation in $\mathbb{E}_2^4$ by means of spacelike split vectors

Now we consider another representation of R_{qp}^t. Let R_{qp}^t be defined by

$$R_{qp}^t(y) = e^{v_1\varphi_1 t} y e^{v_2\varphi_2 t}, \quad (v_1)^2 = 1, \ (v_2)^2 = 1, \tag{6.5}$$

such that $v_1 \wedge v_2$ is a non-null vector. To each quaternion y we associate a curve $\widetilde{y} : \mathbb{R} \to \mathbb{H}'$ defined by $\widetilde{y}(t) = R_{qp}^t(y)$. As the first derivative of $\widetilde{y}$ we obtain

$$\frac{d\widetilde{y}}{dt}(t) = v_1\varphi_1 e^{v_1\varphi_1 t} y e^{v_2\varphi_2 t} + e^{v_1\varphi_1 t} y e^{v_2\varphi_2 t} v_2\varphi_2 = v_1\varphi_1 \widetilde{y}(t) + \widetilde{y}(t) v_2\varphi_2. \tag{6.6}$$

Differentiating the left hand and right hand sides of (6.6), at the same time using (6.6) to eliminate first order derivatives of $\widetilde{y}(t)$, we have

$$\frac{d^2\widetilde{y}}{dt^2}(t) = \left[(\varphi_1)^2 + (\varphi_2)^2\right] \widetilde{y}(t) + 2\varphi_1\varphi_2 v_1 \widetilde{y}(t) v_2. \tag{6.7}$$

Now if it happened that

$$v_1 \widetilde{y}(t) v_2 = \mu(t) \widetilde{y}(t), \tag{6.8}$$

where μ is a real valued function, then $\widetilde{y}$ would satisfy a linear homogeneous second order ordinary differential equation with real coefficients. By Proposition 6.1, the image of $\widetilde{y}$ would lie in a two dimensional subspace and necessarily the span of $y{=}\widetilde{y}(0)$ and $\widetilde{y}(1)$ would be an invariant subspace. Note however that $\|v_1\widetilde{y}(t)\,v_2\| = \|\widetilde{y}(t)\|$, because v_1 and v_2 are unit split vectors. Thus if equation (6.8) were to hold, then μ would have to be either the constant function 1 or the constant function -1. In fact, we can simplify the condition of equation (6.8) with $\mu = \pm 1$.

Proposition 6.7. *For y in $\mathbb{H}'$, $v_1 y v_2 = \pm y$ if and only if $v_1\widetilde{y}(t)\,v_2 = \pm y$ holds for all t.*

Proof. Similarly to Proposition 6.4, it can be proofed. □

Now we look for those y in $\mathbb{H}'$ that satisfy one the linear equations $v_1 y \pm y v_2 = 0$ such that $v_1 \wedge v_2$ is a non-null vector. To do this we introduce a basis for $\mathbb{H}'$. A natural choose is the set consisting of 1, v_1, v_2, and $v_1 v_2$. Of course, this is not a basis if $v_1 = \pm v_2$. We first consider the case where $v_1 = \pm v_2$ and look for solutions to $v_1 y - y v_1 = 0$. From Theorem 5.4, 1 and v_1 are solutions to $v_1 y - y v_1 = 0$. Moreover, the solutions to $v_1 y + y v_1 = 0$ are the pure quaternions y that are orthogonal to v_1. Thus the solution spaces to the two equations $v_1 y \pm y v_1 = 0$ give a decomposition of $\mathbb{H}'$ into the sum of two 2-dimensional orthogonal subspaces.

In this case $v_1 \neq \pm v_2$, from Theorem 5.5, $(v_1 + v_2)$ and $(v_1 v_2 + 1)$ are solutions to $v_1 y - y v_2 = 0$. Using (6.6), we see that for $y = v_1 + v_2$

$$\frac{d\widetilde{y}}{dt}(0) = v_1\varphi_1 (v_1 + v_2) + (v_1 + v_2)\, v_2\varphi_2 = (\varphi_1 + \varphi_2)\,(v_1 v_2 + 1)\,.$$

Therefore $\widetilde{y}(0) = v_1 + v_2$ is orthogonal to $\widetilde{y}'(0)$ and thus to $v_1 v_2 + 1$. Furthermore, we can show that $v_2 - v_1$ and $v_1 v_2 - 1$ are orthogonal solution of $v_1 y + y v_2 = 0$. Finally, it is easy to check that each pair of vectors is orthogonal to the other pair. Thus the vectors $u_1 + u_2$, $u_1 u_2 + 1$, $u_2 - u_1$ and $u_1 u_2 - 1$ constitute an orthogonal basis for $\mathbb{H}'$.

Assume that $v_1 \neq -v_2$, let $y = v_1 + v_2$, and recall that this y satisfy $v_1 y - y v_2 = 0$ (or equivalently $v_1 y v_2 = y$). For this y, (6.7) becomes

$$\frac{d^2\widetilde{y}}{dt^2}(t) - (\varphi_1 + \varphi_2)^2\,\widetilde{y}(t) = 0.$$

Hence

$$|\varphi_1 + \varphi_2|^{-1}\left\|\frac{d\widetilde{y}}{dt}(0)\right\| = \|\widetilde{y}(0)\|\,, \quad \left\langle \widetilde{y}(0)\,, \frac{d\widetilde{y}}{dt}(0)\right\rangle = 0.$$

The same kind of results hold in the remaining cases when $v_1 \neq v_2$.

We consolidate what we have learned into our theorem:

Theorem 6.8. *Let $q = e^{v_1\varphi_1}$ and $p = e^{v_2\varphi_2}$, where $(v_1)^2 = 1$, $(v_2)^2 = 1$ and $v_1 \wedge v_2$ is a non-null vector. The semi orthogonal transformation R^t_{qp} of $\mathbb{H}'$ is a product of two rotations in orthogonal planes. If $v_1 \neq \pm v_2$, then R^t_{qp} rotates the plane spanned by $v_1 + v_2$ and $v_1 v_2 + 1$ through the angle $|\varphi_1 + \varphi_2|$ and the plane spanned by $v_2 - v_1$ and $v_1 v_2 - 1$ through the angle $|\varphi_1 - \varphi_2|$. If $v_1 = \pm v_2$, then the invariant planes are the span of 1, v_1 and its orthogonal complement.*

Proposition 6.9. *Let* $(v_1)^2 = 1$, $(v_2)^2 = 1$, *and* $v_1 \wedge v_2$ *is a non-null vector. Then the subspace spanned by* $\{v_1 + v_2, v_1v_2 + 1\}$ *is the timelike plane of index 1 and the subspace spanned by* $\{v_2 - v_1, v_1v_2 - 1\}$ *is timelike plane of index 1.*

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Levent Kula
Department of Mathematics
Faculty of Science
Ankara University
Tandogan Ankara 06100, Turkey
E-mail address: kula@science.ankara.edu.tr

Yusuf Yayli
Department of Mathematics
Faculty of Science
Ankara University
Tandogan Ankara 06100, Turkey
E-mail address: yayli@science.ankara.edu.tr