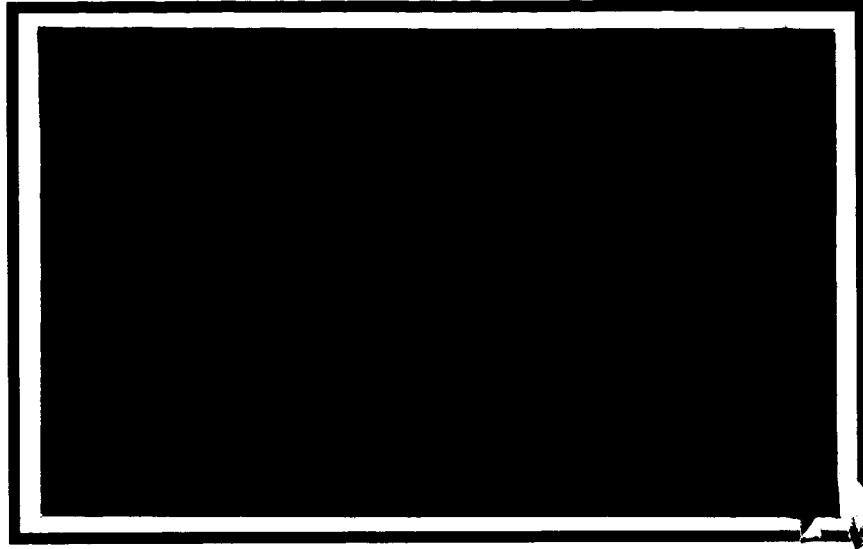# Center for Research on Economic and Social Theory
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Successful Takeovers without Exclusion

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Section 1. Introduction.

For many years, a common view of takeovers was that if a firm is inefficiently run, then an individual (whom we shall refer to as a raider) can, if he is more efficient than current management, offer more than the current value of the stock, buy out the firm, run it more efficiently and thus raise the value of the stock. Then reselling the stock would earn the raider a profit.[1] More recently,[2] it has been argued that individual stockholders have an incentive to free ride on the improvements brought about by the raider. If the holding of each stockholder is a small proportion of the firm, no stockholder will recognize the effect his decision has on whether or not the raid succeeds. Hence, assuming stockholders have rational expectations, if the raid is going to succeed, no stockholder will sell unless he is offered at least the post-takeover value of his stock. Consequently, the raider cannot purchase any share unless he pays at least what it is worth to him if the raid succeeds. If he does so, he cannot earn profits by taking over the firm.[3]

For this reason, these authors conclude that raids can never succeed without mechanisms which prevent minority stockholders from receiving all of the gains in the value of their shares.[4] That is, since stockholders will not tender for less than the expected post-takeover value of their shares to them, a divergence is created between this value and the value of these shares to the raider. The idea behind these mechanisms is that "The only way to create proper incentives for the production of a public good (*i.e.*, guaranteeing that the firm is efficiently run) is to exclude non-payers (minority stockholders) from enjoying the benefits of the public good."[5]

A key element in the argument above is that no shareholder perceives that he may affect the outcome. We will show that, with a finite number of stockholders, this is not true—that is, some stockholders must be pivotal. Making some stockholders pivotal is crucial because it forces them to choose whether or not the raid succeeds. Hence, they cannot free ride. Consequently, exclusion is not necessary for successful takeovers.

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[1] See, for example, Manne [1965].

[2] See, for example, Grossman and Hart [1980], Bradley [1980], Vermaelen [1981], and Bradley and Kim [1984].

[3] This form of the argument is taken from Grossman and Hart [1980].

[4] Vishny and Shleifer [1984] propose one alternative solution to the free rider problem. They posit the existence of a single "large" stockholder who has enough at stake to assume much of the burden of maintaining the efficiency of management.

[5] Grossman and Hart, page 59. The details in parentheses have been added.

This result differs from those of previous writers because we view the free rider problem differently. This difference is much like the change in views in the Public Finance literature on this subject. Prior to 1972, the accepted view of free rider problems was that the ability of agents to free ride to avoid sharing the burden of providing some public good implies that private provision of the public good will be suboptimal. Since then, the literature on incentive compatible mechanisms have shown that there are ways to structure the environment (through the design of a mechanism) to ameliorate the free rider problem. This is done by making some agents pivotal. Thus, we are led to ask whether there are mechanisms, superior in some respects to exclusion, which the raider himself can implement.

To answer this question, we explicitly analyze the equilibria in a takeover game when there is a finite number of stockholders each of whom owns a noninfinitesimal share of the firm. In this context, the raider's strategy choice induces a subgame among the stockholders. This is analogous to the way a mechanism induces a game among the agents in the economy. Our approach is to consider alternative strategy sets for the raider to determine which strategies enable him to design an efficient mechanism, *i.e.*, take over the firm when he is more efficient than current management. Since the alternative strategy sets can represent various possible legal restrictions on takeover bids, our results have implications for the ongoing debate on this subject.[6]

As in all of the papers cited above, we focus on games of complete information. Also as in these papers, we assume that current management has no ability to fight the takeover.[7] Finally, we focus on the case of a single raider, rather than the more complex case of multiple raiders.[8]

After defining the game more precisely in Section 2, we turn to an analysis of bids without exclusion. In Section 3A, we analyze the most restrictive strategy set we consider. In this section, we only allow the raider to make so–called any–and–all bids. These bids specify a fixed price and have the raider offering to buy every share tendered to him at this price. In this case, all equilibria of the game involve a strictly positive probability of a successful raid if the raider is more efficient than current management. In fact, some equilibria have the takeover succeeding with probability 1.

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[6] Discussed in detail in Bagnoli and Lipman [1985].

[7] For discussion of some mechanisms by which managers can fight takeover attempts, see Bradley and Rosenzweig [1985].

[8] For a discussion of the effect multiple raiders may have, see Schwartz [1985].

In Section 3B, we broaden the raider's strategy set to allow him what we refer to as conditional bids.[9] Here the raider offers a price per share *if* at least some specified number of shares are tendered. If fewer are tendered, the raider purchases no shares. If more are tendered, the raider purchases at least as many as he originally offered to buy at the specified price. Since the raider can specify one share and since he can choose to buy every share that is offered, this strategy set includes any–and–all bids. With this broader strategy set for the raider, there is a unique equilibrium in which the raider takes over with probability 1 if he is more efficient than current management. Furthermore, this is an equilibrium for any enlargement of the raider's strategy set so long as exclusion is prohibited. Any other equilibrium with such a strategy set must have exactly the same outcome as this one.

In Section 4, the raider has the ability to exclude minority stockholders from some of the benefits from the takeover. Recall that previous writers have asserted that exclusion is necessary for successful takeovers and, hence, is socially desirable. By contrast, in our model, if conditional bids are allowed, exclusion creates the possibility of takeovers by raiders who are less efficient than current managers. This is not true of all equilibria when exclusion is allowed. In fact, exclusion has positive effects from a social point of view when the raider is restricted to any–and–all bids.

In the final section, we show that the social and private rankings of these alternatives differ. However, because we only consider preferences over a limited range of alternatives, this does not imply that the socially optimal restrictions on the raider's strategy set will necessarily differ from the privately optimal restrictions. We offer a few concluding remarks on this subject, one which we are continuing to research.

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[9] These are commonly referred to as partial bids. However, we avoid this terminology as we allow "partial" bids for 100% of the shares.

Section 2. Notation and Definitions.

We consider a two-stage game with I stockholders who own all N shares of stock in the firm. The ith stockholder has h_i shares of stock where h_i is an integer. We assume that $0 < h_i < N - K$ for all i, where $K < N$ is the number of shares needed to control the corporation. Otherwise, some shareholder can prevent the takeover from succeeding even if everyone else sells, as $N - h_i < K$ for some i. We will let p_0 be the value of a share of stock when the firm is run by the current management and p_1 the value of a share if the firm is run by the raider. If the raider is more efficient than current management, then $p_1 > p_0$.

In the first stage of the game, the raider will choose a strategy t from his strategy set $\mathcal{T}$. Because, as noted in Section 1, the raider's strategy set will differ in each of the following sections, its precise definition will be postponed.

Given any t, the second stage is a subgame played by the stockholders in which each shareholder simultaneously chooses (possibly via a mixed strategy) a number of shares to tender to the raider. We require that the ith shareholder offer an integer number of shares between 0 and h_i. A pure strategy by shareholder i will be denoted σ_i and we will let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_I)$. A mixed strategy for shareholder i, denoted F_i, will be a choice of a probability distribution over the set of integers between 0 and h_i. Let F be the vector $(F_1, F_2, \ldots, F_I)$.

We take all players to be risk neutral. Hence the expected payoff to the ith stockholder is the expected value of his h_i shares. The payoff to the raider is his expected profit on the takeover bid ignoring any costs associated with actually making a bid.[10] We assume that if the takeover bid fails, each share continues to be worth p_0. Furthermore, we will take p_0 to be constant throughout the analysis. In particular, we do not consider how the threat of takeovers influences managerial efficiency.[11]

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[10] One may wonder if the fact that we ignore costs associated with making a bid significantly affects our conclusions on the viability of takeover bids. This is not the case. Previous writers conclude that without exclusion, raiders will at best break even if these costs are zero. We conclude that raiders will always earn strictly positive profits if these costs are zero. Furthermore, it is straight-forward to include these costs and is easy to see that their inclusion would simply limit the raiders who make bids to those who are sufficiently better than the current management at running the firm.

[11] This issue is a central feature of Grossman and Hart's model. While we do not consider these effects, their inclusion would complicate the mathematics but not change the basic results.

We consider the subgame perfect equilibria of this game. In Section 3B, we will refine this concept slightly (for reasons which we will explain there) and consider what we will call an r-equilibrium (or refined equilibrium). An equilibrium is a vector $\{F(t), t\}$ such that

(*i*) $F_i(t)$ maximizes the expected payoff to shareholder *i* given $F_j(t)$ for $j \neq i$ for each *t* (*i.e.*, $F_i(t)$ is a best reply for shareholder *i*) and

(*ii*) *t* maximizes the raider's expected profits (over the set of allowed strategies for the raider) given $F(t)$.

That is, $F(t)$ is an equilibrium in the subgame induced by the raider's choice of *t* and the raider's choice of *t* is optimal for him given the set of equilibria induced by the possible choices of *t*. An r-equilibrium is an equilibrium such that for all *t*, $F(t)$ constitutes a trembling-hand perfect equilibrium in the subgame induced by *t*. A complete explanation of this concept is provided in Appendix B where the concept is used in Theorem 2. Also, an explanation for why this refinement is utilized may be found in Section 3B.

Section 3A. Any–and–All Bids without Exclusion.

In this section, we consider equilibria in our two-stage game when the raider's strategy set $\mathcal{T}$ is restricted to so–called any-and-all bids. In this case, $\mathcal{T}$ will be taken to be the real line. The raider's strategy choice, t, will be a bid price, b, a price per share which will be offered for any and all shares the stockholders tender to him. Previous writers have claimed that no raid can succeed with $b < p_1$. In our model, there are many equilibria in the subgame induced by such a b in which the raid may succeed even if b is made arbitrarily close to p_0. Before presenting our general characterization of equilibria in this game, we present three examples of equilibria in the subgame induced by a given $b \in (p_0, p_1)$.

First, many asymmetric pure strategy equilibria exist in which exactly K shares are tendered to the raider and so the raid succeeds with probability 1. In fact, any σ such that $\sum_j^I \sigma_j = K$ and $\sigma_i \leq h_i$ for all i is an equilibrium. Since the raid succeeds, a shareholder who tenders an additional share foregoes p_1 to earn $b < p_1$. Since the raid would fail if fewer shares are tendered a shareholder who tenders fewer shares earns p_0 on each rather than $b > p_0$. Therefore no shareholder has an incentive to deviate from his proposed equilibrium strategy. Furthermore, there is no pure strategy equilibrium in which the number of shares tendered is not equal to K. If less than K are sold in some proposed equilibrium, then any shareholder choosing $\sigma_i < h_i$ prefers to choose a larger σ_i. If more than K are sold, then any shareholder choosing $\sigma_i > 0$ prefers to choose a smaller σ_i. Hence, in general, no symmetric pure strategy equilibria exist.[12]

Notice that, because exactly K shares are sold in these equilibria, each seller is pivotal. By this we mean that if any of them tenders any fewer shares, the raid will fail. Thus, given the strategies of the other shareholders, each seller, in effect, must choose whether or not the raid succeeds. Hence, no seller can free ride.

There are symmetric equilibria in mixed strategies if all shareholders have the same number

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[12] There is one special case in which there exists a symmetric pure strategy equilibrium in the subgame. In the event that $\frac{K}{I}$ is an integer and every shareholder owns at least $\frac{K}{I}$ shares, then every stockholder offering $\frac{K}{I}$ is a symmetric pure strategy equilibrium in the subgame.

of shares.[13] As an example, suppose that $h_i = 1$ for all i. Suppose

$$F_j(\sigma_j) = \begin{cases} 1 - \gamma & \sigma_j = 0, \\ 1 & \sigma_j = 1, \end{cases}$$

for all $j \neq i$. That is, each shareholder other than i sells his share with probability γ. Then the ith shareholder will also randomize if and only if

$$\text{(1)} \qquad b = \sum_{j=0}^{K-1} \binom{N-1}{j} \gamma^j (1-\gamma)^{N-1-j} p_0 + \sum_{j=K}^{N-1} \binom{N-1}{j} \gamma^j (1-\gamma)^{N-1-j} p_1.$$

For any b strictly between p_0 and p_1, the right-hand side of the above is strictly smaller than b at $\gamma = 0$ and strictly larger at $\gamma = 1$. Since the right-hand side is continuous in γ and strictly increasing in γ for all γ strictly between zero and one, there is a unique $\gamma \in (0,1)$ satisfying equation (1). Therefore, an equilibrium has each stockholder j choosing $F_j(\sigma_j)$ as given above where γ satisfies (1).

Clearly if the asymmetric pure strategy equilibrium is the one which occurs in the subgame, then the raider earns strictly positive profits for any $b \in (p_0, p_1)$. It is less obvious but also true that this holds if the symmetric mixed strategy equilibrium ensues instead. To see this, note that the raider's expected profits are

$$\sum_{j=0}^{K-1} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} j p_0 + \sum_{j=K}^{N} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} j p_1 - N\gamma b$$

Substituting for b from (1) and rearranging shows that this is in fact equal to

$$\text{(2)} \qquad \binom{N}{K} \gamma^K (1-\gamma)^{N-K} \left(p_1 - p_0 \right) K > 0.$$

This expression has an interesting interpretation because $\binom{N}{K} \gamma^K (1-\gamma)^{N-K}$ is the probability that exactly K shares are tendered. Thus, the raider's profits are proportional to the probability that each seller is pivotal. As with the asymmetric pure strategy equilibria, the raider's ability to

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[13] It is not clear how to define a symmetric equilibrium in mixed strategies when stockholders have different numbers of shares.

earn profits hinges on his ability to make each stockholder pivotal with positive probability.[14]

We can also explicitly analyze one particularly tractable class of asymmetric mixed strategy equilibria. Suppose at least $K+1$ stockholders own exactly one share. Suppose $S > K+1$ of these stockholders choose γ as given by equation (1) where S replaces N. Also, suppose that each of the remaining shareholders chooses not to tender any shares. This set of strategies constitutes an asymmetric mixed strategy equilibrium. Note that the equation defining γ implies that each of the S shareholders is indifferent between selling and not selling. This means that the stockholders who choose a nonzero probability of selling act to set b equal to the expected value of a share, where the probability of a successful raid is a calculated conditional on the event that they do not sell. Clearly, then, the expected value of a share when we do not condition on this event is larger than b. Hence all the stockholders who choose a probability of zero of selling their shares are, in fact, choosing optimally. Hence this is an equilibrium.[15] The raider's profits would be given by equation (2) where S replaces N. Hence, again, his profits are proportional to the probability that each seller is pivotal.

So we have shown that in three broad classes of equilibria in the induced subgame, the raider earns strictly positive expected profits and the raid succeeds with strictly positive probability for any b between p_0 and p_1. As discussed above, this occurs because the raider is able to make each seller pivotal (with positive probability). As the following lemma shows, the raider is always able to make at least one seller pivotal. In Theorem 1, we show that this ability insures that the raider earns strictly positive expected profits and that the raid succeeds with strictly positive probability in equilibrium. Lemma 1 and Theorem 1 are proved in Appendix A.

Lemma 1: *When the raider is restricted to any–and–all bids, in every equilibrium in the subgame induced by any $b \in (p_0, p_1)$, at least one seller is pivotal with strictly positive probability.*

Theorem 1: *When the raider is restricted to any-and-all bids, in every equilibrium, (i) $p_0 < b < p_1$, (ii) the raider earns strictly positive expected profits, and (iii) the raid*

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[14] Interestingly, if the symmetric mixed strategy equilibrium arises in the subgame induced by each b, the raider chooses b to maximize (2). This is equivalent to choosing γ to maximize (2). This means that he chooses γ to maximize the probability that each seller is pivotal. For additional details, see analagous calculations in Section 5.

[15] Palfrey and Rosenthal [1983] refer to equilibria with this form as "quasi-symmetric."

succeeds with probability strictly greater than zero as long as the raider is more efficient than current management.

It is important to point out that this result does require the raider to be more efficient than current management—i.e., that $p_1 > p_0$. Otherwise, the raid cannot succeed, as there is no $b \in (p_0, p_1)$. Though this condition seems obvious, we will see in Section 4 that given some strategy sets for the raider, there are equilibria in which less efficient raiders take over. In fact, we could restate Theorem 1 as: if the raider is restricted to any–and–all bids, he has a nonzero probability of taking over and will earn strictly positive profits in the attempt if and only if he is more efficient than current management.

Notice that the raider's expected profit is strictly positive for any I and N. Thus for any finite number of stockholders, the raid can succeed and earn profits for the raider. By contrast, previous writers seem to have considered the case of an infinite number of shareholders each owning an infinitesimal fraction of the firm. They have asserted that no equilibrium in such a game can have the raid succeed and earn the raider positive profits. A question of some importance, then, is what the set of equilibria converges to as the number of shareholders goes to infinity. That is, as the number of stockholders becomes large, do the probability of a successful raid and the expected profits of the raider go to zero? If so, then for sufficiently large N, our results would be insignificantly different from those of previous writers. In Appendix C, we show that under certain conditions, neither the probability of success nor the raider's expected profits go to zero. Because of this, the infinite number of stockholders approach does not appear to adequately approximate the large numbers case.

Section 3B. Conditional Bids without Exclusion.

In this section, we will consider the situation in which T contains conditional bids as well as any–and–all bids. With such a bid, the raider chooses three objects. First, he chooses a number b which he will pay for shares offered to him. Second, he chooses an integer number of shares he wishes to acquire; that is, he announces he will pay b for κ shares. If fewer than that number is offered to the raider, he buys no shares at all. If that number is offered, he buys all of them and pays b for each. The third object he chooses is a rationing device which determines how many he buys from whom in the event that more than κ shares are offered. We always require him to choose devices which specify that at least κ shares are purchased in this event. As an example of such a device, the Williams Act requires that in such an event, the raider must buy the same percentage of shares from each person who offers shares to him and that he buy κ shares in total. This is commonly referred to as purchasing on a *pro rata* basis. We will not restrict the raider to this specific device. However, the results we prove will *not* require that he has devices other than this to choose from. Notice that if $\kappa = 1$ and the raider chooses to buy all shares offered above κ, then the conditional bid is in fact identical to an any–and–all bid. Hence the strategy set considered here strictly contains the set considered in the last section.

Consider the following strategy for the raider. Suppose he offers infinitesimally more than p_0 for all N shares, i.e., $\kappa = N$. Clearly, one equilibrium in the induced subgame has all share-holders offerring all their shares. Any deviation from the proposed equilibrium strategy causes the shareholder to earn only $h_i p_0$, while selling all shares earns h_i times a number strictly larger than p_0. Notice also that another equilibrium has no one offering any shares for sale. If any shareholder other than i chooses to withhold some shares, then i is indifferent between offerring all his shares and offerring any number less since the raider will not purchase regardless of what i does.

This latter subgame equilibrium is not stable in the following sense. Suppose that i thinks that there is some tiny chance that all other stockholders will sell all of their stock. No matter how small that probability is, he will choose to sell all his stock. Thus the only stable equilibrium in the subgame is where all stockholders sell all their stock. More precisely, this is the only trembling–hand

10

perfect equilibrium in the subgame induced when the raider chooses this strategy.

Suppose we focus on equilibria in the two-stage game which have trembling–hand perfect equilibria in the subgame induced by any strategy choice of the raider. These are what we have defined as r–equilibria. In this case, if the raider offers the bid defined above, he will take over the firm with probability one and earn profits which can be made arbitrarily close to $(p_1 - p_0)N$. Thus he can obtain an amount arbitrarily close to the increase in the value of the firm. As is intuitively clear, this is the best he can do unless he is able to somehow take some of the firm's initial value from the stockholders–that is, unless exclusion is allowed. So if exclusion is not allowed, we have the following theorem which is proven more rigorously in the Appendix B.

Theorem 2:   *When the raider is restricted to conditional bids, the unique r-equilibrium has $t = (b^*, \kappa^*, \cdot)$, where (i) $b^* = p_0 + \delta$ for δ made arbitrarily close to but strictly greater than 0,[16] (ii) $\kappa^* = N$,[17] and (iii) the raid succeeds with probability 1 if the raider is more efficient than current management.*

This theorem implies that if conditional bids are allowed, then the raid succeeds with probability 1 if and only if the raider is more efficient than current management. This is, of course, stronger than the result of the last section. Notice also that this equilibrium is independent of I or the "size" of each shareholder relative to the firm. Hence it continues to be the unique equilibrium as each stockholder "becomes small" relative to the firm.

The strategy described in Theorem 2 yields profits for the raider that are arbitrarily close to the increase in the value of the firm. Clearly, this is the best he can do unless he is able to take some of the firm's initial value from the minority stockholders. A necessary condition for this is that he be able to exclude.

Corollary: *If the raider is allowed any strategy set containing conditional bids but not exclusion, an equilibrium of the game is the equilibrium given in Theorem 2. Further-*

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[16] Technically, the raider's best strategy is undefined, since he must choose δ strictly greater than zero but always is better off choosing it smaller. Since there is no smallest number strictly larger than zero, this leaves δ undefined. We will ignore this problem in all that follows and will simply write $b^* = p_0 + \delta$ where it is understood that $\delta > 0$, but can be taken to be arbitrarily small.

[17] The device is irrelevant because $\kappa^* = N$ implies that it is impossible to have more than κ^* shares offered for sale to the raider.

more, any other r–equilibrium has the same outcome.[18]

Notice that, unlike in Section 3A, there is a unique equilibrium when the strategy set is enlarged to permit conditional bids but bars any form of exclusion. A conditional bid with $\kappa = N$ makes *every* stockholder pivotal and thus willing to sell even for $p_0 + \delta$. The ability to make every stockholder pivotal enables the raider to extract all of the gains from better management of the firm. In Section 3A, the raider's strategy set was restricted to any–and–all bids. While the raider was able to make at least one seller pivotal with strictly positive probability, he was unable to extract all of the gains. Unlike when conditional bids are allowed, every stockholder selling every share was not an equilibrium because no one was pivotal in that situation.

Section 4. Bids with Exclusion.

Earlier authors have suggested that no takeover bid less than p_1 can succeed because no individual shareholder can affect the probability that the raid will succeed. Thus, given rational expectations, if a bid will succeed, no shareholder will tender for anything less than the value of his shares given success. Hence, they conclude that successful takeovers require that the value of a share to a minority stockholder be less than the value of that share to the raider if the raid succeeds. Different authors have considered different exclusionary devices.

Grossman and Hart suggest that dilution is an exclusionary device. The idea of dilution is that prior to a takeover the shareholders voluntarily accept a dilution of their property rights in the event of a takeover by adopting rules which allow the raider to exclude them from some of the increase in the value of a share. For example, the raider may do this by selling some of the firm's assets to another firm he controls at less than the market value of these assets. This tactic effectively transfers some of the value of the firm to him, preventing minority shareholders from receiving the full value of their shares in the event of a takeover.

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[18] With this strategy, the raider acquires all N shares at $p_0 + \delta$ per share. With an enlarged strategy set, there may be other strategies which induce the same outcome, so that the raider may choose one of these strategies instead. However, any strategy which does not induce this outcome must yield lower profits.

Bradley [1980] and Bradley and Kim [1984] focus on front–loaded two–tier bids as an exclusionary device. These bids are conditional bids specifing two prices, the front–end and the back–end price. The former is paid on the κ shares the raider seeks to acquire. If he takes over, then the minority stockholders are forced to sell their shares for the back–end price.[19] As long as the back–end price is less than p_1, the raider is able to exclude minority stockholders from some of the increase in the value of the firm. Consequently, this acts like dilution.

In the last section, we showed that, with any finite number of shareholders and with any distribution of those shares,[20] allowing conditional bids without exclusion is sufficient to guarantee that a takeover occurs whenever the raider is more efficient than current management. Hence exclusion is unnecessary for this purpose. This does not imply, of course, that exclusion cannot have any advantages. As we will show in this section, exclusion has beneficial effects from the point of view of society if any–and–all bids are the only bids allowed, but has harmful effects if conditional bids are allowed.

We will focus on dilution as an exclusionary device. Let ψ be the amount of dilution. Following Grossman and Hart, this means that in the event of a takeover a minority shareholder receives $p_1 - \psi$ rather than p_1. Thus if the raider takes over the firm by purchasing, say, k shares at a price of b, then his profits are $(p_1 - b)k + (N - k)\psi$. Notice that setting the back–end price equal to $p_1 - \psi$ and allowing conditional bids extends our analysis to front–loaded two–tier bids.[21]

First, consider the effect of dilution when the raider is restricted to any–and–all bids. For any $b > p_0$, if ψ is large enough so that $p_1 - \psi \leq p_0$, it is a dominant strategy in the subgame for each shareholder to sell all his shares. To see this, note that selling a share earns b while holding it earns p_0 if the raid fails and $p_1 - \psi$ if it succeeds. Since both of these are strictly less than b by assumption, each shareholder will tender all h_i shares. Since this is true for any $b > p_0$, clearly, the raider will choose the smallest b in this range. If he chooses b outside this range—that is, he sets $b \leq p_0$, then the raid will not succeed as no one selling is an equilibrium.[22] Notice also that

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[19] The Williams Act requires that the back–end price of such a bid be no smaller than p_0.

[20] By any distribution of those shares, we mean only to include distributions which satisfy $K > N - h_i$ for all i.

[21] Since the raider takes ψ as given, our treatment has the raider constrained to choose a particular back–end price. If the raider's best front–loaded two–tier bid has the back–end price as low as legally allowed, then this formulation entails no loss of generality.

[22] Technically, we cannot conclude that he will not take over with such a bid. It is also an equilibrium in the subgame for everyone to sell. As Grossman and Hart argue, this is a fairly unreasonable equilibrium. Because of this, we will focus

the raider's profits, which are arbitrarily close to $N(p_1 - p_0)$, will only be positive if he is more efficient than current management. If the raider is less efficient than current management, we must have $p_1 - \psi < p_0$. Thus less efficient raiders cannot takeover. Hence we have proven the following theorem and corollary.

Theorem 3: *If the raider is restricted to any-and-all bids and $p_1 - \psi \leq p_0$, then the unique equilibrium has*[23] $b = p_0 + \delta$ *and each shareholder tenders all h_i shares, if the raider is more efficient than current management.*

Corollary: *If a raider is restricted to any-and-all bids and is less efficient than current management, then the unique equilibrium has no bid made.*

Notice that this equilibrium is identical to the equilibrium discussed in Theorem 2 in *every* detail. In both equilibria, each stockholder sells all his shares and receives $p_0 + \delta$ for each. Thus the raid succeeds with probability 1 if the raider is more efficient than current management and the raid fails otherwise. Hence there is no difference to society or to the shareholders between allowing conditional bids without dilution and restricting the raider to any-and-all bids with sufficient dilution, *i.e.*, $\psi \geq p_1 - p_0$.

If $p_1 - \psi > p_0$, then the effects of permitting dilution are not as pronounced. In this case, it is simply as if p_1 had been made smaller from the point of view of shareholders. Thus, if the raider chooses $b \in (p_0, p_1 - \psi)$, then all the possibilities for equilibria in the induced subgame discussed in Section 3A are still possibilities here. If the raider chooses $b \in [p_1 - \psi, p_1)$, then the raider will take over with probability 1.

In Grossman and Hart's model, dilution enables the raider to make a bid of the maximum of $p_1 - \psi$ and p_0, which will succeed with probability 1. Any $\psi > 0$ enables the raider to choose a $b < p_1$ and to make tendering a dominant strategy for every shareholder. In our model, the same is true. However, raids are possible and profitable even if $\psi = 0$ because the raider can make some sellers pivotal. For any $\psi > 0$, the raider can still choose such a strategy and will do so unless ψ is

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only on the equilibrium in which no one sells in this subgame.

[23] Where δ is viewed in the same way as the δ of Theorem 2.

14

such that it is more profitable to make tendering a dominant strategy.[24]

The effects of allowing conditional bids and dilution are very different. To see this, suppose that $p_1 - \psi > p_0$. If conditional bids are allowed, dilution will have *no* effect at all in this range. Recall from Theorem 2 that the raider can make a bid t^* which causes all N shares to be tendered to him. In this event, his profits are (approximately) $(p_1 - p_0)N$. In general, if the raider chooses a bid of b and $\kappa \geq K$ shares are tendered, then his profits are $\kappa(p_1 - b) + (N - \kappa)\psi$. Given that $b \geq p_0$, this expression is maximized at $b = p_0$. Also, if $\psi < p_1 - p_0$, this is maximized at $\kappa = N$. Hence the equilibrium of Theorem 2 would still be the unique r–equilibrium in the game. Consequently, if ψ is in this range, dilution has no effect.

Now consider the case in which $\psi \geq p_1 - p_0$. Then an equilibrium will have the raider taking advantage of dilution. This equilibrium is presented and characterized in the following theorem. First, we will define a particular class of rationing devices which may be used by the raider in the event that more shares are tendered than he sought. An encouraging device will be defined as one such that for all i, if $\sum_{j \neq i} \sigma_j \geq \kappa$, then the ith shareholder's best strategy is $\sigma_i = h_i$. In other words, if the number of shares tendered by the other stockholders is at least the number requested by the raider, then each stockholder's best strategy is to tender all his shares.[25]

Theorem 4: *If the raider can make conditional bids and if $p_1 - \psi \leq p_0$, then an equilibrium has the raider offering the conditional bid[26] $b = p_0 + \delta$ for $\kappa = K$ shares with any encouraging device. In this equilibrium, the raid succeeds with probability 1.*

Proof: First, we will show that given the raider's bid, it is an equilibrium in the induced subgame for all stockholders to tender all their shares. This follows because if all shareholders other than i tender all their shares, then at least $N - h_i$ shares will be tendered. Since $N - h_i > K = \kappa$, this means that the number of shares tendered exceeds the number the raider will purchase. Since the rationing device is an encouraging device, by definition, this implies that the ith stockholder will choose $\sigma_i = h_i$.

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[24] This is not meant to imply that ψ has no effect on the equilibrium in this case. This issue is discussed in Section 5.

[25] It is easy to show that an example of such a device is precisely the device required by the Williams Act, namely, purchasing on a *pro–rata* basis.

[26] Again, see our earlier discussion on the treatment of δ.

Given this equilibrium in the subgame, the raider's expected profit from this strategy is arbitrarily close to $K(p_1 - p_0) + (N - K)\psi$. But the raider cannot earn any higher profit than this from any alternative strategy. ∎

It is important to note that this equilibrium is not unique because the equilibrium in the subgame induced by the raider's choice of this strategy is not unique. For example, as with the bid made in Theorem 2, another equilibrium in the subgame would have all shareholders refusing to tender any shares. Unlike the situation in that section, though, this equilibrium can be trembling-hand perfect. There, the shareholder would gain δ per share if everyone offered all their shares and would gain nothing otherwise. Hence, with even a tiny probability that everyone else tenders all their shares, the remaining shareholder would tender all of his. In this section, if everyone offers all their shares, each shareholder receives $p_0 + \delta$ for some of his shares and the smaller amount $p_1 - \psi$ for each of his remaining shares. Hence he is worse off if everyone offers all their shares and so the instability of the analogous equilibrium in Section 3B is not present here.

One other important characteristic of this equilbrium is that it can be an equilibrium even if the raider is less efficient than current management. The following theorem elucidates this point.

Theorem 5: *If the raider is allowed conditional bids, then for any $\psi > 0$, there exists a $\varrho(\psi) < p_0$ such that for any $p_1 > \varrho(\psi)$, there is an equilibrium in which the raid succeeds with probability 1. In other words, there always exists a lower bound on p_1 which is strictly smaller than p_0 for which the raid succeeds with probability 1 in an equilibrium.*

Proof: Since $p_1 - \psi < p_0$, the conditions for Theorem 4 are met.[27] The raider's profit in the equilibrium discussed there is arbitrarily close to $K(p_1 - p_0) + (N - K)\psi$. This is strictly positive if and only if

$$p_1 > p_0 - [(N - K)/K]\psi.$$

The right-hand side is strictly smaller than p_0 for any $\psi > 0$. Hence one can define $\varrho(\psi)$ to be the right-hand side, which will complete the proof. ∎

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[27] Note that if $p_1 < p_0$, then $p_1 - \psi < p_0$ for any $\psi > 0$.

Notice that this implies that with any nonzero level of dilution and conditional bids, there is an equilibrium in which a raider who is less efficient than current management takes over. If the raider is constrained to any–and–all bids or if the allowed level of dilution is zero, this is never true.

Section 5. Stockholders' and Society's Rankings.

In order to rank the outcomes implied by various strategy sets for the raider, we must specify the criteria used. We present social and private rankings of these outcomes as viewed prior to the observation of p_1. This is as if, in a "stage 0," we rank corporate charters which restrict the set of takeover bids that will be considered. At this time, the raider is not known and hence p_1 is not known. Thus the stockholders and society will be taking expectations with respect to p_1 using some exogenously given prior distribution.[28]

We assume that society wishes to have the firm run by the most efficient manager possible, be that current management or the raider, and that society is indifferent about the distribution of wealth between the stockholders and the raider.[29] Given this goal, we immediately see that society prefers conditional bids with no dilution to conditional bids with any positive level of dilution. Recall that if the raider is allowed conditional bids with no dilution, the raid succeeds with probability 1 if the raider is more efficient than current management and fails with probability 1 otherwise. By contrast, if dilution is allowed, then there are equilibria in which inefficient raiders succeed with probability 1. While there are other equilibria, allowing dilution cannot improve the set of outcomes and can lead to inefficient ones. Thus, regardless of which equilibria society expects to obtain, no dilution is preferred.

Notice, though, that the equilibrium when dilution is sufficiently large and the raider is

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[28] The assumption that the priors are exogenous is important here. For example, we could imagine the following process "generating" p_1. Suppose that a raider observes the firm and can choose to undertake some expenditure to find out how efficiently he could run the firm—that is, to find out his p_1. Then the profit he could earn for any given p_1 would crucially affect whether or not he would undertake the expenditure. Since the provisions in the corporate charter would be a key determinant of this, these provisions would affect the probability of observing a raider at all. Furthermore, if the raider has incomplete information about the current value of the firm at the time he must make his decision, then the provisions of the charter may provide information which would affect his decision.

[29] This is analogous to maximizing the sum of consumer's surplus and profits.

restricted to any–and–all bids is identical to the equilibrium with conditional bids and no dilution. Hence society is indifferent between conditional bids with no dilution and any–and–all bids with infinite dilution.[30] Since both permit the raider to capture all of the increment in the value of the firm and *no more*, the raider achieves the socially efficient outcome in either situation. Hence the socially optimal level of dilution cannot be defined independently of the restrictions on the raider's strategy set.

If the raider is restricted to any–and–all bids, society prefers infinite dilution to any finite level. For any finite ψ, there is a strictly positive probability that $p_1 - \psi > p_0$. If so, there are equilibria with a strictly positive probability of failure of an efficient takeover.

Society's preferences over finite values of ψ, given a restriction to any–and–all bids, is more complicated. Because of the multiplicity of equilibria, it is not possible to unambiguously compare any equilibrium with one value of ψ to any equilibrium with a different value of ψ. That is, consider a fixed p_1 and suppose $\psi = 0$. Consider an equilibrium where the b chosen by the raider induces a pure strategy equilibrium in the subgame. In this equilibrium, the raid succeeds with probability 1. For the same p_1 and some value of $\psi > 0$, we can focus on a mixed strategy equilibrium. In such an equilibrium, the probability that the raid will succeed is strictly less than one. But clearly, we cannot conclude that increases in ψ lower the probability of a successful takeover since we could focus on the mixed strategy equilibrium at $\psi = 0$ and the pure strategy equilibrium for the strictly positive ψ and reverse this conclusion.

However, we can analyze society's preferences within a class of equilibria. That is, it is clear that society is indifferent about ψ given that, for any value of ψ under consideration, we focus only on pure strategy equilibria. This is true because the probability of success for an efficient raider is always 1 and the probability of failure for an inefficient raider is also 1. Suppose, instead, we focus on symmetric mixed strategy equilibria. In this case, in comparing two values of ψ, society always prefers the larger value. To see this, note that γ is now determined by

$$(3) \qquad b = \sum_{j=0}^{K-1} \binom{N-1}{j} \gamma^j (1-\gamma)^{N-1-j} p_0 + \sum_{j=K}^{N-1} \binom{N-1}{j} \gamma^j (1-\gamma)^{N-1-j} (p_1 - \psi),$$

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[30] If the distribution of p_1 has a finite support, we could replace "infinite dilution" with "ψ equal to the maximum possible $p_1 - p_0$" in this statement.

and the raider's profits $\pi(b)$ are given by

$$\sum_{j=0}^{K-1} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} j p_0 + \sum_{j=K}^{N} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} [j p_1 + (N-j)\psi] - N\gamma b$$

$$= \sum_{j=0}^{K-1} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} j p_0 + \sum_{j=K}^{N} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} j (p_1 - \psi) - N\gamma b$$

$$+ N\psi \sum_{j=K}^{N} \binom{N}{j} \gamma^j (1-\gamma)^{N-j}.$$

Substituting for b and rearranging,

$$\binom{N}{K} \gamma^K (1-\gamma)^{N-K} (p_1 - \psi - p_0) + N\psi \sum_{j=K}^{N} \binom{N}{j} \gamma^j (1-\gamma)^{N-j}$$

$$= \binom{N}{K} \gamma^K (1-\gamma)^{N-K} (p_1 - p_0)$$

$$+ \psi [N \sum_{j=K}^{N} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} - \binom{N}{K} \gamma^K (1-\gamma)^{N-K} K].$$

By assumption, any b chosen induces the symmetric mixed strategy equilibrium. Thus, any b chosen induces a γ in the subgame so that the profit–maximizing bid corresponds to a profit–maximizing γ. That is, the raider will choose γ to maximize this expression.[31]

The derivative of (3) with respect to ψ, is

$$N \sum_{j=K}^{N} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} - K \binom{N}{K} \gamma^K (1-\gamma)^{N-K},$$

which may be rewritten as

$$\sum_{j=0}^{N} \binom{N}{j} \gamma^j (1-\gamma)^{N-j} f(j),$$

where

$$f(j) = \begin{cases} 0 & \text{for } 0 \leq j \leq K-1, \\ N-K & \text{for } j = K, \\ N & \text{for } K+1 \leq j \leq N. \end{cases}$$

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[31] In other words, equation (3) above defines a functional relationship between γ and b. The raider chooses the point on this function that maximizes his profit. All we are doing is substituting the constraint into the objective function and maximizing with respect to γ.

Hence it is the expected value of a function increasing in j. Since an increase in γ shifts the distribution of j to the right in the sense of first-order stochastic dominance, this must increase the expected value of any increasing function of j. Thus, this expression is strictly increasing in γ.

This implies that the cross partial of profits with respect to γ and ψ is strictly positive. Using this condition, the application of the implicit function theorem immediately implies that the raider's choice of γ is increasing in ψ. Since the equilibrium value of γ increases with ψ, the probability of a successful takeover increases with ψ. Thus society is always made better off by a larger ψ given the restriction to any-and-all bids.[32]

Finally, we cannot determine society's preference between any-and-all bids with finite dilution and conditional bids with dilution. Recall that we have shown that when conditional bids and dilution are allowed, there is an equilibrium in which a raider takes over with probability 1 whether he is more or less efficient than current management. We have also shown that with any-and-all bids with dilution, there are equilibria in which efficient takeovers succeed with probability 1, others in which they succeed with positive probability (less than 1), but none in which an inefficient takeover succeeds with nonzero probability. Consequently, without being able to specify which equilibrium is attained, we cannot say which set of restrictions society prefers. Our problem is that there is no accepted procedure in game theory to determine what outcome is obtained when there exist multiple equilibria.

The natural way to analyze the preferences of the stockholders as a group would be to use the Pareto ordering. In general, this is not possible because the stockholders may value equilibria differently. However, if the stockholders can sell their stock to one another, then the preferred set of restrictions will be that set which maximizes the total expected revenue to the stockholders. To see this, suppose that we are comparing two alternative restrictions, A and B. Let the former lead to an equilibrium in which the total returns to stockholders is smaller. If B is Pareto preferred to A, then there is no difficulty ranking these alternatives. If, however, B is not Pareto preferred, then some stockholder, say i, receives a higher expected payment when A is chosen. But since the total expected payment is lower in this case, it must be true that some other stockholder, say j, receives a lower expected payment and that his loss from the choice of A is greater than i's gain.

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[32] Note that this result will extend, in a straightforward manner to the class of asymmetric mixed strategy equilibria discussed in Section 3A. We conjecture that it extends to all asymmetric mixed strategy equilibria.

Then j would be willing to purchase i's stock for more than its value to i under option A in order to ensure that B be chosen instead. Hence the relevant ranking of these two alternatives from the stockholders' perspective would certainly be that B is preferred to A.

Given these criteria, we see immediately that the rankings of the stockholders differ from the rankings of society. In particular, stockholders prefer requiring any–and–all bids with no dilution to either conditional bids with no dilution or any–and–all bids with infinite dilution. This is true because in the only equilibrium with either of the latter two strategy sets, total receipts to stockholders are arbitrarily close to Np_0. However, with any–and–all bids without dilution, total receipts will always be higher because no share earns less than p_0 and with positive probability some shares earn p_1.

As noted in the analysis of society's preferences, the multiplicity problem implies that we cannot unambiguously compare any two equilibria with any–and–all bids and different values of ψ. Again, we can focus on comparisons within a class of equilibria. Here, however, difficulties remain. Consider first the asymmetric pure strategy equilibria. Since total receipts to stockholders must be $Kb + (N - K)(p_1 - \psi)$, stockholders prefer lower values of ψ. Consider now the symmetric mixed strategy equilibrium. Note that the expected payoff to each shareholder in such an equilibrium is b. It is easy to construct examples in which the ψ which maximizes b for certain values of p_1 is strictly positive and finite. Consequently, if these values of p_1 are very likely *ex ante*, the privately optimal ψ is strictly positive and finite. Clearly, given the multiplicity of equilibria, it is impossible to characterize the privately optimal ψ when the raider is resriced to any–and–all bids.

Of all of the alternatives we consider, the stockholders' least preferred alternative is conditional bids with dilution. To see this, recall that the raider can guarantee $N(p_1 - p_0)$ for himself by using the strategy discussed in Theorem 2. Hence he will only choose a strategy which yields at least the expected payoff from this strategy. Since the raid succeeds with probability 1 when the raider uses this strategy, the total payoffs to all players are as large as they can possibly be. Thus since no alternative strategy can increase total payoffs, any strategy which makes the raider better off must lower expected payments to the stockholders. In short, not surprisingly, the stockholders would prefer to give the raider less ability to take wealth from them.

Thus, we see that the stockholders' prefer a socially inefficient outcome (any–and–all bids without dilution) to the socially efficient outcomes (any–and–all bids with sufficient dilution or conditional bids without dilution). The social and private rankings are summarized in the table below where we have abbreviated "any–and–all" by A, and abbreviated "conditional bids" by C. Entries in either of the column A's are listed so that moving up the column gives strictly better outcomes. An entry in column B is noncomparable to the entry in column A appearing in the same row. It is preferred to an entry in any lower row of column A. Thus, for example, C, $\psi > 0$ in the same row as A, $\psi \geq 0$ for society means that we cannot compare the two but that they are both worse than A, $\psi = \infty$ and C, $\psi = 0$.

Society		Stockholders	
A	B	A	B
$A, \psi = \infty;\ \ C, \psi = 0$ $A, \psi \geq 0$	$C, \psi > 0$	$A, \psi = 0$ $C, \psi = 0;\ A, \psi = \infty$ $C, \psi > 0$	$A, \psi > 0$

Our results concerning exclusion differ from the conclusions of previous writers. Since we have shown that exclusion is not necessary for takeover bids to succeed, exclusion is not necessarily socially optimal. In particular, exclusion may allow inefficient raiders to take over. Also, the trade–off for stockholders is different from that perceived by other writers. Grossman and Hart, for example, suggest that an increase in ψ is beneficial in that it makes the takeover more probable and is harmful in that it reduces the raider's bid. Consequently, they argue that the privately optimal ψ is finite and strictly positive. In our model, increases in ψ may make the takeover more probable, but may not.[33] Furthermore, increases in ψ may actually increase the bid. Hence, neither of the consequences of an increase in ψ identified by Grossman and Hart necessarily result.

Finally, we noted that the stockholders' ranking differed from the social ranking. This occurs because in the socially efficient solutions presented, all of the gains from trade accrue to the raider. Stockholders prefer restrictions on the raider which alter the distribution of the gains

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[33] Obviously, the probability of takeover cannot increase if it is already 1.

from trade in their favor. The fact that stockholders' preferences over these alternatives differ from society's does not necessarily imply that private corporate charters will be socially sub–optimal. Since we have only considered a very restricted class of possible strategy sets for the raider, we cannot conclude that there is no strategy set which is both socially and privately optimal.[34]

We believe that further analysis of this issue requires the inclusion of details that have been omitted from this model. These include, especially, incomplete information, defensive efforts by the current management, and competing raiders. We believe that an explicit multi–stage game would facilitate a clearer understanding of the issue. For example, conditional bids with $\kappa = N$ can be blocked by a competing raider who has been able to acquire some shares. Given this, society may prefer restricting the raider to any–and–all bids with infinite dilution. We are continuing to research this issue.

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[34] To make this point forcefully, consider the following example. Suppose that stockholders can write into their corporate charter that only conditional bids with $\kappa = N$, $b = p_1 - \delta$, and no dilution will be considered. Such a restriction is socially optimal because every efficient raider will takeover and no inefficient ones will. Also, this restriction is the best that shareholders can do for themselves. Obviously, it seems unreasonable to allow such provisions in the model since this relies heavily on the rather poor assumption of complete information.

Lemma 1: *When the raider is restricted to any-and-all bids, in every equilibrium in the subgame induced by some $b \in (p_0, p_1)$, at least one seller is pivotal with strictly positive probability.*

Proof. Suppose not. Then the probability that any given shareholder is pivotal in equilibrium is zero. This implies that we cannot have the raid succeed with probability 1. If it did, because no shareholder is pivotal, each would prefer to sell no shares as $p_1 > b$. Also, we cannot have the raid fail with probability 1. Since no shareholder is pivotal, in this event, each would prefer to sell all shares because $b > p_0$, another contradiction.

So the raid succeeds with probability strictly between zero and one. Let s_i be the smallest number of shares and ℓ_i be the largest number of shares that the ith shareholder tenders with nonzero probability in equilibrium.[35] Since the raid may fail, $\sum_i s_i \leq K - 1$. Further, since no shareholder is pivotal,

$$(A.1) \qquad \sum_{j \neq i} s_j + \ell_i \leq K - 1 \qquad \forall i.$$

That is, since the probability that i affects the outcome is zero, then i cannot affect the outcome in the event that each other shareholder sells s_j shares. Further, since the raid may succeed, $\sum_i \ell_i \geq K$. Again, since no shareholder is pivotal,

$$(A.2) \qquad \sum_{j \neq i} \ell_j + s_i \geq K \qquad \forall i.$$

More generally, for any set of possible choices by the other shareholders, the choice of the ith shareholder cannot affect the outcome. To develop the implications of this, we let $I = \{1, 2, \ldots, I\}$ be the set of shareholders and let $I \backslash \{i\}$ be the set of shareholders other than i. In equilibrium,

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[35] If i chooses a pure strategy, then $s_i = \ell_i$.

for any partition of $I\setminus\{i\}$ into two sets, there is a nonzero probability that for each j in one set, j chooses s_j, and for each j in the other set, j chooses ℓ_j. A typical partition is denoted by I_i^s and I_i^ℓ, where the former is the set of j who choose s_j. Then the fact that i cannot be pivotal implies that for any such partition, for all i, either

$$(A.3) \qquad \sum_{j \in I_i^s} s_j + \sum_{j \in I_i^\ell} \ell_j \leq K - 1 - \ell_i$$

or

$$(A.4) \qquad \sum_{j \in I_i^s} s_j + \sum_{j \in I_i^\ell} \ell_j \geq K - s_i.$$

We will now show by induction that (A.3) holds for all partitions.

First consider I_i^ℓ containing a single element, say $\{k\}$. Suppose (A.3) does not hold so that (A.4) must. Then

$$\sum_{j \neq i,k} s_j + \ell_k \geq K - s_i$$

which implies $\sum_{j \neq k} s_j + \ell_k \geq K$. But this contradicts (A.1). Therefore, (A.3) must hold for I_i^ℓ containing one member.

Now we will show that if (A.3) holds for all I_i^ℓ containing n members for all i, it holds for all I_i^ℓ containing $n+1$ members for all i. By hypothesis, for any I_k^ℓ containing n members,

$$\sum_{j \in I_k^s} s_j + \sum_{j \in I_k^\ell} \ell_j \leq K - 1 - \ell_k \quad \forall k.$$

Choose $i \in I_k^s$. Rearranging,

$$\sum_{j \in I_k^s, j \neq i} s_j + \sum_{j \in I_k^\ell} \ell_j + \ell_k \leq K - 1 - s_i.$$

Since

$$\sum_{j \in I_k^\ell} \ell_j + \ell_k = \sum_{j \in I_i^\ell} \ell_j,$$

25

for a set $I_i^\ell = I_k^\ell \cup \{k\}$, we can write this as

$$\sum_{j \in I_i^s} s_j + \sum_{j \in I_i^\ell} \ell_j \le K - 1 - s_i < K - s_i.$$

Obviously, this contradicts (A.4) for this partition so that (A.3) must hold. Since I_i^ℓ contains $n+1$ members and since every set containing $n+1$ members can be constructed similarly, we have that (A.3) holds for every partition.

In particular, (A.3) must hold for all I_k^ℓ containing $I-2$ members. Hence,

$$s_i + \sum_{j \ne i,k} \ell_j \le K - 1 - \ell_k.$$

This implies that

$$s_i + \sum_{j \ne i} \ell_j \le K - 1 < K$$

which contradicts (A.2). Therefore, it is impossible to have no one pivotal when the raid succeeds with probability strictly between zero and one. Since we have already shown that it is impossible to have no one pivotal otherwise, we must have at least one shareholder pivotal with strictly positive probability in equilibrium.∎

Theorem 1: *When the raider is restricted to any-and-all bids, in every equilibrium, (i) $p_0 < b < p_1$, (ii) the raider earns strictly positive expected profits, and (iii) the raid succeeds with probability strictly greater than zero as long as the raider is more efficient than current management.*

Proof: Note that the raider earns zero profits by offering a bid less than or equal to p_0.[36] Clearly, $b \le p_1$, since otherwise the raider loses money on every share purchased. If $b = p_1$, the raider earns negative profits in the event that the raid fails and zero profits if it succeeds. Hence $b = p_1$ or $b \le p_0$ will only be chosen if he cannot earn strictly positive expected profits with some

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[36] Also, a bid equal to p_0 cannot succeed since no shareholder will sell at that price unless the probability of success of the raid is zero.

other bid. We now show that *any* $b \in (p_0, p_1)$ must earn strictly positive expected profits and so some b in this range will be chosen.

First, notice that if $b > p_0$, then the probability that the raid succeeds must be strictly positive. Otherwise, all shareholders will face a choice between b per share and p_0 per share. If so, all shareholders would choose to sell all their shares and the takeover would succeed with probability 1, a contradiction.

We have already seen that the only pure strategy equilibria possible in the subgame have exactly K shares sold. In these equilibria, the raid succeeds with probability 1 and the raider earns profits of $K(p_1 - b)$. Hence the statement of the theorem is clearly true if this is the equilibrium in the induced subgame. So suppose instead that we have a mixed strategy equilibrium in the subgame.

We will now show that in such an equilibrium, at least one of the shareholders who randomizes is pivotal with strictly positive probability. By Lemma 1, some shareholder must be pivotal with strictly positive probability. Suppose that every pivotal stockholder is one choosing a pure strategy. Therefore, the outcome is independent of the choice of any other stockholder. Thus, the raid either succeeds with probability 1 or fails with probability 1. In neither case will any other shareholder randomize.

Let i be a shareholder who randomizes between selling two distinct numbers of shares and who is pivotal with strictly positive probability. Let s be the smallest and ℓ be the largest number of shares he tenders with positive probability. Define $\phi_i(k)$ as the probability that the shareholders other than i offer at least $K - k$ shares. That is, $\phi_i(k)$ is the probability that the raid will succeed given that i sells k shares.

Since i associates nonzero probability with selling s shares and ℓ shares, he must receive the same expected payoff from either choice. Hence we must have

$$\ell b + (h_i - \ell)[\phi_i(\ell)p_1 + (1 - \phi_i(\ell))p_0] = s b + (h_i - s)[\phi_i(s)p_1 + (1 - \phi_i(s))p_0]$$

which can be rewritten as

$$(3) \qquad (\ell - s)[b - \phi_i(s)p_1 - (1 - \phi_i(s))p_0] + (h_i - \ell)[\phi_i(\ell) - \phi_i(s)](p_1 - p_0) = 0$$

Since ℓ must be no larger than h_i and since $p_1 > p_0$, the second term must be nonnegative if $\phi_i(s) < \phi_i(\ell)$.

Note that

$$\phi_i(k) = 1 - \text{Prob}\Big[\sum_{j \neq i} \sigma_j < K - k\Big] = 1 - G_i(K - k),$$

where $G_i(\cdot)$ is the distribution function for $\sum_{j \neq i} \sigma_j$. But since the distribution function must be nondecreasing in its argument, we have $G_i(K - k)$ nonincreasing in k so that $\phi_i(k)$ is nondecreasing in k. Hence the fact that $s < \ell$ implies that $\phi_i(s) \leq \phi_i(\ell)$. Further, suppose that they are equal. If so, then there is no event with strictly positive probability in which i's choice changes the outcome. But then i is not pivotal, a contradiction. Thus, $\phi_i(s) < \phi_i(\ell)$. Therefore, the second term in (3) must be nonnegative and so the first term is nonpositive. That is,

$$b \leq \phi_i(s)p_1 + (1 - \phi_i(s))p_0.$$

Now, let $\rho_i(k) = F_i(k) - F_i(k - 1)$ which is the probability that i sells exactly k shares. Also let Φ be the probability that the raid succeeds. Then by the relationship between joint, marginal, and conditional distributions,

$$\Phi = \sum_{k=s}^{\ell} \phi_i(k)\rho_i(k).$$

That is, Φ is a convex combination of $\phi_i(k)$ for values of k that i chooses with nonzero probability. Since $\phi_i(\ell) > \phi_i(s)$, we have $\Phi > \phi_i(s)$ which immediately implies

$$\Phi p_1 + (1 - \Phi)p_0 - b > 0.$$

That is, the raider's expected profit per share acquired is strictly positive. Since $\Phi > 0$, it must also be true that the expected number of shares acquired is strictly positive.

These two statements imply that the raider's expected profits are strictly positive. To see

this, let n be the number of shares the raider acquires. Let $p(n)$ be the value of a share given that the raider acquires n shares. Of course, $p(n) = p_1$ if $n \geq K$, and is p_0 otherwise. Thus $p(n)$ is an increasing function of n. We have shown that $E(p(n) - b) > 0$ and that $E(n) > 0$. Since $p(n)$ and thus $p(n) - b$ is increasing in n, one can show (see Tong [1980]) that $\text{Cov}(p - b, n) > 0$. But then $E[(p - b)n] > E(p - b)E(n)$. Since the latter is strictly positive, the former, which is the raider's expected profit, is also strictly positive. ∎

Theorem 2: *When the raider is restricted to conditional bids, the unique r-equilibrium has $t = (b^*, \kappa^*, \cdot)$, where (i) $b^* = p_0 + \delta$ for δ made arbitrarily close to but strictly greater than 0, (ii) $\kappa^* = N$, and (iii) the raid succeeds with probability 1 if the raider is more efficient than current management.*

Proof: An ϵ-equilibrium in the subgame induced by the raider's choice of t is defined as an equilibrium in which each stockholder is restricted to choosing each of his possible pure strategies with probability no less than $\epsilon > 0$. A trembling–hand perfect equilibrium in the subgame is any equilibrium which is the limit of ϵ-equilibria for some sequence of ϵ's converging to 0. We proceed by first showing that the only ϵ-equilibrium in the subgame induced by this strategy for the raider has each stockholder putting ϵ weight on each possible pure strategy other than selling all shares. This immediately implies that the only trembling–hand perfect equilibrium has each stockholder tendering all his shares.

Given that every shareholder other than i has put nonzero probability on every one of his possible pure strategies, it is clear that the probability of any specific number[37] of shares being tendered by these stockholders is strictly positive. If any total number of shares other than $N - h_i$ is offered by these shareholders, then the raid will not take place even if i sells all of his shares. Thus, the ith stockholder is indifferent between all his possible strategies. But if the number of shares tendered by the other stockholders is exactly $N - h_i$, then i strictly prefers selling all his shares as he receives $b > p_0$ for each if he sells all and p_0 otherwise. Thus the expected payoff to i from selling all his shares is strictly larger than the expected payoff to selling any other number of shares. Hence his best strategy, given that he must choose each possible pure strategy with positive probability in an ϵ-equilibrium, is to put ϵ probability on each pure strategy other than selling all his shares. Since this is true for each shareholder, the unique trembling-hand perfect equilibrium in this subgame has each shareholder tendering all his shares.

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[37] Between 0 and $N - h_i$, of course.

This implies that if the raider chooses the strategy specified in the theorem, then in any r–equilibrium, the raid must succeed with probability 1. Hence the raider's profit from this strategy must be arbitrarily close to $N(p_1 - p_0)$. If a strategy is to earn higher profits, then, because exclusion is prohibited, it would have to be true that stockholders sold some of their shares at a price less than p_0, which they would never choose to do.

To show that this is the unique equilibrium, we must establish that no other strategy yields as high a payoff for the raider. It is clear that any strategy yielding as high a payoff for the raider must have all N shares sold to the raider with probability 1 for $p_0 + \delta$ per share. Thus any such strategy must have the same b as the strategy specified. Furthermore, any such strategy must have $\kappa = N$. To see this, suppose that the raider chooses $b = p_0 + \delta$ and $\kappa < N$. Then there is no pure strategy equilibrium in the induced subgame with all N shares tendered. If every other stockholder tenders all his shares, then stockholder i can withhold up to $N - \kappa$ shares and earn $p_1 > b$ per share. Thus no other strategy earns profits as high as this one for the raider. Therefore, this is the unique r–equilibrium. ∎

APPENDIX C: Convergence of Equilibrium with Any–and–All Bids.

As discussed in the text, previous writers have argued that without exclusion, the free rider problem prevents successful takeovers when there is an infinite number of "small" stockholders.

We examine the limit of the equilibria in our two–stage game as the number of stockholders goes to infinity. The basic question to be answered is the following: Is the outcome with an infinite number of stockholders a reasonable approximation to some equilibrium outcome with a finite but "large" number of stockholders? If so, then differences between our conclusions and those of previous writers would become irrelevant when considering very large corporations—firms which provide much of the interest in takeovers. We will show that, under certain conditions, the probability of success and the expected profits of the raider will not converge to zero. As a result, the large, finite case is not similar to the infinite case considered previously. We will use the phrase "convergence fails" to indicate this.

To show this, we will analyze how certain equilibria with some finite number of stockholders change as the number of stockholders increases without bound. The multiplicity of subgame equilibria makes an analysis of convergence in the two–stage game difficult. For this reason, we focus on equilibria in which the equilibrium in each subgame is of a particular type. For example, we consider the case in which the asymmetric pure strategy equilibrium obtains in the subgame induced by each b for each number of shareholders. We consider three cases corresponding to the three examples in Section 3A.

Let $p_0(N)$ be the value of p_0 when there are N shares of stock and define $p_1(N)$ analogously. Since the number of shares of stock in the firm will be increasing, the total value of the firm ($N p_0(N)$ or $N p_1(N)$, depending on which is relevant) will be approaching infinity if $p_0(N)$ and $p_1(N)$ do not approach 0 sufficiently quickly. If one argues that the value of an individual's holdings should not be getting small as N increases, then $p_0(N)$ and $p_1(N)$ should not decline with N. An alternative

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[38] We focus on any–and–all bids for two reasons. First, as noted in Section 3B, the analysis of convergence with conditional bids is trivial. Second, this is the case considered by Grossman and Hart [1979].

procedure would be to keep the value of the firm constant as N increases. However, it is clear that a large firm typically is a more valuable firm—that is, firms with a larger number of shares of stock outstanding tend to be worth more money in total. Therefore, when examining how the equilibrium changes as the number of shareholders increases, we should allow the value of the firm to increase as well. We will show that the equilibrium outcome will not converge to the outcome previous writers have identified in general if $Np_0(N)$ and $Np_1(N)$ rise sufficiently quickly with N. In fact, some cases will only require that these terms be constant in N.

We will let $K(N)$ be the smallest integer larger than αN, where α is some fixed fraction of the firm which must be acquired for the raider to gain control. Thus this fraction is not a function of N, though, of course, the total number of shares to be acquired will be.

Consider first the equilibria in the two–stage game when we restrict attention to the asymmetric pure strategy equilibria in the induced subgame. In this case, for any $b \in (p_0, p_1)$, the raid succeeds with probability one and the raider's profits are $K(N)(p_1(N) - b)$. Hence, the raider chooses $b = p_0(N) + \delta$. Clearly, since the probability that the raid succeeds is one for all N, the limit of the probability of success as the number of shareholders goes to infinity is one. Furthermore, the raider's profits converge to a strictly positive number if $K(N)(p_1(N) - p_0(N))$ does not converge to zero. Since $K(N)$ is of order N, this will certainly hold if the value of the firm under either management is non–decreasing in N. In other words, so long as $p_1(N)$ and $p_0(N)$ are at least of order N^{-1}, the raider's profits do not converge to zero. Therefore, this type of equilibrium fails to converge.

Now, we restrict our attention to the symmetric mixed strategy equilibrium in the induced subgame. In the text, we showed that the raider's profits for any N as a function of the equilibrium value of γ are

$$\binom{N}{K} \gamma^K (1 - \gamma)^{N-K} K(p_1 - p_0).$$

Clearly, for any fixed N, we would replace K with $K(N)$, etc., to obtain the correct expression for profits in the notation of this Appendix:

$$\binom{N}{K(N)} \gamma^{K(N)} (1 - \gamma)^{N-K(N)} K(N)(p_1(N) - p_0(N)).$$

As argued in the Section 5, we can analyze the raider's choice of γ to maximize this expression. The first-order condition for this maximization problem is

$$K(N)\gamma^{K(N)-1}(1-\gamma)^{N-K(N)} - (N-K(N))\gamma^{K(N)}(1-\gamma)^{N-K(N)-1} = 0.$$

Clearly, the raider chooses $\gamma = \frac{K(N)}{N}$ which converges to α as $N \to \infty$. It is straightforward to use standard central limit theorem arguments to show that this implies that the probability that the takeover succeeds does not go to zero.

The raider's equilibrium expected profits are

$$(C.1) \qquad \binom{N}{K(N)}\left(\frac{K(N)}{N}\right)^{K(N)}\left(\frac{N-K(N)}{N}\right)^{N-K(N)} K(N)(p_1(N) - p_0(N)).$$

Note that the first term is the probability that exactly $K(N)$ shareholders tender, i.e., the probability that every seller is pivotal in equilibrium. The second term is the increase in value of $K(N)$ shares. With an infinite number of stockholders, the probability that any stockholder is pivotal is zero. This does not imply that the raider's profits go to zero because the second term goes to infinity if the value of the firm goes to infinity. We now show that the first term only goes to 0 as fast as $\sqrt{N}$ goes to infinity.[39]

Stirling's formula (see, e.g., Billingsley [1979]) states that

$$n! \approx \sqrt{2\pi}\, n^{n+(1/2)} e^{-n}.$$

Substituting into (C.1) and rearranging yields

$$\left(\frac{1}{1-\frac{K(N)}{N}}\right)^{\frac{1}{2}}\left(\frac{\sqrt{K(N)}(p_1(N) - p_0(N))}{\sqrt{2\pi}}\right).$$

The first term converges to the constant, $\sqrt{\frac{1}{1-\alpha}}$. Since $K(N)$ goes to infinity as fast as N, this requires $p_1(N)$ and $p_0(N)$ to go to zero no faster than $\sqrt{N}$ goes to infinity. That is, if the value of the firm rises at least as fast as $\sqrt{N}$, then the raider's profits will not converge to zero as $N \to \infty$.

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[39] This result is a generalization of a result in Chamberlain and Rothschild [1981].

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