THE LOCAL INDEX FORMULA

IN NONCOMMUTATIVE GEOMETRY

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Abstract. *In noncommutative geometry a geometric space is described from a spectral point of view, as a triple $(A, \mathcal{H}, D)$ consisting of a $*$-algebra $\mathcal{A}$ represented in a Hilbert space $\mathcal{H}$ together with an unbounded selfadjoint operator D, with compact resolvent, which interacts with the algebra in a bounded fashion. This paper contributes to the advancement of this point of view in two significant ways: (1) by showing that any pseudogroup of transformations of a manifold gives rise to such a spectral triple of finite summability degree, and (2) by proving a general, in some sense universal, local index formula for an arbitrary spectral triple of finite summability degree, in terms of the Dixmier trace and its residue-type extension.*

Many of the tools of the differential calculus acquire their full power only when formulated at the level of variational calculus, where the original space X one is dealing with is replaced by a functional space $\mathcal{F}(X)$ of functions or fields on X. The space X itself is involved only indirectly in $\mathcal{F}(X)$, for instance to write down the right hand side $F(\varphi)$ of a nonlinear evolution equation,

$$\frac{d\varphi}{dt} = F(\varphi), \qquad \varphi \in \mathcal{F}(X),$$

with the right hand side usually involving the pointwise product of functions φ on X and partial differentiation.

The essence of noncommutative geometry is the existence of many situations in which $\mathcal{F}(X)$ makes perfectly good sense, while X itself is no longer an ordinary space, described set-theoretically by means of points $p \in X$ and coordinates. When X is given by a set, the basic structure on the space $\mathcal{F}(X)$ of (real or complex valued) functions on a set X is the pointwise product of functions. Given two functions f_1, f_2 one forms a new function $f_1 f_2$ by:

$$(1) \qquad (f_1 f_2)(p) = f_1(p)\, f_2(p), \qquad \forall p \in X \ .$$

In noncommutative geometry one still has a product on $\mathcal{F}(X)$ but the commutativity property of (1)

$$(2) \qquad\qquad f_1 f_2 = f_2 f_1, \qquad \forall f_j \in \mathcal{F}(X) \, .$$

is dropped. It is precisely this commutativity property which signals that X is an ordinary set. When dropped, one no longer deals with just a set X, but essentially with a set endowed with relations between different points. For instance, if one considers a set Y consisting of two points $\{1, 2\}$ and the relation which identifies 1 and 2, then $\mathcal{F}(Y, \mathrm{rel})$ is the space $M_2(\mathbb{C})$ of 2×2 complex matrices with the product

$$(3) \qquad\qquad (f_1 f_2)(i, j) = \sum f_1(i, k) \, f_2(k, j)$$

i.e. the usual product of matrices.

In this simple example the ordinary space $\{1, 2\}$, given by the two points without any relation, is described by the subalgebra of diagonal matrices. It is the "off-diagonal" matrices, such as $e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ or $e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, which describe the relation. This type of construction of an algebra $\mathcal{F}(X)$ is rather general. It extends to a pseudogroup of transformations of a manifold and also to the holonomy pseudogroup of a foliation (see [Co]). The resulting noncommutative algebra encodes the structure of the "space with relations". We shall later discuss in detail the case of a smooth manifold together with its full diffeomorphism group.

As another simple example we can consider the case of a single point divided by a discrete group Γ. Then the corresponding algebra $\mathcal{F}$ is the group ring attached to Γ, whose elements f are functions (with finite support) on Γ,

$$(4) \qquad\qquad g \to f_g \in \mathbb{C},$$

with the product given by linearization of the group law $g_1, g_2 \to g_1 g_2$ in Γ:

$$(5) \qquad\qquad (f_1 f_2)_g = \sum_{g_1 g_2 = g} f_{1, g_1} \, f_{2, g_2} \, .$$

So far, in describing the functional space $\mathcal{F}(X)$ associated to an ordinary space X we have ignored the degree of regularity of the elements $f \in \mathcal{F}(X)$ as functions of $p \in X$. To various degrees of regularity correspond various branches of the general theory of noncommutative

2

associative algebras. The latter are assumed to be algebras over $\mathbb{C}$, which moreover are involutive, i.e. endowed with an antilinear involution

$$(6) \qquad f \to f^* \quad , \quad (f_1 f_2)^* = f_2^* \, f_1^* \; .$$

The two kinds of regularity assumptions for which the corresponding algebraic theory is satisfactory are:

measurability, which corresponds to the theory of von Neumann algebras;

continuity, which corresponds to the theory of C^*-algebras.

In both theories the Hilbert space plays a key role. Indeed, both types of algebras are faithfully representable as algebras of operators in Hilbert space with suitable closure hypothesis. One can trace the role of Hilbert space to the simple fact that *positive* complex numbers are those of the form

$$(7) \qquad \lambda = z^* z \; .$$

In any of the above algebras, functional analysis provides the existence, via Hahn-Banach arguments, of sufficiently many linear functionals L which are positive

$$(8) \qquad L(f^* f) \geq 0$$

From such an L, one easily constructs a Hilbert space together with a representation, by left multiplication, of the original algebra.

Next, many of the tools of *differential topology*, such as the de Rham theory of differential forms and currents, the Chern character etc..., are well captured (see [Co]) by cyclic cohomology applied to *pre C^*-algebras*, i.e. to dense subalgebras of C^*-algebras which are stable under the holomorphic functional calculus:

$$(9) \qquad f \to h(f) = \frac{1}{2i\pi} \int \frac{h(z)}{f - z} \, dz$$

where h is holomorphic in a neighbourhood of $\mathrm{Spec}(f)$. The prototype of such an algebra is the algebra $C^\infty(X)$ of smooth functions on a manifold X. The cyclic cohomology construction then recovers the ordinary differential forms, the de Rham complex of currents and so on. More significantly, this construction also applies to the highly noncommutative example of group rings, in which case the group cocycles give rise to cyclic cocycles with direct application to the Novikov conjecture on the homotopy invariance of the higher signatures of non-simply connected manifolds with given fundamental group. (For a more thorough discussion, see [Co]).

If one wants to go beyond differential topology and reach the geometric structure itself, including the metric and the real analytic aspects, it turns out that the most fruitful point of view is that of *spectral geometry*. More precisely, while our measure theoretic understanding of the space X was encoded by a (von Neumann) algebra of operators $\mathcal{A}$ acting in the Hilbert space $\mathcal{H}$, the *geometric* understanding of the space X will be encoded, not by a suitable subalgebra of $\mathcal{A}$, but by an operator in Hilbert space:

$$(10) \qquad D = D^*, \quad \text{selfadjoint unbounded operator in } \mathcal{H} .$$

In the compact case, i.e. X compact, the operator D will have discrete spectrum, with (real) eigenvalues λ_n, $|\lambda_n| \to \infty$, when $n \to \infty$.

Formulating the precise conditions to which the triples $(\mathcal{A}, \mathcal{H}, D)$ should be subjected is tantamount to devising the axioms of noncommutative geometry. If we let F and $|D|$ be the elements of the polar decomposition of D,

$$(11) \qquad D = F|D| \quad , \quad |D|^2 = D^2 \quad , \quad F = \text{Sign } D$$

then the operators F and $|D|$ play a similar role to the measurements of angles and, respectively, of length in Hilbert's axioms of geometry. In particular the operator $F = \text{Sign } D$ captures the conformal aspect while D describes the full geometric situation.

Considering F alone, the quantized calculus was developed (cf. [Co]) based on the following dictionary.

Classical	*Quantum*
Complex variable	Bounded operator in Hilbert space $\mathcal{H}$
Real variable	Selfadjoint operator
Infinitesimal	Compact operator
Infinitesimal of order $\alpha > 0$	Compact operator whose characteristic values μ_n satisfy $\mu_n = O(n^{-\alpha})$
Differential	$dT = [F, T] = FT - TF$
Integral of infinitesimal of order 1	Dixmier trace $\text{Tr}_\omega(T)$

We refer to Appendix A for a thorough treatment of the Dixmier trace. For a host of applications of the quantized calculus, including Julia sets, the quantum Hall effect and

4

the analysis of group rings, the reader is referred to [Co]. A further application, namely the construction of a 4-dimensional conformal invariant analogue of the 2-dimensional Polyakov action, is discussed in [C_2].

Our goal in the present paper is to use the quantized calculus to develop geometry from a spectral point of view. In more precise terms, our initial datum will be a triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra represented in the Hilbert space $\mathcal{H}$ and D is a selfadjoint operator in $\mathcal{H}$ with compact resolvent, which almost commutes with any $a \in \mathcal{A}$, to the extent that

$$[D, a] \text{ is bounded for any } a \in \mathcal{A} .$$

The basic example of such a triple is provided by the Dirac operator on a closed Riemannian (Spin) manifold. In that case, $\mathcal{H}$ is the Hilbert space of L^2 spinors on the manifold M, $\mathcal{A}$ is the algebra of (smooth) functions acting in $\mathcal{H}$ by multiplication operators and D is the (selfadjoint) Dirac operator. One can easily check that no information has been lost in trading the geometric space M for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. Indeed (see [Co]), one recovers

(i) the space M, as the spectrum $\mathrm{Spec}(\overline{\mathcal{A}})$, of the norm closure of the algebra $\mathcal{A}$ of operators in $\mathcal{H}$;

(ii) the geodesic distance d on M, from the formula:

$$d(p, q) = \mathrm{Sup} \left\{ |f(p) - f(q)| \; ; \; \|[D, f]\| \leq 1 \right\} \quad , \quad \forall p, q \in M .$$

The right hand side of the above formula continues to make sense in general and the simplest non-Riemannian example where it applies is the 0-dimensional situation in which the geometric space is finite. In that case both the algebra $\mathcal{A}$ and the Hilbert space $\mathcal{H}$ are finite dimensional, so that D is a selfadjoint matrix. For instance, for a two-point space, one lets $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ act in the 2-dimensional Hilbert space $\mathcal{H}$ by

$$f \in \mathcal{A} \rightarrow \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix} ,$$

and one takes $D = \begin{bmatrix} 0 & \mu \\ \mu & 0 \end{bmatrix}$. The above formula gives $d(a, b) = 1/\mu$.

As a slightly more involved 0-dimensional example, one can consider the algebraic structure provided by the elementary Fermions, i.e. the three families of quarks. Thus, one lets $\mathcal{H}$ be the finite dimensional Hilbert space with orthonormal basis labelled by the left-handed and right-handed elementary quarks such as u_L^r, u_R^b, ... The algebra $\mathcal{A}$ is $\mathbb{C} \oplus \mathbb{H}$, where

the complex number λ in $(\lambda, q) \in \mathcal{A}$ acts on the right-handed part by λ on "up" particles and $\overline{\lambda}$ on "down" particles. The isodoublet structure of the left-handed (up, down) pairs allows the quaternion q to act on them by the matrix

$$\begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} \qquad q = \alpha + \beta j \quad ; \quad \alpha, \beta \in \mathbb{C} \, .$$

Then the Yukawa coupling matrix of the standard model provides the selfadjoint matrix D.

In [C-L] the theory of matter fields was developed in the above framework, under the finite-dimensionality hypothesis that the characteristic values of D^{-1} are $O(n^{-1/d})$, for some finite d.

This allows to define the action functional of Quantum Electrodynamics at the same level of generality (cf. [Co]). The striking fact there is that if one replaces the usual picture of space-time by its product by the above 0-dimensional example, the QED action functional gives the Glashow-Weinberg-Salam standard model Lagrangian with its Higgs fields and symmetry breaking mechanism. In the development of this theory, the tools of the quantized calculus, in particular the Dixmier trace as the substitute for the Lebesgue integral, played an essential role.

The matter field Lagrangian involves the metric $g_{\mu\nu}$ but does not involve any derivative of $g_{\mu\nu}$. This indicates that the difficulty involved in developing the analogue of gravity in the above context is of a different scale. In order to overcome it, one needs both a good list of examples of spectrally defined spaces and a difficult mathematical problem to solve. By a spectrally defined space we mean a triple $(\mathcal{A}, \mathcal{H}, D)$ as above; the involutive algebra $\mathcal{A}$ is not necessarily commutative. We shall also refer to them as *spectral triples*.

Before running through the list of examples, let us state the mathematical problem:

compute by a local formula the cyclic cohomology Chern character of $(\mathcal{A}, \mathcal{H}, D)$.

More specifically, the representation of $\mathcal{A}$ in $\mathcal{H}$ together with the operator D allows to set up an index problem:

$$\operatorname{Ind}_D : K_j(\mathcal{A}) \to \mathbb{Z}$$

where $j = 0$ in the $\mathbb{Z}/2$-graded (or even case) and $j = 1$ otherwise. The index map turns out to be polynomial and given, in the above generality, by the pairing of $K_j(\mathcal{A})$ with the following cyclic cocycle

$$\tau(a^0, a^1, \ldots, a^n) = \operatorname{Trace}\left(a^0[F, a^1] \ldots [F, a^n]\right), \qquad \forall a^j \in \mathcal{A}$$

6

where n has the same parity as j and $n > d - 1$. In the even case, one replaces the trace by the supertrace, i.e. one uses the $\mathbf{Z}/2$-grading γ of $\mathcal{H}$ to write

$$\tau(a^0, a^1, \ldots, a^n) = \operatorname{Trace}\left(\gamma a^0 [F, a^1] \ldots [F, a^n]\right), \qquad \forall a^j \in \mathcal{A} .)$$

The class of τ in the cyclic cohomology $HC^n(\mathcal{A})$ is the Chern character of $(\mathcal{A}, \mathcal{H}, D)$. We refer to [Co] for more details as well as for the appropriate normalizations.

The general problem is to compute the class of τ by a *local formula*. A partial answer to this problem was already obtained in [Co], by means of a general local formula for the *Hochschild class* of τ as the Hochschild n-cocycle:

$$(12) \qquad \varphi(a^0, \ldots, a^n) = \operatorname{Tr}_\omega\left(a^0 [D, a^1] \ldots [D, a^n] |D|^{-n}\right), \qquad \forall a^j \in \mathcal{A},$$

where n is as above and, in the even case, with γ inserted in front of a^0 .

In the above formula Tr_ω is the Dixmier trace, which when evaluated on a given operator T only depends upon the asymptotic behavior of its eigenvalues. More precisely, for $T \geq 0$, with $\mu_n(T)$ the nth eigenvalue of T in decreasing order, one has (cf. Appendix A):

$$\operatorname{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_0^N \mu_n(T);$$

this is insensitive to the perturbation of μ_n by any sequence $\varepsilon_n = o\left(\frac{1}{n}\right)$, i.e. such that $n\varepsilon_n \to 0$, $n \to \infty$.

For a classical pseudodifferential operator P with distributional kernel $k(x, y)$ the Dixmier trace is given by the Wodzicki residue

$$\operatorname{Tr}_\omega(T) = \int a(x)$$

where $k(x, y)$ has an asymptotic expansion near the diagonal of the form

$$k(x, y) = a(x) \log(d(x, y)) + b(x, y),$$

with b bounded.

In particular, when one evaluates Tr_ω on a product $T_1 \ldots T_n$ of such operators the result is expressed as an integral *in a single variable* x of a local quantity. This is in sharp contrast with what happens for the ordinary trace, which when evaluated on $T_1 \ldots T_n$ involves a multiple integral, of the form

$$\int k_1(x_1, x_2) \, k_2(x_2, x_3) \ldots k_n(x_n, x_1),$$

where the x_j's vary arbitrarily in the manifold.

While the expression (12) of the Hochschild cocycle φ is local in full generality, it only accounts for the Hochschild class of the Chern character of $(\mathcal{A}, \mathcal{H}, D)$, which is not sufficient to recover the index map. In the manifold case for instance, it only gives the index of D with coefficients in the Bott K-theory class supported by an arbitrarily small disk

In Section II of this paper we shall obtain a general local formula for all the components of the cyclic cocycle τ. This will be achieved by adapting the Wodzicki residue, the unique extension of the Dixmier trace to pseudodifferential operators of arbitrary order, to all our examples. For spectrally defined spaces $(\mathcal{A}, \mathcal{H}, D)$, we shall see that the usual notion of dimension is replaced by a *dimension spectrum*, which is a subset of $\mathbb{C}$. Under the assumption of simple discrete dimension spectrum, the Wodzicki residue makes sense and defines a trace on the algebra of the pseudodifferential operators of $(\mathcal{A}, \mathcal{H}, D)$. The latter algebra is obtained by analysing the one-parameter group $\sigma_t = |D|^{it} \cdot |D|^{-it}$ in a manner very similar to Tomita's analysis of the modular automorphism group of von Neumann algebras. When the dimension spectrum is discrete but not simple, the analogue of the Wodzicki residue is no longer a trace; it satisfies, however, cohomological identities which relate it to higher residues.

Under the sole hypothesis of discreteness of the dimension spectrum, we shall obtain a *universal local formula for the Chern character* of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, expressing the components of the Chern character in terms of finite linear combinations, with *rational coefficients*, of higher residues applied to products of iterated commutators of D^2 with $[D, a^j]$, $a^j \in \mathcal{A}$. A noteworthy feature of the proof is the use of renormalization group techniques to remove the transcendental coefficients which arise when the dimension spectrum has multiplicity. In the manifold case, this formula reduces, of course, to the classical local index formula. In general however it is necessarily more intricate, in several respects, because of its large domain of applicability, which encompasses for instance the diffeomorphisms-equivariant situation described in Section I.

We conclude the introduction with a list of spectral triples corresponding to geometric or group-theoretic spaces.

1. *Riemannian manifolds* (with some variations allowing for Finsler metrics and also for the replacement of $|D|$ by $|D|^\alpha$, $\alpha \in \,]0, 1]$).

2. *Manifolds with singularities.* For this, the work of J. Cheeger on conical singularities is very relevant. In fact, the spectral triples are stable under the operation of "coning", which is easy to formulate algebraically.

8

3. *Discrete spaces and their product with manifolds* (as in the discussion in [Co] of the standard model). The spectral triples are of course stable under products.

4. *Cantor sets.* Their importance lies in the fact that they provide examples of dimension spectra which contain complex numbers (cf. Section II).

5. *Nilpotent discrete groups.* The algebra $\mathcal{A}$ is the group ring of the discrete group Γ, and the nilpotency of Γ is required to ensure the finite-summability condition $D^{-1} \in \mathcal{L}^{(p,\infty)}$. We refer to [Co] for the construction of the triple for subgroups of Lie groups.

6. *Transverse structure for foliations.* This example, or rather the intimately related example of the *Diff*-equivariant structure of a manifold will be treated in detail in Section I of this paper.

I. Diffeomorphism invariant geometry

1. Diffeomorphism invariant geometric structures

Let W be a smooth manifold and $\mathrm{Diff}(W)$ its group of diffeomorphisms. Given an arbitrary subgroup Γ of $\mathrm{Diff}(W)$ one can form the crossed product of the algebra of functions on W by the action of Γ (cf. [Co]) and describe in this way the measure theory and topology of the noncommutative space encoding the identifications of points in W by the action of Γ.

The basic idea which we developed in [C_1], in order to obtain invariants of K-theory of the C^*-algebra $C_0(W) \rtimes \Gamma$, is to relate the general case of an arbitrary Γ acting on W to a "type II" situation, in which the action of Γ preserves a certain G-structure which we shall describe and use at great length below.

In general, and for instance if we take $\Gamma = \mathrm{Diff}(W)$, the action of Γ on W preserves no structure at all. Thus, if we take $W = S^1$ there is no Γ-invariant measure in the Lebesgue measure class, and even at the measure theory level the crossed product $L^\infty(S^1) \rtimes \Gamma$ is of type III. The basic structure theory of type III factors, as crossed products of type II by an action of $\mathbb{R}_+^*$, is easy to interpret in this example (cf. e.g. [Co]) as the replacement of the manifold $W = S^1$ by the total space P of the $\mathbb{R}_+^*$-principal bundle of (positive) 1-densities on S^1.

On this total space P the group $\mathrm{Diff}(W)$ is still acting and now there is on P a tautological invariant measure for the action of $\mathrm{Diff}(W)$, so that the crossed product of P by Γ is of type II. Furthermore the group $\mathbb{R}_+^*$ acts on this crossed product and gives back (up to Morita equivalence) the original crossed product of W by Γ.

Even though Γ acting on P preserves a natural density, it does not preserve a Riemannian geometry, since it would then have to be contained in the Lie group of isometries of that geometry. Let us describe the natural geometric structure on P preserved by the action of $\mathrm{Diff}(S^1)$. By construction, P is an $\mathbb{R}_+^*$ principal bundle over S^1 and we let $\pi : P \to S^1$ be the canonical projection. Given $x \in S^1$, a point $p \in \pi^{-1}\{x\}$ is the same thing as a unit of length in the tangent space $T_x(S^1)$. Moreover, the canonical action of $\mathbb{R}_+^*$ on P also specifies a unit of length in $T_p\left(\pi^{-1}\{x\}\right)$, given by the vertical vector field $\frac{d}{dt}\left(e^t\,p\right)$.

We thus have a natural integrable subbundle V of the tangent bundle of P together with Euclidean metrics on both V and $N = T/V$, where at any $p \in P$ we use the identification of N_p with $T_x(S^1)$, $x = \pi(p)$, to define the metric at p.

Since this construction is completely canonical it is invariant under the action of $\mathrm{Diff}(S^1)$.

10

If we make the (non canonical) choice of section $d\theta$ of P we can label the points of P by (θ, λ), $\theta \in S^1$, $\lambda \in \mathbb{R}_+^*$ or equivalently (θ, s), $s \in \mathbb{R}$, $\lambda = e^s$. In these coordinates, the vertical metric is ds^2 and the transverse one is $(e^s \, d\theta)^2$. A diffeomorphism φ of S^1 acts in the obvious way, namely:

$$\varphi(\theta, s) = (\varphi(\theta), s + \log \varphi'(\theta)) \ .$$

If we perform the measure theory construction of the principal $\mathbb{R}_+^*$ bundle in higher dimension, we obtain the correct description of the corresponding type II algebra but since we use only the (absolute value of the) determinant of the Jacobian matrix of φ we only control the volume distortion by φ but not the geometric distortion.

To describe the latter, we take for P the bundle over W whose fiber P_x at each $x \in W$ is the space of all Euclidean metrics on the vector space $T_x(W)$. Thus, a point p of P is given by a point $x \in W$ together with a non-degenerate quadratic form, $g_{\mu\nu} \, dx^\mu \, dx^\nu$ in local coordinates, on $T_x(W)$. Equivalently, we can describe P as the quotient of the frame bundle of W, whose fiber at $x \in W$ is the space of linear isomorphisms $\mathbb{R}^n \to T_x \, W$, by the action of the subgroup $O(n) \subset GL(n, \mathbb{R})$. On the symmetric space $GL_n/O(n)$ we use the natural invariant Riemannian metric, which on the tangent space at the unit matrix, identified with the space of symmetric matrices, is given by the Hilbert-Schmidt norm. Once transported to the fiber P_x, $x \in W$, this metric gives an Euclidean structure on the vertical bundle $V \subset TP$. Given a vertical path $p(t)$, $p(0) = p$ its square length at $t = 0$ is simply the trace of $(p^{-1} \, \dot{p})^2$.

Also, exactly as above, we can identify the transverse space $N_p = T_p \, P/V_p$ with the tangent space $T_x(W)$, $x = \pi(p)$, so that the quadratic form p provides us with a natural Euclidean structure on N_p. In order to have a convenient terminology, we introduce the following definition:

Definition I.1. *By a triangular structure on a manifold M we mean an integrable subbundle V of the tangent bundle TM together with Euclidean metrics on both V and $N = T/V$.*

We can summarize the above discussion as follows:

Proposition I.1. *Given a manifold W, the space P of all metrics on W, defined above, has a canonical triangular structure, invariant under the action of $\mathrm{Diff}(W)$.*

This construction was used in $[C_1]$ to prove analytic properties of cyclic cocycles such as the transverse fundamental class or Gelfand-Fuchs cohomology classes. We refer to

11

$[C_1]$ for purely geometric corollaries of this technique. To obtain them, it was crucial to relate the K-theory of the C^*-algebras obtained (from crossed products by subgroups Γ of Diff W) using W and using P. This followed from the "dual Dirac" construction of a bivariant Kasparov class (cf. $[C_1]$). In [H-S] M. Hilsum and G. Skandalis went further and constructed the transverse fundamental class in K-homology for the space P, using hypoelliptic operators. They did this at the level of homotopy classes of Fredholm modules and the central theme of the first part of this paper will be to refine their construction in order to describe the geometry of the crossed products (of P by Γ) by a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfying all the conditions of our general spectral geometry.

Thus, we shall now free ourselves from the particular features of the example P above and discuss, in the context of triangular manifolds, the construction of the corresponding spectral triple.

One of the new features will be the *quartic* aspect of the discussion, as opposed to the quadratic feature of Riemannian geometry.

Finally one should keep in mind that the crossed product of P by a given subgroup Γ of Diff(W) is only the "type II" counterpart of the crossed product of W by Γ. To obtain the latter from the former, one needs to take a crossed product by "$GL_n/O(n)$", operation in which the non amenability of the Lie group GL_n, $n > 1$ comes into play. In this paper we shall content ourselves with the type II discussion.

2. The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ of a triangular structure on a manifold

We let M be a smooth (not necessarily compact) manifold together with an integrable subbundle V of its tangent bundle. We let $N = TM/V$ be the transverse bundle and assume that both N and V are oriented Euclidean even dimensional vector bundles.

Our first aim is to construct a hypoelliptic differential operator Q corresponding to the signature of M, which modulo lower order only depends upon the Euclidean structures of both V and N but not upon a choice of Riemannian metric on M. This will be done by combining a longitudinal signature operator of order 2 with the usual signature operator in the transverse direction. Then, using Q, we shall define a first-order operator D by the equation

$$(1) \qquad\qquad Q = D|D| \ .$$

Let us first see how one can replace the usual signature operator by an equivalent operator of order 2.

α) *Second order signature operator*

Let V be a smooth (and, for simplicity, compact) oriented even-dimensional Riemannian manifold. On the bundle $\wedge T_{\mathbb{C}}^*$ of exterior differential forms on V one has a natural $\mathbb{Z}/2$-grading γ, $\gamma^2 = 1$, $\gamma = \gamma^*$, given by the $*$-operation, such that

$$(2) \qquad\qquad d^* = -\gamma\, d\, \gamma\,;$$

thus, the signature operator $d + d^*$ anticommutes with γ. Let

$$\Delta = (d + d^*)^2 = dd^* + d^*d.$$

It commutes with both d and d^* and we can thus consider, for $\lambda \in [0,1]$, the operators

$$(3) \qquad\qquad U_\lambda = \Delta^{1/2} + \lambda(d - d^*) \quad , \quad U_\lambda^* = \Delta^{1/2} + \lambda(d^* - d) \ .$$

One has

$$U_\lambda U_\lambda^* = \Delta - \lambda^2(d - d^*)^2 = (1 + \lambda^2)\Delta$$

and similarly

$$U_\lambda^* U_\lambda = (1 + \lambda^2)\Delta.$$

Lemma I.1. 1) U_λ *commutes with the $\mathbb{Z}/2$-grading γ.*

2) $U_\lambda(d + d^*)U_\lambda^* = 2\Delta^{1/2}(dd^* - d^*d)$ *for $\lambda = 1$.*

Proof. 1) Both Δ and $d - d^* = d + \gamma d\gamma$ commute with γ.

One has

$$\left(\Delta^{1/2} + (d - d^*)\right)(d + d^*)\left(\Delta^{1/2} - (d - d^*)\right) =$$

$$\Delta(d + d^*) + (d - d^*)(d + d^*)\Delta^{1/2} + \Delta^{1/2}(d + d^*)(d^* - d) - (d - d^*)(d + d^*)(d - d^*) \ .$$

On the other hand,

$$(d - d^*)(d + d^*) = -d^*d + dd^*, \ (d + d^*)(d^* - d) = -d^*d + dd^*$$

and

$$-(d - d^*)(d + d^*)(d - d^*) = (dd^* - d^*d)(d^* - d) = -dd^*d - d^*dd^* = -\Delta(d + d^*). \quad \blacksquare$$

13

It follows, ignoring finite rank operators and using the operators $U_\lambda \, \Delta^{-1/2}$ which are bounded, that one gets a homotopy between the signature operator and the operator $\Delta^{-1/2}(dd^* - d^*d)$ with the same $\mathbb{Z}/2$ grading. This latter operator is an elliptic pseudo-differential operator of order 1 defined by the equation:

$$(4) \qquad\qquad D|D| = dd^* - d^*d \ .$$

The second order operator $dd^* - d^*d$, with the $\mathbb{Z}/2$-grading γ, thus represents the signature class on M.

Let us now combine it with $d + d^*$ in the above context.

β) *The mixed signature operator*

We let M, $V \subset TM$ and $N = TM/V$ be as above. We consider over M the hermitian vector bundle E with fiber

$$(5) \qquad\qquad E = \wedge^* \, V_{\mathbb{C}}^* \otimes \wedge^* \, N_{\mathbb{C}}^* \ .$$

Its metric comes from the metrics of V and N, together with the orientations these yield $\mathbb{Z}/2$-grading operators γ_V and γ_N. It also yields a natural volume element, i.e. a section of

$$\wedge^v \, V^* \otimes \wedge^n \, N^* = \wedge^d \, T^*M$$

where $v = \dim V$, $n = \dim N$, $d = \dim M = v + n$.

Thus the Hilbert space $L^2(M, E)$ of sections of E has a natural inner product, independent of any additional choice.

Using the canonical *flat* connection of the restriction of the bundle N to the leaves of the foliation by V, we can define the longitudinal differential d_L, as an operator of degree $(1, 0)$ with respect to the obvious bigrading, satisfying

$$(6) \qquad\qquad d_L^2 = 0 \ .$$

The operator d_L^* is, by definition, the adjoint of d_L. It is of the form

$$(7) \qquad\qquad d_L^* = -\gamma_V \, d_L \, \gamma_V + \text{Order } 0$$

with the additional term of order 0 uniquely prescribed without any extra choice.

14

This means that the following operator is a well defined longitudinal elliptic operator:

$$(8) \qquad\qquad Q_L = d_L d_L^* - d_L^* d_L \ .$$

By the discussion of section α) this operator describes at the K-theory level the longitudinal signature class. To obtain the full signature of M we need to combine it with a transverse signature operator, which is of order 1 as a differential operator.

Our next step will thus be to define the operator $d_H + d_H^*$, where d_H is of degree $(0,1)$ in the bigrading of E and corresponds to transverse differentiation.

This step will require an additional choice of a (non integrable) subbundle H of TM transverse to V, $\dim H = n$. It is crucial that such a choice does not affect the principal symbol of the operator as a hypoelliptic operator (see below).

The choice of H provides a natural isomorphism

$$(9) \qquad\qquad j_H : \ \wedge V_x^* \otimes \wedge N_x^* \to \wedge T_x^*, \qquad \forall x \in M,$$

and for $\omega \in C^\infty(M, \wedge^r V^* \otimes \wedge^s N^*)$ we define $d_H(\omega)$ as the component of bidegree $(r, s+1)$ of

$$(10) \qquad\qquad j_H^{-1} \ d(j_H \ \omega) \ .$$

To understand the ambiguity in the choice of H we consider locally a function f which is leafwise constant, i.e. $d_L f = 0$. Then $d_H f$ is independent of H and given as a section of N^*. We can then define the transverse symbol of d_H using its commutation with such f:

$$(11) \qquad\quad d_H(f\omega) - f d_H(\omega) = df \wedge \omega \qquad \forall f, d_L \ f = 0;$$

in the right hand side we use the natural algebra structure for $\wedge V^* \otimes \wedge N^*$.

Thus, (11) means that the transverse symbol is independent of the choice of H. We let:

$$(12) \qquad\qquad Q_H = d_H + d_H^*$$

where the $*$ is taken relative to the inner product in $L^2(M, E)$. We now combine Q_L and Q_H, using the parity $(-1)^{\partial_N}$ in the transverse direction which commutes with Q_L and anticommutes with Q_H, to define

$$(13) \qquad\qquad Q = Q_L \ (-1)^{\partial_N} + Q_H \ .$$

15

We should remark that one can use $(-1)^{\partial_N}$ instead of $1 \otimes \gamma_N$, without changing the homotopy class of the operators, since these two gradings are homotopic among operators which anticommute with Q_H.

The selfadjoint operator D is now uniquely defined by the equation:

$$(14) \qquad\qquad D|D| = Q .$$

Note that Q is formally selfadjoint by construction. We shall not discuss here the problem of selfadjointness of Q in the noncompact case. This issue will need to be adressed eventually, in connection with our main example, i.e. the total space P in I.1.

The following theorem shows that the operator D constructed above gives rise to a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ for the crossed product $\mathcal{A} = C_c^\infty(M) \rtimes \Gamma$, where Γ is any group of diffeomorphisms preserving the triangular structure.

Theorem I.1. 1) *$[D, f]$ is bounded for any $f \in C_c^\infty(M)$, and both f and $[D, f]$ belong to $\cap_{n \geq 1} \mathrm{Dom}\ \delta^n$, where $\delta = [|D|, \cdot]$.*

2) *If M is compact D has compact resolvent, in all cases*

$$f(D - \lambda)^{-1} \quad is\ compact \quad f \in C_c^\infty(M) , \ \lambda \notin \mathbb{R} .$$

3) *Changing the choice of H only affects D and $|D|$ by bounded operators (locally in the noncompact case).*

4) *Let $\varphi \in \mathrm{Diff}(M)$ preserve the foliation V and be isometric on both V and N. Let U_φ be the corresponding unitary in $\mathcal{H}$. Then $[D, f U_\varphi]$ is bounded, and both $f U_\varphi$ and $[D, f U_\varphi]$ belong to $\cap_{n \geq 1} \mathrm{Dom}\ \delta^n$ for all $f \in C_c^\infty(M)$.*

We shall also give the precise summability in 2) by showing that $f(D - \lambda)^{-1} \in \mathcal{L}^{(p,\infty)}$, $p = v + 2n$, and compute the Dixmier trace of the product $f|D|^{-p}$ (cf. [Co]).

3. Preliminaries on the $\Psi DO'$ calculus

As a technical tool in the proof of Theorem I.1, we shall describe the pseudodifferential calculus which is adapted to the situation. It is just a special case of the pseudodifferential calculus on Heisenberg manifolds (cf. [B-G]), which is however sufficiently different from the ordinary ΨDO calculus to deserve a careful treatment. The reader familiar with [B-G] can skip this section.

16

Recall that M is foliated by the integrable subbundle V. We shall only use charts, i.e. local coordinates x^i, which are foliation charts, that is

(15)
$$V \quad \text{is generated by} \quad \frac{\partial}{\partial x^j} = \partial_j \ , \ j = 1, \ldots, v \ .$$

Thus the plaques, i.e. the leaves of the restriction of the foliation, are $\mathbb{R}^v \times$ pt. In such coordinates we shall use the ordinary formula to pass from a symbol $\sigma(x, \xi)$ to the corresponding operator:

(16)
$$P_\sigma = (2\pi)^{-m} \int e^{i\langle(x-y),\xi\rangle} \, \sigma(x,\xi) \, d^m\xi \qquad m = v + n \ .$$

One has $\xi \in (\mathbb{R}^v \times \mathbb{R}^n)^* = \mathbb{R}_v \times \mathbb{R}_n$ and one defines a (coordinate dependent) notion of homogeneity of symbols using

(17)
$$\lambda \cdot \xi = (\lambda \, \xi_v, \lambda^2 \, \xi_n) \quad \text{for} \quad \xi = (\xi_v, \xi_n) \ , \ \lambda \in \mathbb{R}_+^* \ .$$

The natural length for ξ which is homogeneous of degree 1 is

(18)
$$\|\xi\|' = \left(\|\xi_v\|^4 + \|\xi_n\|^2\right)^{1/4} \ .$$

Let us start with a symbol σ, smooth on $\mathbb{R}^m \times \mathbb{R}_m \backslash \{0\}$ and homogeneous of degree q for the dilations (17), i.e.

(19)
$$\sigma(x, \lambda \cdot \xi) = \lambda^q \, \sigma(x, \xi) \ .$$

In order to control the operator P_σ defined in (16), one needs to control the partial derivatives

$$\partial_x^\alpha \, \partial_\xi^\beta \, \sigma(x, \xi) \ .$$

When we apply ∂_x^α, the homogeneity property (19) is preserved. The action of $\partial_{\xi_v}^i$, $i = 1, \ldots, v$, lowers q by 1 while the action of $\partial_{\xi_n}^i$, $i = 1, \ldots, n$, lowers it by 2. Thus, if we let

(20)
$$\langle \beta \rangle = \sum_1^v \beta_i + 2 \sum_1^n \beta_{v+j},$$

we see that $\partial_x^\alpha \, \partial_\xi^\beta \, \sigma$ is homogeneous of degree $q - \langle \beta \rangle$. It follows that, for x in a compact subset of $\mathbb{R}^m$ and α, β fixed,

(21)
$$\left| \partial_x^\alpha \, \partial_\xi^\beta \, \sigma(x, \xi) \right| \leq C_{\alpha, \beta} \, (1 + \|\xi\|')^{q - \langle \beta \rangle} \ .$$

17

To employ the usual classes of ΨDO one needs to relate the right hand side to the expression

$$(1 + \|\xi\|)^{q - \frac{1}{2}|\beta|} .$$

With $a = \|\xi_v\|$, $b = \|\xi_n\|$, one has

$$\|\xi\|^2 = a^2 + b^2, \qquad \|\xi\|'^4 = a^4 + b^2,$$

so that

$$\|\xi\|' \leq \left(1 + \|\xi\|^2\right)^{1/2} \leq 1 + \|\xi\| \ , \ 1 + \left(\|\xi\|'\right)^4 \geq \|\xi\|^2,$$

$$(22) \qquad\qquad \|\xi\|^{1/2} \leq 1 + \|\xi\|' \leq 2 + \|\xi\| .$$

It thus follows that, if $q \geq 0$, a homogeneous symbol σ of degree q is of class $S^q_{0,1/2}$, i.e.

$$(23) \qquad\qquad \left| \partial_x^\alpha \, \partial_\xi^\beta \, \sigma(x,\xi) \right| \leq C_{\alpha,\beta} \left(1 + \|\xi\|\right)^{q - \frac{1}{2}|\beta|},$$

while for $q < 0$ it is of class $S^{q/2}_{0,1/2}$.

This implies in particular that for $q \leq 0$ the operator P_σ is bounded in L^2 (cf. [B-G]).

We can now introduce the relevant class of symbols for the proof of Theorem I.1. We consider symbols σ such that:

$$(24) \qquad \begin{array}{c} \text{there exists a sequence } (\sigma_q) \text{ of homogeneous symbols,} \\ \sigma_q \text{ of degree } q, \text{ with } \sigma \sim \sum_{q \leq q_0} \sigma_q \end{array}$$

where $\sim$ means that for any N the difference

$$\sigma_N = \sigma - \sum_{N < q \leq q_0} \sigma_q$$

satisfies the inequalities (21) for $q = N$.

Definition I.2. *We shall say that an operator P is a $\Psi DO'$ if it is a P_σ, with σ as in* (24).

Let us now describe the composition $P_\sigma \circ P_{\sigma'}$ of two $\Psi DO'$. Denoting as usual

$$(25) \qquad\qquad D_x^\alpha = (-i)^{|\alpha|} \, \partial_x^\alpha,$$

18

the action of P_σ can be formally expanded as

$$P_\sigma = \sum \frac{1}{\alpha!} \, \partial_\xi^\alpha \, \sigma(x,\xi) \, D_x^\alpha \ .$$

Let us consider the expansion:

$$(26) \qquad \qquad \sum \frac{1}{\alpha!} \, \partial_\xi^\alpha \, \sigma_{q_1}(x,\xi) \, D_x^\alpha \, \sigma'_{q_2}(x,\xi) \ .$$

The degree of homogeneity of each of the terms is

$$q_1 + q_2 - \langle \alpha \rangle,$$

where $\langle \alpha \rangle$ is defined in (20).

This shows that (26) makes sense as an asymptotic expansion and corresponds to the product $P_\sigma \circ P_{\sigma'}$.

Note that every differential operator P is a $\Psi DO'$; the only difference is in the notion of degree, since while $\partial/\partial x_j = \partial_j$ has degree 1 for $j = 1,\ldots,v$ it has degree 2 for $j = v+1,\ldots,v+n$. It is not true however that an ordinary ΨDO is a $\Psi DO'$, even in the order 0 case.

We need to define the notions of principal symbol and ellipticity for $\Psi DO'$. To this end we shall first consider what happens under a change of foliation chart.

Let φ be such a change of charts. It defines a local $\mathbb{R}^n$ diffeomorphism φ_n, in such a way that, with the notations $x = (x_v, x_n)$, $y = (y_v, y_n)$, one has

$$(27) \qquad \qquad \varphi(x) = (\varphi_v(x_v, x_n), \varphi_n(x_n)) \ .$$

Similarly, $\psi = \varphi^{-1}$ is of the same form, $\psi = (\psi_v, \psi_n)$. Given a covector $\xi \in T_x^*$ the corresponding covector at $y = \varphi(x)$ is

$$(28) \qquad \qquad \eta = \psi_*^t(\xi)$$

i.e. $\langle \eta, Y \rangle = \langle \xi, \psi_*(Y) \rangle, \qquad \forall \, Y \in T_y.$

With $\sigma = \sigma(y, \eta)$ a homogeneous symbol, let us consider the composition

$$(29) \qquad \qquad \widetilde{\sigma}(x,\xi) = \sigma\left(\varphi(x), \psi_{*,\varphi(x)}^t(\xi) \right) \ .$$

19

To compare it with $\widetilde{\sigma}(x, \lambda \cdot \xi)$ we just need to understand the linear map ψ_*^t at $\varphi(x) = y$. This map, L, preserves the natural subspace $N^* \subset T^*$ of covectors orthogonal to the leaves. On the subspace $N^* = \{(0, \xi_n)\}$ one has $\lambda \cdot \xi = \lambda^2 \xi$. Thus

$$L = \begin{bmatrix} L_{vv} & 0 \\ L_{nv} & L_{nn} \end{bmatrix}$$

$$L(\lambda \cdot \xi) = \left(\lambda \, L_{vv}(\xi_v), \lambda^2 \, L_{nn}(\xi_n) + \lambda \, L_{nv}(\xi_v) \right) \; .$$

With obvious notation, we have

$$\sigma(L(\xi)) = \sigma \left(L_{vv}(\xi_v), L_{nn}(\xi_n) + L_{nv}(\xi_v) \right) = \sum \frac{1}{\alpha!} \, \partial_{\eta v}^\alpha \, \sigma \left(L_{vv}(\xi_v), L_{nn}(\xi_n) \right) \left(L_{n,v}(\xi_v) \right)^\alpha,$$

which gives the desired expansion of $\sigma \circ L$ as a sum of homogeneous symbols. This shows that formula (29) defines a transformation of symbols.

To obtain the symbol of the operator P_σ in the new coordinates one writes its kernel as

$$(30) \qquad k(x, x') = (2\pi)^{-m} \int e^{i\langle \varphi(x) - \varphi(x'), \eta \rangle} \, \sigma(\varphi(x), \eta) \, d^m\eta \; .$$

One first changes variables from η to ξ using (28) and the fact that k is a 1-density, so that no Jacobian enters in the change of variables, due to the invariance of the symplectic Liouville measure. Thus,

$$(31) \qquad k(x, x') = (2\pi)^{-m} \int e^{i(\langle x - x', \xi \rangle + \alpha(x, x', \xi))} \, \widetilde{\sigma}(x, \xi) \, d^m\xi,$$

where $\alpha(x, x', \xi)$ is the non linearity at x of the map φ:

$$(32) \qquad \langle \varphi(x) - \varphi(x'), \eta \rangle = \langle x - x', \xi \rangle + \alpha(x, x', \xi) \; .$$

The variable ξ appears linearly in α and the coefficient of ξ_n only invokes $\varphi_n(x_n) - \varphi_n(x'_n)$ and not x_v or x'_v. By construction, α and its first derivatives in x' vanish at $x' = x$. This implies that the Taylor expansion at $x' = x$ of $e^{i\alpha(x, x', \xi)}$ is of the form

$$(33) \qquad e^{i\alpha(x, x', \xi)} = \sum P_\beta(x, \xi)(x - x')^\beta,$$

where $P_\beta(x, \xi)$ is a polynomial in ξ whose degree, in the sense that ξ^α has degree $\langle \alpha \rangle$, is smaller than $\frac{1}{2}\langle \beta \rangle$. Thus, using $\partial_\xi^\beta \, e^{i\langle x - x', \xi \rangle} = i^{|\beta|} \, (x - x')^\beta \, e^{i\langle x - x', \xi \rangle}$ and integrating by parts, one gets the full symbol for P_σ:

$$(34) \qquad \sigma(x, \xi) = \sum i^{-|\beta|} \, \partial_\xi^\beta \, \left(P_\beta(x, \xi) \, \widetilde{\sigma}(x, \xi) \right) \; .$$

20

In particular, at a given x the value of $\sigma(x, \xi)$ only involves $\tilde{\sigma}$ restricted to T_x^* and not its restriction to any T_y^*, $y \neq x$. To this restriction $\tilde{\sigma}$ one applies a differential operator with polynomial coefficients.

Let us now turn to the *principal symbol*. Let σ be a symbol of order q; then its principal part is

$$(35) \qquad \lim_{\lambda \to \infty} \lambda^{-q} \, \sigma(x, \lambda \cdot \xi) \; .$$

We see that it gives a well defined function on the bundle

$$V^* \oplus N^*,$$

the direct sum of the subbundle $N^* \subset T^*$ and of $V^* = T^*/N^*$. The above limit exists and under a change of coordinates it behaves, in view of the above formulae, as a function on $V^* \oplus N^*$.

As an example, let us compute the principal symbol of

$$[|D|, P_\sigma]$$

where σ is of order 0 and where the principal symbol of $|D|$ is

$$(36) \qquad \sigma_1(x, \xi) = \|\xi\|' \qquad (\text{cf. } 18) \; .$$

We can use formula (26) in local coordinates. It gives

$$(37) \qquad \sum_{\langle \alpha \rangle = 1} \left(\partial_\xi^\alpha \, \sigma_1(x, \xi) \, D_x^\alpha \, \sigma(x, \xi) - \partial_\xi^\alpha \, \sigma(x, \xi) \, D_x^\alpha \, \sigma_1(x, \xi) \right) \; .$$

Note that since $\langle \alpha \rangle = 1$, this formula only involves the longitudinal differentiation $D_{x_v}^\alpha$.

4. Proof of Theorem I.1.

The operator D is defined by equation (14), where Q is a selfadjoint differential operator by construction. We shall first show that D is a $\Psi DO'$ of order 1.

As noted before, any differential operator is $\Psi DO'$ in the above sense , but the notions of degree and of principal symbol differ from the usual ones. In particular, Q is *elliptic* of degree 2 and its principal symbol is, for $\xi_v \in V^*$, $\xi_n \in N^*$, the endomorphism of $\wedge V^* \otimes \wedge N^*$

$$\sigma_2(\xi_v, \xi_n) = (e_{\xi_v} \, i_{\xi_v} - i_{\xi_v} \, e_{\xi_v}) \otimes (-1)^\partial + 1 \otimes (i \, e_{\xi_n} + (i \, e_{\xi_n})^*) \; .$$

21

The two sides anticommute, so that when we square σ_2 we get

$$\sigma_2^2(\xi_v, \xi_n) = \left(\|\xi_v\|^4 + \|\xi_n\|^2 \right) \cdot 1,$$

which shows that Q^2 is a $\Psi DO'$ of order 4, elliptic and with principal symbol a multiple of the identity.

As in the ordinary pseudodifferential calculus, this is enough (cf. [Gi] p.52) to construct an asymptotic $\Psi DO'$, $R(\mu)$ which is an asymptotic resolvent for Q^2.

We shall use for D and $|D|$ the following formulas:

$$(38) \qquad\qquad |D| = \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{Q^2}{Q^2 + \mu} \, \mu^{-3/4} \, d\mu$$

$$(39) \qquad\qquad D = \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{Q}{Q^2 + \mu} \, \mu^{-1/4} \, d\mu \ .$$

Let us replace, in these two formulae, $(Q^2 + \mu)^{-1}$ by the asymptotic resolvent $R(\mu)$. Then $R(\mu)(Q^2 + \mu) - 1$ is smoothing of any order, with the corresponding norms controlled by $(1 + \mu)^{-k}$. Thus, when we replace $Q^2/(Q^2 + \mu)$ by $Q^2 R(\mu)$, we use

$$\frac{Q^2}{Q^2 + \mu} \left(1 - (Q^2 + \mu) \, R(\mu) \right) = \frac{Q^2}{Q^2 + \mu} - Q^2 \, R(\mu),$$

which is therefore also smoothing of any order if $\mu \geq 0$. Using the boundedness of

$$\frac{\mu^{1/2} \, Q^2}{Q^2 + \mu} \quad \text{for} \quad \mu \geq 0,$$

a similar statement holds for

$$\mu^{1/2} \left(\frac{Q}{Q^2 + \mu} - Q \, R(\mu) \right) \ .$$

Since the asymptotic symbol of $(Q^2 + \mu)^{-1}$ is, at the principal level,

$$\sigma_{-4} = \left(\|\xi_v\|^4 + \|\xi_n\|^2 + \mu \right)^{-1}$$

one obtains the principal symbols of the operators

$$(40) \qquad\qquad |D|_r = \frac{\sqrt{2}}{2\pi} \int_0^\infty Q^2 \, R(\mu) \, \mu^{-3/4} \, d\mu,$$

22

$$(41) \qquad D_r = \frac{\sqrt{2}}{2\pi} \int_0^\infty Q\ R(\mu)\ \mu^{-1/4}\ d\mu$$

as the integrals

$$(42) \qquad \frac{\sqrt{2}}{2\pi} \int_0^\infty \frac{\|\xi\|'^4}{\|\xi\|'^4 + \mu}\ \mu^{-3/4}\ d\mu = \|\xi\|'$$

$$(43) \qquad \frac{\sqrt{2}}{2\pi} \int_0^\infty \sigma_2(Q) \left(\|\xi\|'^4 + \mu\right)^{-1} \mu^{-1/4}\ d\mu = \frac{\sigma_2(Q)(\xi)}{\|\xi\|'}\ .$$

This is enough to show that both $|D|$ and D are $\Psi DO'$ of order 1 and to give in local coordinates the asymptotic expansion of their symbol.

Since the principal symbol of $|D|$ is $\|\xi\|' \cdot 1$, a multiple of the identity matrix, it commutes with the symbol of any $\Psi DO'$ of order 0. This shows that, with δ denoting the derivation

$$\delta(T) = [|D|, T],$$

one has:

Lemma I.2. *Any $\Psi DO'$ of order 0 belongs to $\underset{n \geq 1}{\cap}$ Dom δ^n.*

This applies to the multiplication operator f as well as $[D, f]$ and proves Theorem I.1 1).

For assertion 2) of the theorem, we shall prove the more precise result

$$(44) \qquad |D|^{-1} \in \mathcal{L}^{(v+2n, \infty)}\ ,$$

which in turn will follow from:

Lemma I.3. *Let P be a $\Psi DO'$ of order $-(v + 2n)$. Then*

$$\mu_k(P) = O(k^{-1})$$

where $\mu_k(P)$ is the kth characteristic value of P.

Proof. It is enough to check this locally, so that one can assume without loss of generality that

$$M = \mathbb{T}^v \times \mathbb{T}^n \quad \text{and} \quad P = \left(\Delta_v^2 \otimes 1 + 1 \otimes \Delta_n + 1\right)^{-\frac{v+2n}{4}}\ .$$

23

One just has to bound the number of eigenvalues $\lambda > \varepsilon$ of P by C/ε for some $C < \infty$. Equivalently, one has to bound the number of eigenvalues $\lambda < \varepsilon^{-\frac{4}{v+2n}}$ of $\Delta_v^2 \otimes 1 + 1 \otimes \Delta_n + 1$ by C/ε. But this number is less than $N_v(E)\ N_n(E)$, where $E = \varepsilon^{-\frac{4}{v+2n}}$, $N_v(E)$ is the number of eigenvalues of Δ_v^2 less than E, while $N_n(E)$ stands for the number of eigenvalues of Δ_n less than E.

One has $N_v(E) \leq C_v\ E^{v/4}$, $N_n(E) \leq C_n\ E^{n/2}$ and we get the required bound. ∎

For Theorem I.1 3), we just note that the choice of H does not affect the principal symbols of either D or $|D|$, while any $\Psi DO'$ of order 0 is bounded.

It remains to prove assertion 4). The operators $U_\varphi\ D\ U_\varphi^{-1}$, $U_\varphi\ |D|\ U_\varphi^{-1}$ are $\Psi DO'$ of order 1, with the same principal symbols as D and $|D|$ respectively. This shows that the following are $\Psi DO'$ of order 0:

$$[D, U_\varphi]\ U_\varphi^{-1}\ ,\ [|D|, U_\varphi]\ U_\varphi^{-1}\ .$$

In particular, they are bounded and belong to $\cap_{n \geq 1}$ Dom δ^n. Thus, $U_\varphi \in$ Dom δ, $\delta(U_\varphi)\ U_\varphi^{-1} \in$ Dom δ, hence $U_\varphi \in$ Dom δ^2. By induction, using $\delta(U_\varphi) \cdot U_\varphi^{-1} \in$ Dom δ^n, one gets $U_\varphi \in \cap_{n \geq 1}$ Dom δ^n and thus $[D, U_\varphi] = ([D, U_\varphi]\ U_\varphi^{-1}) \cdot U_\varphi \in \cap_{n \geq 1}$ Dom δ^n. ∎

5. The Dixmier trace of $\Psi DO'$ of order $-(v + 2n)$

We shall now describe the kernels $k(x, y)$ for operators of order $-(v+2n)$ and compute their Dixmier trace. As we shall see, the relevant question is that of extending a homogeneous symbol $\sigma(\xi)$, $\xi \in \mathbb{R}^{v+n} \backslash \{0\}$,

$$(45) \qquad\qquad \sigma(\lambda \cdot \xi) = \lambda^{-(v+2n)}\ \sigma(\xi) \qquad \forall \lambda \in \mathbb{R}_+^*$$

to a homogeneous distribution.

The degree of homogeneity $q = -(v+2n)$ considered in (45) is the limit case for integrability both near 0 and near ∞. Indeed, the Jacobian of $\xi \to \lambda \cdot \xi$ is λ^{v+2n} and on each orbit of the flow F,

$$F_s(\xi) = e^s \cdot \xi,$$

the measure $\sigma(\xi)\ d^{v+n}\ \xi$ is proportional to $\frac{d\lambda}{\lambda}$. It thus has a logarithmic divergency both at 0 and at ∞. They turn out to be intimately related and we shall investigate the divergency at 0, following closely ([B-G]).

24

We consider the linear form on the space

$$\left\{f \in \mathcal{S}(\mathbb{R}^{v+n}) \; ; \; f(0) = 0\right\} = \mathcal{S}_0$$

given by

$$L(f) = \int f(\xi) \, \sigma(\xi) \, d^{v+n} \, \xi \, .$$

It makes sense because σ is bounded at ∞ (polynomial growth would be enough) and $f(0) = 0$ takes care of the non integrability at 0.

With $f_\lambda(\xi) = f(\lambda^{-1} \cdot \xi)$ the homogeneity of σ means:

$$(46) \qquad\qquad L(f_\lambda) = L(f) \qquad \forall f \in \mathcal{S}_0 \, .$$

By the Hahn-Banach theorem the linear form L extends from the hyperplane $\mathcal{S}_0$ to all of $\mathcal{S}$ as a continuous linear form and we get a one dimensional affine subspace of $\mathcal{S}'$:

$$(47) \qquad\qquad E = \{\tau \in \mathcal{S}' \; ; \; \tau|\mathcal{S}_0 = L\} \, ,$$

the corresponding linear space is the space of multiples of δ_0 the Dirac mass at 0.

The dilations θ_λ,

$$\langle \theta_\lambda(\tau), f \rangle = \langle \tau, f_\lambda \rangle$$

act on E; since they act trivially on the associated linear space, their action on E is given, for some constant c, by

$$(48) \qquad\qquad \theta_\lambda \, \tau = \tau + c \log \lambda \, \delta_0 \, , \qquad \forall \tau \in E \, , \; \lambda \in \mathbb{R}_+^* \, .$$

To determine c, let $\psi \in C_c^\infty([0, \infty[)$ be identically 1 near 0, and let τ be given by

$$(49) \qquad \tau(f) := L\left(f - f(0) \, \psi\left(\| \; \|'\right)\right) = \int \left(f(\xi) - f(0) \, \psi\left(\|\xi\|'\right)\right) \, \sigma(\xi) \, d\xi \, .$$

One has,

$$\begin{aligned}
\tau(f_\lambda) - \tau(f) &= \int \left(f(\lambda^{-1} \cdot \xi) - f(0) \, \psi\left(\|\xi\|'\right)\right) \, \sigma(\xi) \, d\xi \\
&\quad - \int \left(f(\xi) - f(0) \, \psi\left(\|\xi\|'\right)\right) \, \sigma(\xi) \, d\xi \\
&= f(0) \int \left(\psi\left(\|\xi\|'\right) - \psi\left(\lambda\|\xi\|'\right)\right) \, \sigma(\xi) \, d\xi \\
&= f(0) \, c_\sigma \int_0^\infty \left(\psi(\mu) - \psi(\lambda\mu)\right) \, \frac{d\mu}{\mu}
\end{aligned}$$

25

where c_σ is obtained as the pairing between any transversal cycle $\|\xi\|' = $ constant to the foliation of $\mathbb{R}^{v+n}\backslash\{0\}$ by the orbits of the flow F and the closed de Rham current obtained as the contraction $i_e(\sigma\, d\xi)$ of the differential form $\sigma\, d\xi$ by the vector field $e = \lambda^{-1}\, d/d\lambda$ generating the flow.

Letting $\mu = e^u$, $\lambda = e^s$ one gets

$$\int_0^\infty (\psi(\mu) - \psi(\lambda\mu))\ \frac{d\mu}{\mu} = \int_{-\infty}^\infty \left(\psi(e^u) - \psi(e^{u+s})\right)\ du = s = \log\lambda\ .$$

Thus we have shown that $c = c_\sigma$ is exactly the obstruction to extending σ as a homogeneous distribution on $\mathbb{R}^{v+n}$ (cf. [B-G]).

We can write, in a formal way,

$$(50) \qquad\qquad c_\sigma = \int_{\|\xi\|'=1} i_e\, \sigma(\xi)\, d\xi\ .$$

Let us now relate this obstruction to the behavior of the inverse Fourier transform of σ

$$(51) \qquad\qquad \check{\sigma}(y) = (2\pi)^{-(v+n)} \int e^{i\langle y,\xi\rangle}\, \sigma(\xi)\, d\xi\ .$$

We first need to relate the oscillatory integral definition (51) to the Fourier transform for tempered distributions.

For $y \neq 0$, the oscillatory integral is defined as the value of the convergent integral

$$(52) \qquad\qquad (2\pi)^{-v+n} \int e^{i\langle y,\xi\rangle}\, (P^k\, \sigma(\xi))\, d\xi \qquad k \geq 1$$

where the symbol $\sigma(\xi)$ has been smoothed for ξ small, and P_y is a differential operator of degree 1 in $\frac{\partial^\alpha}{\partial\xi^\alpha}$ such that

$$P_y^t(e^{i\langle y,\xi\rangle}) = e^{i\langle y,\xi\rangle}\ .$$

The smoothing of σ near $\xi = 0$ introduces an ambiguity of the addition of arbitrary elements of $(C_c^\infty)\check{\ } \subset \mathcal{S}$. But this does not affect the behavior at $y = 0$, which we are after.

Having chosen an extension τ of σ as a distribution, we need to check that the inverse Fourier transform $\check{\tau}$, a tempered distribution, is represented by a locally integrable tempered function h whose behavior at $y = 0$ is the same as for the oscillatory integral.

Since the Fourier transform of a distribution with compact support is represented by a nice smooth function, we just need to know that if σ_1 is a symbol, smooth on all of $\mathbb{R}^{v+n}$,

26

then its Fourier transform as a tempered distribution is given by the oscillatory integral expression (52).

To check this, let $P\left(y, \frac{\partial}{\partial \xi}\right) = \frac{1}{i} \sum y^j \frac{\partial}{\partial \xi^j}$; then

$$P \; e^{i\langle y, \xi \rangle} = \|y\|^2 \; e^{i\langle y, \xi \rangle} \; .$$

With $f \in \mathcal{S}(\mathbf{R}_{n+v})_0$, we can write:

$$\int \check{f}(\xi) \; \sigma_1(\xi) \; d\xi = (2\pi)^{-(n+v)} \int \int f(y) \; e^{i\langle y \cdot \xi \rangle} \; dy \; \sigma_1(\xi) \; d\xi$$

$$= (2\pi)^{-(n+v)} \int \int P(e^{i\langle y \cdot \xi \rangle}) \; \|y\|^{-2} \; f(y) \; \sigma_1(\xi) \; dy \; d\xi$$

$$= (2\pi)^{-(n+v)} \int \int e^{i\langle y, \xi \rangle} \; \|y\|^{-2} \; f(y) \; (P\sigma_1)(\xi) \; dy \; d\xi$$

$$= (2\pi)^{-(n+v)} \int \int e^{i\langle y, \xi \rangle} \; \|y\|^{-2} \; P\sigma_1(\xi) \; d\xi \; f(y) \; dy \; .$$

Thus, we know that the distribution $\check{\tau}$ is represented outside 0 by a smooth function with tempered growth. This function is then unique and the homogeneity property (48) implies, using $\langle \lambda \cdot \xi, y \rangle = \langle \xi, \lambda \cdot y \rangle$,

(53)
$$\check{\tau}(\lambda^{-1} \cdot y) = (\theta_\lambda \; \tau)^\vee(y) = \check{\tau}(y) + c' \log \lambda \; .$$

Thus,

$$\check{\tau}(y) = \check{\tau}(y/\|y\|') - c' \log(\|y\|') \; , \; c' = (2\pi)^{-(n+v)} \; c \; .$$

We are now ready to deal with the Dixmier trace of $\Psi DO'$ of order $-(v + 2n)$.

We let (M, V) be as above, with M compact.

Proposition I.2. *Let T be a $\Psi DO'$ of order $-(v + 2n)$. Then*

1) T is measurable, $T \in \mathcal{L}^{(1,\infty)}$, with its Dixmier trace $\mathrm{Tr}_\omega(T)$ independent of ω and given by

$$\mathrm{Tr}_\omega = (2\pi)^{-(n+v)} \; \frac{1}{v + 2n} \int_{\|\xi\|'=1} \sigma(x, \xi) \; i_e \; (dx \; d\xi) \; ,$$

where e is the generator of the flow

$$F_s(\xi_v, \xi_n) = (e^s \; \xi_v, e^{2s} \; \xi_n)$$

and the choice of transversal $\|\xi\|' = 1$ is irrelevant. Also $dx \; d\xi$ corresponds to the symplectic volume form.

2) *The kernel $k(x, y)$ for T has the following behavior near the diagonal*

$$k(x, y) = c(x) \, \log(\|x - y\|') + 0(1) \, ,$$

where the 1-density $c(x)$ is given by the formula

$$c(x) = (2\pi)^{-(n+v)} \int_{\|\xi\|'=1} \sigma(x, \xi) \, i_e \, d\xi \, .$$

Remark I.1. Before beginning the proof, let us note that 1) reduces to the usual formula (cf. [Co]) for the Dixmier trace of ordinary ΨDO when either $n = 0$ or $v = 0$; in the latter case the exponent $2s$ accounts for the $2n$. Note also that in 2) the choice of the local distance function $\|x - y\|'$ has no effect on the value of $c(x)$. The statement 2) is a special case of [B-G].

Proof. 1) By Lemma I.2, one has $T \in \mathcal{L}^{(1,\infty)}$ so that $\mathrm{Tr}_\omega(T)$ is well defined. Since operators of lower order are (by the argument of Lemma I.2) of trace class, it follows that $\mathrm{Tr}_\omega(T)$ only depends upon the principal symbol $\sigma(T)$. The map

(54) $$\sigma \to \mathrm{Tr}_\omega(T_\sigma)$$

is then a positive linear form on the space $\mathcal{F}$ of homogeneous symbols of order $-(v + 2n)$. It is therefore given by a positive measure on the (non canonical) unit sphere

(55) $$S^* \left\{ (x, \xi) \in V^* \oplus N^* \; ; \; \|\xi\|' = 1 \right\} \, .$$

The unitary invariance of Tr_ω together with the use of translations in a foliation chart, show that this measure is absolutely continuous with respect to the smooth measure dx on M. The diffeomorphism invariance of the $\Psi DO'$-calculus shows that the conditional measures on the fibers of

(56) $$p : S^* \to M$$

must be, in the appropriate sense, invariant under all maps

$$(\xi_v, \xi_n) \to (L_v \, \xi_v, L_n \, \xi_n)$$

with both L_v and L_n invertible.

Such maps do not act transitively on $V_x^* \oplus N_x^*$. The only two invariant subspaces are $V_x^* \oplus 0$, $0 \oplus N_x^*$. But they do not carry any measure with the correct homogeneity.

This implies that there exists a constant $a(\omega)$ such that

$$(57) \qquad \mathrm{Tr}_\omega(T) = a(\omega) \int_{\|\xi\|'=1} \sigma(x,\xi) \, i_e \, dx \, d\xi \ .$$

To show that this constant $a(\omega)$ does not depend on ω and to determine it, one just needs to compute $\mathrm{Tr}_\omega(T)$ for one specific example for each value of v and n. With the notations of Lemma I.2, we take:

$$(58) \qquad T = \left(\Delta_v^2 \otimes 1 + 1 \otimes \Delta_n + 1 \right)^{-\left(\frac{v+2n}{4} \right)} \ .$$

In order to compute $\mathrm{Tr}_\omega(T)$, we just use the following general fact (cf. Appendix A).

(59) *Let Δ be a positive (unbounded) operator such that $\Delta^{-1} \in \mathcal{L}^{(p,\infty)}$ for some $p \geq 1$, and*

$$t^p \ \mathrm{Trace} \left(e^{-t\Delta} \right) \underset{t \to 0}{\to} L \ .$$

Then $\mathrm{Tr}_\omega(\Delta^{-p})$ is independent of ω and given by

$$\mathrm{Tr}_\omega(\Delta^{-p}) = \frac{L}{\Gamma(p+1)} \ .$$

We take $\Delta = \Delta_v^2 \otimes 1 + 1 \otimes \Delta_n + 1$, $p = \frac{v+2n}{4}$. To compute L it is enough to determine

$$\lim_{t \to 0} t^{p_v} \ \mathrm{Trace} \left(e^{-t\Delta_v^2} \right) = L_v \ , \quad p_v = v/4 \ ,$$

$$\lim_{t \to 0} t^{p_n} \ \mathrm{Trace} \left(e^{-t\Delta_n} \right) = L_n \ , \quad p_n = n/2$$

which then gives $L = L_v \, L_n$.

Using (59) one has $L_v = \Gamma(p_v + 1) \, \mathrm{Tr}_\omega(\Delta_v^{-v/2})$ and $L_n = \Gamma(p_n + 1) \, \mathrm{Tr}_\omega(\Delta_n^{-n/2})$. Choosing the standard metric $\sum d\theta_j^2$ on both $\mathbb{T}^v$ and $\mathbb{T}^n$ gives, with $|S^k|$ the volume of the k-dimensional sphere,

$$(60) \qquad L_v = \Gamma\left(\frac{v}{4} + 1 \right) \frac{1}{v} \, |S^{v-1}| \quad , \quad L_n = \Gamma\left(\frac{n}{2} + 1 \right) \frac{1}{n} \, |S^{n-1}|$$

and

$$\mathrm{Tr}_\omega(\Delta^{-p}) = \Gamma\left(\frac{v+2n}{4} + 1 \right)^{-1} L_v \, L_n \ .$$

29

The principal symbol of Δ^{-p} is

(61)
$$\sigma(x, \xi) = (\|\xi\|')^{-(v+2n)} \ .$$

If we let

$$|S|' = \int_{\|\xi\|'=1} i_e \ (d\xi),$$

we have the equalities

(62)
$$\int_{\|\xi\|'=1} \sigma(x, \xi) \ i_e \ (dx \ d\xi) = (2\pi)^{n+v} \ |S|'$$

and

(63)
$$\int f(\|\xi\|') \ d\xi = |S|' \int_0^\infty f(\rho) \ \rho^{v+2n} \ \frac{d\rho}{\rho} \ ,$$

whenever both sides make sense.

Using $f(\rho) = e^{-\lambda \rho^4}$ yields

(64)
$$|S|' = \frac{1}{2} \ \Gamma \left(\frac{v+2n}{4} \right)^{-1} \ \Gamma \left(\frac{v}{4} \right) \ |S^{v-1}| \ \Gamma \left(\frac{n}{2} \right) \ |S^{n-1}| \ .$$

Together with (60) and (62), this gives the normalization:

(65)
$$\mathrm{Tr}_\omega(\Delta^{-p}) = (2\pi)^{-(n+v)} \ (v+2n)^{-1} \int_{\|\xi\|'=1} \sigma(x, \xi) \ i_e \ (dx \ d\xi) \ .$$

2) The kernel $k(x, y)$ of T is given by

$$k(x, y) = (2\pi)^{-(v+n)} \int e^{i \langle x-y, \xi \rangle} \ \sigma(x, \xi) \ d\xi \ ,$$

so that, for fixed x, it is, as a function of $y - x$, the Fourier transform of $\sigma(x, \xi)$. Thus, using (53) we get the required answer. For a more detailed proof, the reader is referred to [B-G] . ∎

Proposition I.2 extends immediately to non scalar operators, with $\sigma(x, \xi)$ replaced by its trace $\mathrm{tr}(\sigma(x, \xi))$ taken in the fiber over x. We can therefore apply it to the operator $|D|^{-(v+2n)}$ of Theorem I.1, whose symbol is the identity matrix (for $\|\xi\|' = 1$) on a space of dimension $2^{(v+n)}$. We get:

(66)
$$\mathrm{Tr}_\omega \left(f \ |D|^{-(v+2n)} \right) = \pi^{-(n+v)} \ (v+2n)^{-1} \ |S|' \int f(x) \ dx$$

30

where $|S|'$ is given by (64) and dx is the volume form on M corresponding to the given Euclidean structures on both V and N.

6. The analogue of the Wodzicki residue for $\Psi DO'$ operators

Let us now go back to the obstruction c_σ (cf. (50)), and exploit its definition involving the behavior of T near 0 rather than (the ultraviolet behavior) near ∞.

Lemma I.4. ([B-G]) *Let* $\sigma \in C^\infty(\mathbb{R}^{n+v}\backslash\{0\})$ *be homogeneous of order* $q, i.e. \sigma(\lambda \cdot \xi) = \lambda^q \sigma(\xi)$ *, $\forall \xi \neq 0$, $\forall \lambda \in \mathbb{R}_+^*$, with* $(q \in \mathbb{C})$.

a) *If* $q \notin \{-(v+2n) - k \ ; \ k \in \mathbb{N}\}$ *then* σ *extends to a homogeneous distribution on* $\mathbb{R}^{n+v}$.

b) *If* $q = -(v+2n) - k$, *then the obstruction to homogeneous extension is given by the* $c_{\xi^\alpha \sigma}, |\alpha| = k$.

Proof. Let ψ be as in (48) and k the integral part of

$$-\text{Re}(q) - (v + 2n) = a \ .$$

The size of $\sigma(\xi)$ for ξ small is comparable to $(\|\xi\|')^{\text{Re}q} = \|\xi\|'^{-(v+2n+a)}$. Thus $\sigma(\xi) \, \xi^\alpha$ is locally integrable if $|\alpha| > k$ and the following is an extension of σ:

$$(67) \qquad \tau(f) = \int \left(f(\xi) - \sum_{|\alpha| \leq k} \frac{\xi^\alpha}{\alpha!} f^{(\alpha)}(0) \, \psi(\xi) \right) \sigma(\xi) \, d\xi \ .$$

One gets then,

$$\tau(f_\lambda) - \lambda^{q+(v+2n)} \, \tau(f) = \int \left(f(\lambda^{-1} \cdot \xi) - \sum_{|\alpha| \leq k} \frac{\lambda^{-|\alpha|}\xi^\alpha}{\alpha!} f^{(\alpha)}(0) \, \psi(\xi) \right) \sigma(\xi) \, d\xi$$

$$- \lambda^{q+(v+2n)} \int \left(f(\xi) - \sum_{|\alpha| \leq k} \frac{\xi^\alpha}{\alpha!} f^{(\alpha)}(0) \, \psi(\xi) \right) \sigma(\xi) \, d\xi$$

$$= \lambda^{q+(2n+v)} \sum_{|\alpha| \leq k} \frac{f^\alpha(0)}{\alpha!} \left(\int_{\|\xi\|'=1} \xi^\alpha \sigma(\xi) \, i_e \, d\xi \right) \rho_\alpha$$

$$\rho_\alpha = \int_0^\infty (\psi(\mu) - \psi(\lambda\mu)) \, \mu^{q+(v+2n)+|\alpha|} \, d\mu/\mu \ .$$

31

To prove a), it is enough to choose ψ so that the ρ_α all vanish. With $\psi(\mu) = h(\log \mu)$, we thus look for $h' \in C_c^\infty(\mathbb{R})$, $\int h'(s)\, ds = -1$, such that

(68)
$$\int_{-\infty}^{\infty} h'(s)\, e^{bs}\, ds = 0 \qquad \forall b = q + (v + 2n) + |\alpha|\ ,\ 0 \le |\alpha| \le k\ .$$

One just lets $h' = \Pi(d/ds + b)f$, $f \in C_c^\infty(\mathbb{R})$, with $\int f(s)\, ds = (\Pi b)^{-1}$. This is possible since $b \ne 0$ by hypothesis.

Assertion b) can be proved in a similar fashion, since for $q = -(v + 2n) - k$ one can get $\rho_\alpha = 0$ for any α, $|\alpha| < k$. ∎

The ambiguity in the extension of σ is, a priori, of the form $\sum\limits_{|\alpha| \le k} a_\alpha\, \partial^\alpha\, \delta_0$. By a), if $q \notin \{-(v + 2n) - \mathbb{N}\}$, the homogeneous extension exists and is unique, except when $q \in \mathbb{N}$.

Corollary I.1. ([B-G]) *Let $\sigma \in C^\infty(\mathbb{R}^{n+v})$ have an asymptotic expansion $\sigma \sim \sum\limits_{k \le q} \sigma_k$ in homogeneous symbols (for $\lambda\cdot$). Then the Fourier transform $\breve{\sigma}$ (of σ smooth at 0) has a behavior at $y = 0$ of the form:*

$$\breve{\sigma}(y) = \sum_{-(q+(v+2n))}^{0} a_j(y) + c \log \|y\|' + 0(1)\ ,$$

where a_j is homogeneous of degree j.

Proof. We can neglect the difference $\sigma - \sum\limits_{-(v+2n) \le k \le q} \sigma_k$ since it is integrable at ∞ and yields a $0(1)$ contribution for small y. By Lemma I.4, each σ_k, $k > -(v + 2n)$ extends to a homogeneous distribution, so that the Fourier transform $\breve{\sigma}_k$ of σ_k (smoothed at 0) has the indicated divergency at 0. The case $k = v + 2n$ follows from (53). ∎

We see that

(69) *the coefficient of $\log \|y\|'$ only depends on $\sigma_{-(v+2n)}$ and is equal to*

$$(2\pi)^{-(n+v)} \int_{\|\xi\|'=1} \sigma_{-(v+2n)}\, (\xi)\, i_e\, d\xi\ .$$

One can check directly, using (29) and (34), that the density

(70)
$$(2\pi)^{-(n+v)} \int_{\|\xi\|'=1} \sigma_{-(v+2n)}\, (x,\xi)\, i_e\, d\xi$$

is invariantly defined under the action of diffeomorphisms on the $\Psi DO'$ calculus. The obtained density is the coefficient of $\log \|x - y\|'$, in the expansion near the diagonal, of the kernel $k(x, y)$ for $P = P_\sigma$.

Proposition I.3. *The Dixmier trace* Tr_ω *has a canonical extension to a trace on the algebra of* $\Psi DO'$ *operators of arbitrary order. It is given globally by the equality*

$$\mathrm{Tr}_\omega(T) = \frac{1}{v + 2n} \int_M c(x) \, ,$$

where $c(x)$ is the 1-density occuring in the expansion of the kernel k of T near the diagonal, as a coefficient of $\log \|x - y\|'$. In local coordinates it is given by $(v + 2n)^{-1}$ times the expression (70).

The fact that the asserted extension is a trace is proved in greater generality in Section II (cf. Prop. II.1).

We proceed to show that there is a natural extension, similar to [K-V], of the ordinary trace of operators to $\Psi DO'$ of complex order.

To this end, we go back to the space of symbols, where we need to define what is meant by a holomorphic map $z \to \sigma_z$ with values in the space of symbols. We want the order $f(z)$ to be holomorphic in z, the bounds in the asymptotic expansion $\sigma(z) \sim \sum \sigma_{f(z)-p}$ to be uniform and the pointwise values of $\sigma(z)(\xi)$, $\sigma(z) \in C^\infty(\mathbb{R}^{v+n})$, to be holomorphic in the variable z. Then each $\sigma_{f(z)-p}(\xi)$ is also holomorphic in z. The functional

$$(71) \qquad L(\sigma) = \check{\sigma}(0) = (2\pi)^{-(v+n)} \int \sigma(\xi) \, d\xi$$

is well defined on symbols of order $< -(v + 2n)$ (for the real part of the order) and it is holomorphic inasmuch as $L(\sigma_z)$ is a holomorphic function of z for any holomorphic map $z \to \sigma_z$.

Lemma I.5. *The functional L has a unique holomorphic extension $\widetilde{L}$ to the space of symbols of non integral order $(z \notin \mathbb{Z})$. The value of $\widetilde{L}$ on $\sigma \sim \sum \sigma_{z-p}$ is given by*

$$\widetilde{L}(\sigma) = (2\pi)^{-(v+n)} \int \left(\sigma - \sum_0^N \tau_{z-p} \right)(\xi) \, d\xi \quad , \quad N \geq \mathrm{Re}(z) + (v + 2n)$$

where τ_{z-p} is the unique homogeneous extension of σ_{z-p}.

33

This unique extension τ of σ is given by Lemma I.4 a).

Proof. First, the value of N used in Lemma I.5 is irrelevant, since the Fourier transform of a homogeneous distribution such as τ_{z-p} vanishes at 0 if $p > \mathrm{Re}z + (v + 2n)$.

Also the uniqueness is clear since any σ of order z can be connected to integrable order by a holomorphic path.

It remains to show that if $z \to \sigma(z)$ is holomorphic, with order $f(z) \notin \mathbb{Z}$ then $L(\sigma(z))$ is holomorphic.

For large ξ the pointwise value of $\sigma(z) - \sum_0^N \sigma_{f(z)-p}$ is holomorphic in z and has uniformly integrable behavior at ∞; thus it is enough to control the behavior in z of

$$\int \left(\sigma - \sum_0^N \tau_{f(z)-p} \right) \varphi(\xi)\, d\xi \ ,$$

where φ has compact support. In fact we can consider separately the term $\int \tau_{f(z)-p}\, \varphi(\xi)\, d\xi$, which is holomorphic in z by the very construction of τ (cf. (67) and (68)). ∎

Remark I.2. To a symbol of order z and to any given $p \in \mathbb{N}$ one can assign the number

$$\int_{\|\xi\|'=1} \sigma_{z-p}(\xi)\, i_e\, d\xi$$

and the functional thus obtained is holomorphic, but it does not vanish on σ with integrable order and cannot be added to $\widetilde{L}$ to yield another extension.

Let us now return to the (compact) manifold (M, V) as above, and consider the product $\Psi^0 \times \mathbb{C}$ of the space of $\Psi DO'$ of order 0 by $\mathbb{C}$, endowed with the product structure of complex manifold. We shall adapt the method of [K-V] to our context to obtain:

Proposition I.4. *The function* $(P, z) \to \mathrm{Trace}(P|D|^{-z})$ *is holomorphic on* $\Psi^0 \times \{z \ ; \ \mathrm{Re}z > (v + 2n)\}$ *and extends uniquely to a holomorphic function* TR *on* $\Psi^0 \times (\mathbb{C}\backslash\mathbb{Z})$.

Proof. In a local chart, the trace of a $\Psi DO'$ $P = P_\sigma$ of order $< -(v + 2n)$ is given by

$$(72) \qquad\qquad \mathrm{Trace}(P_\sigma) = (2\pi)^{-(n+v)} \int \sigma(x, \xi)\, dx\, d\xi \ ,$$

where the total symbol σ is smooth.

Thus, in a local chart, the following formula provides the required extension of Trace to $\Psi DO'$ of arbitrary order $z \notin \mathbb{Z}$

$$(73) \qquad \mathrm{TR}(P_\sigma) = \int \widetilde{L}(\sigma(x, \cdot)) dx \ .$$

The ambiguity of smoothing operators does not alter the existence of this extension globally on all $\Psi DO'$ of order $z \notin \mathbb{Z}$ and yields the required extension of Trace, provided one knows that $|D|^{-z}$ is a $\Psi DO'$ of order $-z$ and a holomorphic function of z. This follows from the proof of Theorem I.1 (see (38)), using the asymptotic resolvent of Q^2 (cf.[S]). $\blacksquare$

Let us now relate the value of $\mathrm{Tr}_\omega(P)$, $P \in \Psi DO'$ to the residue at $z = 0$ of the function:

$$(74) \qquad \zeta(z) = \mathrm{TR}\left(P|D|^{-z}\right) \ .$$

We can work first at the level of symbols and consider a fixed symbol σ of integral order q. We let σ_z, $\sigma_z(\xi) = \sigma(\xi)\,(\|\xi\|')^{-z}$ and investigate the behavior of $\widetilde{L}(\sigma_z)$ near $z = 0$. Let $N \geq q + (v + 2n)$, then

$$\widetilde{L}(\sigma_z) = (2\pi)^{-(v+n)} \int \left(\sigma - \sum_0^N \sigma_{q-k}\right)(\xi)\ (\|\xi\|')^{-z}\ d\xi$$

where $\sigma_{q-k}(\xi)\,\|\xi\|^{1-z}$ is replaced near 0 by its unique extension as a homogeneous distribution.

The singularity at $z = 0$ comes from ξ in the neighborhood of 0. When $q - k > -(v + 2n)$, $\sigma_{q-k}(\xi)\,d\xi$ is integrable at 0 and the unique extension of $\sigma_{q-k}(\xi)\,\|\xi\|'^{-z}\,d\xi$ is holomorphic in z at $z = 0$. Thus none of these terms contribute to the singularity of $\widetilde{L}(\sigma_z)$ at $z = 0$.

We can choose $N = q + (v + 2n)$ since, by Lemma I.5, any larger value gives the same answer. We thus need to understand the behavior at $z = 0$ of

$$(75) \qquad \int_{\|\xi\|' \leq a} \sigma_{-(v+2n)}(\xi)\,\|\xi\|'^{-z}\,d\xi$$

where $\sigma_{-(v+2n)}(\xi)\,(\|\xi\|')^{-z}\,d\xi$ is extended uniquely as a homogeneous distribution at $\xi = 0$. But for $\mathrm{Re}\,z < 0$ one has integrability near 0 so that this unique extension is the obvious one and one can write (75) as

$$(76) \qquad \left(\int_{\|\xi\|' = 1} \sigma_{-(v+2n)}(\xi)\,i_e\,d\xi\right)\left(\int_0^a \mu^{-z}\,\frac{d\mu}{\mu}\right) \ .$$

35

As $\int_0^a \mu^{-z} \frac{d\mu}{\mu} = -\left[\frac{\mu^{-z}}{z}\right]_0^a = -a^{-z}/z$, one gets that the singularity of $\widetilde{L}(\sigma_z)$ at $z = 0$ is a simple pole with residue:

$$(77) \qquad (2\pi)^{-(v+2n)} \int_{\|\xi\|'=1} \sigma_{-(v+2n)}(\xi)\ i_e\ d\xi\ .$$

Next, when we investigate (74) the situation is more complicated since in a local (foliation) chart we have an intricate expression for the total symbol of $|D|^{-z}$.

Let us first remark that the above discussion of the behavior of $\widetilde{L}(\sigma_z)$ continues to hold when σ_z is of the form

$$(78) \qquad \sigma_z = \sigma(\xi, z)\ (\|\xi\|')^{-z}\ ,$$

where $z \to \sigma(\cdot, z)$ is a holomorphic map to symbols of fixed order q. Moreover, the residue at $z = 0$ is given by (77), with $\sigma(\xi, 0)$. Using this, we should be able to replace $|D|$ by the operator $|D_1|$ which in the given (foliation) chart involves the flat metric

$$(79) \qquad \|\xi\|'^4 = \|\xi_v\|^4 + \|\xi_n\|^2\ ,$$

independently of $x = (x_v, x_n)$.

Since the total symbol of D_1^4, assumed to be given by (79), does not involve x, the corresponding (differential) operator is translation invariant. So are the complex powers $|D_1|^z$ and their total symbol is given by (a smoothed version of)

$$(80) \qquad \sigma_z(x, \xi) = (\|\xi\|')^z\ .$$

The computation of the total symbol of $P|D_1|^{-z}$ for any $\Psi D0'$, is then obvious, by (26), and the above discussion shows that the function $\zeta_1(z) = \mathrm{TR}\,(P|D_1|^{-z})$ has a simple pole at $z = 0$, which is given by (77). This continues to hold for any holomorphic map $z \to P_z$ to symbols of fixed order q.

We can now write

$$(81) \qquad P|D|^{-z} = P\ U(z)|D_1|^{-z}\ ,\quad U(z) = |D|^{-z}\ |D_1|^z$$

and it just remains to show that $U(z)$ is a holomorphic map to operators of order 0, with $U(0) = 1$.

Thus, the principal symbol of $U(z)$ will be $\sigma(\xi)^{-z/4}\ \sigma_1(\xi)^{z/4}$. By construction, $U(z)$ is the $\Psi D O'$ product of two holomorphic maps and hence is holomorphic. We thus proved the following result:

36

Theorem I.2. *Let P be a $\Psi DO'$ of integral order q. Then the function $z \to \mathrm{Trace}\,(P|D|^{-z})$ is holomorphic for $\mathrm{Re}\,z > q + (v + 2n)$ and admits a (unique) analytic continuation to $\mathbb{C} \backslash \mathbb{Z}$ with at most simple poles at integers $k \leq q + (v + 2n)$. Its residue at $z := 0$ is given by*

$$\mathrm{Res}_{z=0}\,\mathrm{Trace}\,\bigl(P|D|^{-z}\bigr) = (v + 2n)\,\mathrm{Tr}_\omega(P),$$

where Tr_ω is defined in Proposition I.3.

II. The universal local index formula

1. Dimension spectrum

In this section, we shall describe a general local index formula in terms of the Dixmier trace, extended to operators of arbitrary order, for our spectral triples:

$$(1) \qquad (\mathcal{A}, \mathcal{H}, D) .$$

Contrary to the standard practice, we shall focus on the odd case, the point being that in the even case there is a natural obstruction to express the (cyclic cocycle) character (cf. [Co]) of the triple (1) in terms of a residue or Dixmier trace. Indeed, the latter vanishes on any finite rank operator and thus will give the result 0 whenever $\mathcal{H}$ is finite dimensional. Since it is easy to construct finite dimensional (i.e. $\dim \mathcal{H} < \infty$) even triples with $\mathrm{Ind}(D) \neq 0$ one cannot expect to cover this case as well. However, one can convert any even triple into an odd one by crossing it with S^1, i.e. with the triple

$$(2) \qquad \left(C^\infty(S^1) , \ L^2(S^1) , \ D = \frac{1}{i} \frac{\partial}{\partial \theta} \right) .$$

Thus, there is no real loss of generality in treating the odd case only.

The next point is that the usual notion of dimension (cf. [Co]) for spectral triples, provided by the degree of summability

$$(3) \qquad D^{-1} \in \mathcal{L}^{(p,\infty)} ,$$

gives only an upper bound on dimension and cannot detect the dimensions of the various pieces of a space constructed as a union of pieces of different dimensions $(\mathcal{A}_k, \mathcal{H}_k, D_k)$, $k = 1, ..., N$,

$$(4) \qquad \mathcal{A} = \oplus \mathcal{A}_k , \ \mathcal{H} = \oplus \mathcal{H}_k , \ D = \oplus D_k .$$

In [Co] we gave a formula for the p-dimensional Hochschild cohomology class of the character, namely:

$$(5) \qquad \tau(a^0, \ldots, a^p) = \mathrm{Tr}_\omega \left(a^0 [D, a^1] \ldots [D, a^p] \, |D|^{-p} \right) .$$

Clearly, this Hochschild cocycle cannot account for lower dimensional pieces in a union such as (4).

As it turns out, the correct notion of dimension is given not by a single real number p but by a subset

$$(6) \qquad\qquad Sd \subset \mathbb{C}$$

which shall be called the *dimension spectrum* of the given triple. We shall assume that Sd is a discrete subset of $\mathbb{C}$, condition which can be incorporated in the definition of Sd, as follows:

Definition II.1. *A spectral triple* (1) *has discrete dimension spectrum Sd, if $Sd \subset \mathbb{C}$ is discrete and for any element of the algebra $\mathcal{B}$ generated by the $\delta^n(a)$, $a \in \mathcal{A}$, the function*

$$\zeta_b(z) = \mathrm{Trace}\left(b|D|^{-z}\right)$$

extends holomorphically to $\mathbb{C}\backslash Sd$.

Here δ denotes the derivation $\delta(T) = [|D|, T]$ and we assume that $\mathcal{A} \subset \underset{n>0}{\cap} \mathrm{Dom}\, \delta^n$ (see also Appendix B). The operator $b|D|^{-z}$ of Definition II.1 is then of trace class for $\mathrm{Re}\, z > p$, with p as in (3). On the technical side, we shall assume that the analytic continuation of ζ_b is such that $\Gamma(z)\, \zeta_b(z)$ is of rapid decay on vertical lines $z = s + it$, for any s with $\mathrm{Re}\, s > 0$.

It is not difficult to check that Sd has the correct behavior with respect to the operations of sum and product for spectral triples:

$$(7) \qquad\qquad Sd\ (\text{Sum of two spaces}) = \cup Sd(\text{Spaces})$$

$$(8) \qquad\qquad Sd\ (\text{Product of two spaces}) = Sd(\text{Space}_1) + Sd(\text{Space}_2)\ ;$$

more precisely, (8) holds with the exception of $Sd \cap -\mathbb{N}$.

According to Theorem I.2 of Section I, the dimension spectrum of the hypoelliptic triple considered there is contained in

$$(9) \qquad\qquad \{q \in \mathbb{N}\ ;\ q \leq v + 2n\}\ .$$

It is easy to give many examples of spectral triples with discrete dimension spectrum, but we shall now concentrate on the general theory of such spaces.

39

Our first task will be to extend the Wodzicki residue to this general framework, or equivalently, to extend the Dixmier trace to operators $P|D|^{-z}$ of arbitrary order, where P is an element of $\mathcal{B}$. In fact it is more convenient (cf. Appendix B) to introduce the algebra $\Psi^*(\mathcal{A})$ of operators which have an expansion:

$$(10) \qquad P \simeq b_q|D|^q + b_{q-1}|D|^{q-1} + \cdots \quad , \quad b_q \in \mathcal{B} ,$$

where the equality with $\displaystyle\sum_{-N<n\leq q} b_n\ |D|^n$ holds modulo OP^{-N}.

To see that it is an algebra one uses Theorem B.1 of Appendix B, which gives an identity of the form:

$$(11) \qquad |D|^\alpha\ b \simeq \sum_0^\infty c_{\alpha,k}\ \delta^k(b)\ |D|^{\alpha-k} ,$$

where $c_{\alpha,k}$ is the coefficient of ε^k in the expansion of

$$(12) \qquad (1+\varepsilon)^\alpha = \sum_0^\infty \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}\ \varepsilon^k ,$$

with $\varepsilon(b) = \delta(b)\ |D|^{-1}$.

We shall say that the dimension spectrum Sd is *simple*, when the singularities of the functions $\zeta_b(z)$ of Definition II.1 at $z \in Sd$ are at most simple poles. Similarly, we say that Sd has finite multiplicity k when ζ_b has at most a pole of order k. We shall assume for simplicity that Sd has finite multiplicity in this section.

Proposition II.1 *Let $p < \infty$ be the degree of summability of D.*

a) *For $P \in \Psi^*(\mathcal{A})$ the function $h(z) = \text{Trace}\left(P|D|^{-2z}\right)$ is holomorphic for $\text{Re}\,z > \frac{1}{2}(\text{Order}\ P + p)$ and extends to a holomorphic function on the complement of a discrete subset of $\mathbb{C}$.*

b) *Let $\tau_k(P)$ be the residue at 0 of $z^k\ h(z)$, $k \geq 0$; then*

$$\tau_k(P_1P_2 - P_2P_1) = \sum_{n>0} \frac{(-1)^{n-1}}{n!}\ \tau_{k+n}(P_1\ L^n(P_2)),$$

where L is the derivation $L = 2\log(1+\varepsilon)$.

Proof. a) The statement follows immediately from Definition II. 1, for any finite sum of operators $b_n\ |D|^n$. Furthermore, if P is of order less than $-N$ then $h(z)$ is holomorphic if $\text{Re}\,z > \frac{1}{2}(p-N)$, and for any given z this is achieved for N large enough.

b) First, the derivation $L = 2 \log(1 + \varepsilon)$ makes sense as a power series in ε and can be viewed at the formal level as implemented by $\log |D|^2$.

One has, for any P, an expansion near 0

$$(13) \qquad \text{Trace}\left(P|D|^{-2z}\right) = \sum_{k \geq 0} \tau_k(P) \, z^{-(k+1)} + 0(1) \; .$$

We can then write

$$(14) \qquad \text{Trace}\left(P_2 P_1 |D|^{-2z}\right) = \text{Trace}\left(P_1 (1 + \varepsilon)^{-2z} \, (P_2) \, |D|^{-2z}\right)$$

and, since

$$(15) \qquad (1 + \varepsilon)^{-2z} = \exp(-zL) \; ,$$

we get

$$(16) \qquad \text{Trace}\left(P_2 P_1 |D|^{-2z}\right) = \sum \frac{(-z)^n}{n!} \, \text{Trace}\left(P_1 \, L^n(P_2) \, |D|^{-2z}\right) \; .$$

By (13) we can expand:

$$\text{Trace}\left(P_1 \, L^n(P_2) \, |D|^{-2z}\right) = \sum \tau_q\left(P_1 \, L^n(P_2)\right) \, z^{-(q+1)} + 0(1)$$

and, when multiplied by z^n, we see that we get the exponent $z^{-(k+1)}$ for $n - q = -k$. Thus, the coefficient of $z^{-(k+1)}$ in the expansion (16) is

$$\sum_{n=q-k} \frac{(-1)^n}{n!} \, \tau_q\left(P_1 \, L^n(P_2)\right) = \sum \frac{(-1)^n}{n!} \, \tau_{n+k} \left(P_1 \, L^n(P_2)\right) \; .$$

Therefore, we obtain:

$$(17) \qquad \tau_k(P_2 P_1) - \tau_k(P_1 P_2) = \sum_{n > 0} \frac{(-1)^n}{n!} \, \tau_{n+k}\left(P_1 \, L^n(P_2)\right) \; .$$

∎

It follows, of course, that if q is the multiplicity of Sd, i.e. the highest order of poles, then τ_q is a trace.

By Appendix A, in the case of simple spectrum the trace $\tau = \tau_0$ is an extension of the Dixmier trace, the latter being defined only when the operator $P \in \Psi^*(\mathcal{A})$ belongs to OP^{-p}.

2. Local formula for the Chern character

Before giving the general local formula for the Chern character of a triple $(\mathcal{A}, \mathcal{H}, D)$ with discrete dimension spectrum, we need to recall a few basic definitions from [Co].

First the cyclic cohomology $HC^n(\mathcal{A})$ is defined as the cohomology of the complex of cyclic cochains, i.e. those satisfying

$$(18) \qquad \psi(a^1, \dots, a^n, a^0) = (-1)^n \, \psi(a^0, \dots, a^n) \, , \qquad \forall a^j \in \mathcal{A} \, ,$$

under the coboundary operation b given by:

$$(19) \qquad (b\psi)(a^0, \dots, a^{n+1}) =$$

$$\sum_0^n (-1)^j \, \psi(a^0, \dots, a^j \, a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \, \psi(a^{n+1} \, a^0, \dots, a^n) \, , \quad \forall a^j \in \mathcal{A} \, .$$

Equivalently, $HC^n(\mathcal{A})$ can be described in terms of the second filtration of the (b, B) bicomplex of arbitrary (non cyclic) cochains on $\mathcal{A}$, where $B : C^m \to C^{m-1}$ is given by

$$(20) \qquad (B_0 \, \varphi)(a^0, \dots, a^{m-1}) = \varphi(1, a^0, \dots, a^{m-1}) - (-1)^m \, \varphi(a^0, \dots, a^{m-1}, 1) \, ,$$

$$B = AB_0 \quad , \quad (A\psi)(a^0, \dots, a^{m-1}) = \sum (-1)^{(m-1)j} \, \psi(a^j, \dots, a^{j-1}) \, .$$

To an n-dimensional cyclic cocycle ψ one associates the (b, B) cocycle $\varphi \in Z^p(F^q \, C)$, $n = p - 2q$ given by

$$(21) \qquad (-1)^{[n/2]} \, (n!)^{-1} \, \psi = \varphi_{p,q}$$

where $\varphi_{p,q}$ is the only non zero component of φ.

Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with $D^{-1} \in \mathcal{L}^{(p,\infty)}$, its Chern character in cyclic cohomology is obtained from the following cyclic cocycle τ_n, $n \geq p$, n odd,

$$(22) \qquad \tau_n(a^0, \dots, a^n) = \lambda_n \, \mathrm{Tr}' \left(a^0 [F, a^1] \dots [F, a^n] \right) \, , \qquad \forall a^j \in \mathcal{A} \, ,$$

where $F = \mathrm{Sign} D$, $\lambda_n = \sqrt{2i} \, (-1)^{\frac{n(n-1)}{2}} \, \Gamma \left(\frac{n}{2} + 1 \right)$ and

$$(23) \qquad \mathrm{Tr}'(T) = \frac{1}{2} \, \mathrm{Trace}(F(FT + TF)) \, .$$

In [Co] we obtained the following general formula for the Hochschild cohomology class of τ_n in terms of the Dixmier trace :

$$(24) \qquad \varphi_n(a^0, \dots, a^n) = \lambda_n \, \mathrm{Tr}_\omega \left(a^0 [D, a^1] \dots [D, a^n] \, |D|^{-n} \right) \, , \qquad \forall a^j \in \mathcal{A} \, .$$

Our local formula for the *cyclic cohomology* Chern character, i.e. for a cyclic cocycle cohomologous to (22), will be expressed in terms of the (b, B) bicomplex. Bearing this in mind we see that if we want to regard the cochain φ_n of (24) as a cochain of the (b, B) bicomplex, we should use, instead of λ_n, the normalization constant

$$(25) \qquad \mu_n = (-1)^{[n/2]} (n!)^{-1} \lambda_n = \sqrt{2i} \, \frac{\Gamma\left(\frac{n}{2} + 1\right)}{n!} \qquad (\text{for } n \text{ odd}) \, .$$

Let us now state the result. We let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple with discrete dimension spectrum and $D^{-1} \in \mathcal{L}^{(p,\infty)}$. We shall use the following notations:

$$(26) \qquad da = [D, a] \, , \quad \forall a \text{ operator in } \mathcal{H} \, ,$$

$$(27) \qquad \nabla(a) = [D^2, a] \quad ; \quad a^{(k)} = \nabla^k(a) \, , \quad \forall a \text{ operator in } \mathcal{H} \, .$$

Theorem II.1. a) *The following formula defines a cocycle in the (b, B) bicomplex of $\mathcal{A}$:*

$$\varphi_n(a^0, \ldots, a^n) = \sqrt{2i} \sum_{q \geq 0, k_j \geq 0} c_{n,k,q} \, \tau_q \left(a^0 (da^1)^{(k_1)} \ldots (da^n)^{(k_n)} \, |D|^{-(n+2\Sigma k_j)} \right) \, ,$$

where

$$c_{n,k,q} = (-1)^{k_1 + \cdots + k_n} \, (k_1! \ldots k_n!)^{-1} \, \Gamma^{(q)} \left(k_1 + \cdots + k_n + \frac{n}{2} \right) \times$$

$$\frac{1}{q!} \left((k_1 + 1)(k_1 + k_2 + 2) \ldots (k_1 + \cdots + k_n + n) \right)^{-1},$$

with $\Gamma^{(q)}$ the qth derivative of the Γ function.

b) *The cohomology class of the cocycle (φ_n), n odd, in $HC^{\mathrm{odd}}(\mathcal{A})$ coincides with the cyclic cohomology Chern character $ch_*(\mathcal{A}, \mathcal{H}, D)$.*

Before starting the proof let us note that the term $\tau_q(T_{n,k})$ with coefficient $c_{n,k,q}$ in the above sum vanishes when

$$(28) \qquad n + \sum k_j > p \, ,$$

since the operator $T_{n,k}$ is in $\mathcal{L}_0^{(1,\infty)}$ when (28) holds. This implies that for fixed n the sum involved contains only finitely many terms. (We assume that Sd has finite multiplicity so that only finitely many q's are involved.) It also implies that

$$(29) \qquad \varphi_n = 0 \, , \qquad \text{if} \qquad n > p \, .$$

43

Assertion b) is the cyclic cohomology analogue of Theorem IV.2.8 of [Co]. It implies in the same way that if the cyclic homology is Hausdorff then φ_* is cohomologous to $ch_*(\mathcal{A}, \mathcal{H}, D)$. Note also that all the operators $T_{n,k}$ involved in the above formula are *homogeneous of degree* 0 in D, i.e. they are unaffected by the scaling

$$(30) \qquad\qquad D \to \lambda D , \qquad \lambda \in \mathbb{R}_+^* .$$

Finally, let us remark that assertion a), i.e. the equality

$$(31) \qquad\qquad b\,\varphi_n + B\,\varphi_{n+2} = 0 , \qquad \forall n ,$$

is actually a consequence of our proof of b). However, it is an instructive exercise to check it directly. We shall do so by making use of the following properties (with $\tau = \tau_0$) :

$$(32) \qquad\qquad D\,da + da\,D = \nabla(a) , \qquad \forall a \text{ operator in } \mathcal{H}$$

$$(33) \qquad\qquad \tau\left((da)^{(k)}\,|D|^{-q}\right) = 0 , \qquad \forall a \in \mathcal{A} , \ \forall k \geq 0 , \ \forall q$$

$$(34) \qquad\qquad \tau\left(\nabla(T)\,|D|^{-q}\right) = 0 , \qquad \forall T , \ \forall q$$

$$(35) \qquad\qquad D_{(k)}\,b = \sum_0^\infty \frac{(-1)^\ell}{\ell!}\,b^{(\ell)}\,D_{(k+\ell)} , \qquad \forall b \in B ,$$

where by definition $D_{(k)} = \Gamma(k)\,|D|^{-2k}$.

The meaning of (32) is that, if we view the graded commutator with D as a graded derivation in the appropriate way, then $d^2 = \nabla$. The meaning of (33) and (34) is that integration by parts is possible, since both d and ∇ are derivations. Finally (35) follows from Theorem B.1 of Appendix B and will be used only under the trace τ so that only finitely many non zero terms of the sum of the right hand side will appear.

Proof. a) We shall perform the computations under the simplifying assumption of simple spectrum. Only minor modifications will be required to treat the general case.

Let us first show that $B\,\varphi_1 = 0$. One has (up to the overall factor $\sqrt{2i}$ which we shall ignore) :

$$(36) \qquad\qquad (B_0\,\varphi_1)(a) = \sum c_{1,k}\,\tau\left((da)^{(k)}\,|D|^{-1-2k}\right) ,$$

hence, using (33), one gets

$$B_0 \, \varphi_1 = 0 \quad \text{and} \quad B \, \varphi_1 = 0.$$

We shall now compare $b \, \varphi_1$ and $-B \, \varphi_3$. The Leibnitz rule gives, in general,

$$(37) \qquad \frac{1}{q!} \, (b_1 \ldots b_n)^{(q)} = \sum_{\substack{q_j \geq 0 \\ \Sigma q_j = q}} \frac{b_1^{(q_1)}}{q_1!} \cdots \frac{b_n^{(q_n)}}{q_n!} \, .$$

One has

$$(38) \qquad \varphi_1(a^0, a^1) = \sum_k \frac{(-1)^k}{(k+1)!} \, \tau \left(a^0 (da^1)^{(k)} \, D_{\left(k + \frac{1}{2}\right)} \right) \, , \qquad \forall a^j \in \mathcal{A} \, .$$

To compute $b \, \varphi_1$ we need to apply τ, for each k, to

$$\left(a^0 a^1 (da^2)^{(k)} - a^0 \, d(a^1 a^2)^{(k)} \right) D_{\left(k + \frac{1}{2}\right)} + a^0 (da^1)^{(k)} \, D_{\left(k + \frac{1}{2}\right)} \, a^2 \, .$$

Using $d(a^1 a^2) = a^1 \, da^2 + (da^1) a^2$, (37) and (35) we thus get

$$- \sum_{\substack{k_1 + k_2 = k \\ k_1 \neq 0}} k! \, a^0 \, \frac{(a^1)^{(k_1)}}{k_1!} \, \frac{(da^2)^{(k_2)}}{k_2!} \, D_{\left(k + \frac{1}{2}\right)} - \sum_{k_1 + k_2 = k} k! \, a^0 \, \frac{(da^1)^{(k_1)}}{k_1!} \, \frac{(a^2)^{(k_2)}}{k_2!} \, D_{\left(k + \frac{1}{2}\right)}$$

$$+ \sum_{\ell \geq 0} (-1)^\ell \, a^0 \, (da^1)^{(k)} \, \frac{(a^2)^{(\ell)}}{\ell!} \, D_{\left(k + \ell + \frac{1}{2}\right)} \, .$$

Thus, if we introduce the cochains

$$(39) \qquad \tau(1; k_1, k_2) \, (a^0, a^1, a^2) = (-1)^{k_1 + k_2} \, \tau \left(a^0 (a^1)^{(k_1)} \, (da^2)^{(k_2)} \, D_{\left(k_1 + k_2 + \frac{1}{2}\right)} \right) \, ,$$

$$\tau(2, k_1, k_2) \, (a^0, a^1, a^2) = (-1)^{k_1 + k_2} \, \tau \left(a^0 (da^1)^{(k_1)} \, (a^2)^{(k_2)} \, D_{\left(k_1 + k_2 + \frac{1}{2}\right)} \right) \, ,$$

we can express $b \, \varphi_1$ as follows:

$$(40) \qquad b \, \varphi_1 = - \sum_{\substack{k_1 \neq 0 \\ k_j \geq 0}} \frac{(k_1 + k_2 + 1)^{-1}}{k_1! \, k_2!} \, \tau(1, k_1, k_2)$$

$$+ \sum_{k_j \geq 0} \frac{\left((k_1 + 1)^{-1} - (k_1 + k_2 + 1)^{-1} \right)}{k_1! \, k_2!} \, \tau(2, k_1, k_2) \, .$$

We shall now express $B \, \varphi_3$ in a similar manner, as a linear combination of the cochains (39). We have

$$(41) \qquad B_0 \, \varphi_3(a^0, a^1, a^2) = \sum c_{3,k} \, \tau \left((da^0)^{(k_1)} \, (da^1)^{(k_2)} \, (da^2)^{(k_3)} \, |D|^{-(3 + 2|k|)} \right) \, ,$$

45

where $|k| = k_1 + k_2 + k_3$.

The cochain $B \varphi_3$ is the sum of the three cochains obtained from $B_0 \varphi_3$ by cyclic permutations,

(42)
$$B \varphi_3 = B_0 \varphi_3 + (B_0 \varphi_3)' + (B_0 \varphi_3)'' ,$$

where

$$(B_0 \varphi_3)'(a^0, a^1, a^2) = (B_0 \varphi_3)(a^1, a^2, a^0) \quad \text{and} \quad (B_0 \varphi_3)''(a^0, a^1, a^2) = (B_0 \varphi_3)(a^2, a^0, a^1).$$

Using integration by parts (i.e. (33) and (34)), we can express $B_0 \varphi_3$ as

(43)
$$\sum_{k_j \geq 0, \ell \geq 0} (-1)^{k_1+1} \alpha_k \frac{1}{\ell!} \frac{1}{(k_1 - \ell)!} \frac{1}{k_2!} \frac{1}{k_3!} \tau(1, k_2 + 1 + \ell, k_3 + k_1 - \ell)$$
$$+ \sum (-1)^{k_1} \alpha_k \frac{1}{\ell!} \frac{1}{(k_1 - \ell)!} \frac{1}{k_2!} \frac{1}{k_3!} \tau(2, k_2 + \ell, k_3 + 1 + k_1 - \ell) ,$$

where

$$\alpha_k = (k_1 + 1)^{-1} (k_1 + k_2 + 2)^{-1} (k_1 + k_2 + k_3 + 3)^{-1}.$$

Let us now compute $(B_0 \varphi_3)'$:

(44)
$$(B_0 \varphi_3)'(a^0, a^1, a^2) =$$
$$\sum (-1)^{|k|} \alpha_k \frac{1}{k_1!} \frac{1}{k_2!} \frac{1}{k_3!} \tau \left((da^1)^{(k_1)} (da^2)^{(k_2)} (da^0)^{(k_3)} D_{(3/2+|k|)} \right) .$$

In order to obtain terms of the form (39) we must move $(da^0)^{(k_3)}$ in front, using (35), and then integrate by parts. We use the trace property of τ and apply (35), for each term, with

(45)
$$b = (da^1)^{(k_1)} (da^2)^{(k_2)} .$$

Thus, we get

(46)
$$(B_0 \varphi_3)'(a^0, a^1, a^2) =$$
$$\sum (-1)^{|k|+\ell} \alpha_k \frac{1}{k_1!} \frac{1}{k_2!} \frac{1}{k_3!} \frac{1}{\ell!} \tau \left((da^0)^{(k_3)} \left((da^1)^{(k_1)} (da^2)^{(k_2)} \right)^{(\ell)} D_{(\frac{3}{2}+|k|+\ell)} \right) .$$

46

Integration by parts gives:

$$\tau\left((da^0)^{(k_3)} \left((da^1)^{(k_1)} (da^2)^{(k_2)}\right)^{(\ell)} D_{(3/2+|k|+\ell)}\right)$$

$$= (-1)^{k_3} \tau\left(da^0 \left((da^1)^{(k_1)} (da^2)^{(k_2)}\right)^{(\ell+k_3)} D_{(3/2+|k|+\ell)}\right)$$

$$= (-1)^{k_3+1} \tau\left(a^0 \left((a^1)^{(k_1+1)} (da^2)^{(k_2)}\right)^{(\ell+k_3)} D_{(3/2+|k|+\ell)}\right)$$

$$+ (-1)^{k_3} \tau\left(a^0 \left((da^1)^{(k_1)} (a^2)^{(k_2+1)}\right)^{(\ell+k_3)} D_{(3/2+|k|+\ell)}\right) \ .$$

We then use (37) to get

$$\left((a^1)^{(k_1+1)} (da^2)^{(k_2)}\right)^{(\ell+k_3)} = \sum \frac{(\ell+k_3)!}{m!(\ell+k_3-m)!} (a^1)^{(k_1+1+m)} (da^2)^{(k_2+\ell+k_3-m)}$$

$$\left((da^1)^{(k_1)} (a^2)^{(k_2+1)}\right)^{(\ell+k_3)} = \sum \frac{(\ell+k_3)!}{m!(\ell+k_3-m)!} (da^1)^{(k_1+m)} (a^2)^{(k_2+1+\ell+k_3-m)} \ .$$

This gives the following formula for $(B_0 \ \varphi_3)'$:

(47)
$$(B_0 \ \varphi_3)' = \sum_{k_j \geq 0, \ell, m \geq 0} (-1)^{k_3+1} \ \alpha_k \ \frac{1}{k_1!} \ \frac{1}{k_2!} \ \frac{1}{k_3!} \ \frac{1}{\ell!} \ \frac{1}{m!}$$

$$\frac{(\ell+k_3)!}{(\ell+k_3-m)!} \ \tau(1, k_1+1+m, k_2+\ell+k_3-m)+$$

$$\sum_{k_j \geq 0, \ell, m \geq 0} (-1)^{k_3} \ \alpha_k \ \frac{1}{k_1!} \ \frac{1}{k_2!} \ \frac{1}{k_3!} \ \frac{1}{\ell!} \ \frac{1}{m!} \ \frac{(\ell+k_3)!}{(\ell+k_3-m)!}$$

$$\tau(2, k_1+m, k_2+1+\ell+k_3-m) \ .$$

The computation of $(B_0 \ \varphi_3)''$ is completely similar and gives:

(48)
$$(B_0 \ \varphi_3)'' = \sum_{k_j \geq 0, \ell, m \geq 0} (-1)^{k_2+1} \ \alpha_k \ \frac{1}{k_1!} \ \frac{1}{k_3!} \ \frac{1}{\ell!} \ \frac{1}{m!} \frac{1}{(k_2-m)!}$$

$$\tau(1, k_3+1+m, k_1+\ell+k_2-m)+$$

$$\sum (-1)^{k_2} \ \alpha_k \ \frac{1}{k_1!} \ \frac{1}{k_3!} \ \frac{1}{\ell!} \ \frac{1}{m!} \ \frac{1}{(k_2-m)!} \ \tau(2, k_3+m, k_1+1+\ell+k_2-m) \ .$$

In order to compare $b \ \varphi_1$ with $B_0 \ \varphi_3 + (B_0 \ \varphi_3)' + (B_0 \ \varphi_3)'' = B \ \varphi_3$ we just need to compare the coefficients of $\tau(j, k_1, k_2)$ in the formulae (40) and (43) + (47) + (48). The

47

most convenient way to proceed is to introduce generating functions $f_j(x,y)$, $j = 1,2$, in which we replace $\tau(j,k_1,k_2)$ by $x^{k_1} y^{k_2}$.

Let us first compute $f_j(x,y)$ for the expression (40); we get

$$(49) \qquad f_1^b(x,y) = -\sum_{k_1 \neq 0} (k_1 + k_2 + 1)^{-1} \frac{x^{k_1}}{k_1!} \frac{y^{k_2}}{k_2!} \, ,$$

$$f_2^b(x,y) = \sum \left((k_1 + 1)^{-1} - (k_1 + k_2 + 1)^{-1}\right) \frac{x^{k_1}}{k_1!} \frac{y^{k_2}}{k_2!} \, .$$

Thus, $f_1^b(x,y) = -\int_0^1 (e^{u(x+y)} - e^{uy})\, du$, hence

$$(50) \qquad f_1^b(x,y) = \frac{1 - e^{x+y}}{x + y} + \frac{e^y - 1}{y} \, .$$

Similarly,

$$(51) \qquad f_2^b(x,y) = \frac{1 - e^{x+y}}{x + y} + e^y \left(\frac{e^x - 1}{x}\right) \, .$$

Let us next compute f_1 and f_2 for the expression (43); we get

$$(52) \qquad f_1(x,y) = \sum (-1)^{k_1+1} \, \alpha_k \, \frac{1}{\ell!} \, \frac{1}{(k_1 - \ell)!} \, \frac{1}{k_2!} \, \frac{1}{k_3!} \, x^{k_2+1+\ell} \, y^{k_3+k_1-\ell} \, ,$$

$$f_2(x,y) = \sum (-1)^{k_1} \, \alpha_k \, \frac{1}{\ell!} \, \frac{1}{(k_1 - \ell)!} \, \frac{1}{k_2!} \, \frac{1}{k_3!} \, x^{k_2+\ell} \, y^{k_3+k_1+1-\ell} \, .$$

In order to compute them, we introduce the function

$$(53) \qquad f(x,y,z) = \sum \alpha_k \, \frac{x^{k_1}}{k_1!} \, \frac{y^{k_2}}{k_2!} \, \frac{z^{k_3}}{k_3!} \, .$$

With this notation,

$$(54) \qquad f_1(x,y) = -f(-(x+y),x,y)x \, ,$$

$$f_2(x,y) = f(-(x+y),x,y)y \, .$$

With the selfexplanatory notation f_1', f_2' for (47), we have

$$f_1'(x,y) = \sum (-1)^{k_3+1} \, \alpha_k \, \frac{1}{k_1!} \, \frac{1}{k_2!} \, \frac{1}{k_3!} \, \frac{1}{\ell!} \, \frac{1}{m!} \, \frac{(\ell + k_3)!}{(\ell + k_3 - m)!} \, x^{k_1+1+m} \, y^{k_2+\ell+k_3-m}$$

$$(55)$$

$$f_2'(x,y) = \sum (-1)^{k_3} \, \alpha_k \, \frac{1}{k_1!} \, \frac{1}{k_2!} \, \frac{1}{k_3!} \, \frac{1}{\ell!} \, \frac{1}{m!} \, \frac{(\ell + k_3)!}{(\ell + k_3 - m)!} \, x^{k_1+m} \, y^{k_2+1+\ell+k_3-m} \, .$$

It follows that

(56) ,
$$f_1'(x,y) = -e^{(x+y)} f(x,y,-(x+y))x$$
$$f_2'(x,y) = e^{x+y} f(x,y,-(x+y))y .$$

Similarly, for f_1'', f_2'', we have

$$f_1''(x,y) = \sum (-1)^{k_2+1} \alpha_k \frac{1}{k_1!} \frac{1}{k_3!} \frac{1}{\ell!} \frac{1}{m!} \frac{1}{(k_2-m)!} x^{k_3+1+m} y^{k_1+\ell+k_2-m}$$

(57)

$$f_2''(x,y) = \sum (-1)^{k_2} \alpha_k \frac{1}{k_1!} \frac{1}{k_3!} \frac{1}{\ell!} \frac{1}{m!} \frac{1}{(k_2-m)!} x^{k_3+m} y^{k_1+1+\ell+k_2-m}$$

which gives

(58) ,
$$f_1''(x,y) = -e^y f(y,-(x+y),x)x$$
$$f_2''(x,y) = e^y f(y,-(x+y),x)y .$$

We can thus express the generating functions for $B\ \varphi_3$ as follows:

$$f_1^B(x,y) = -\left(f(-(x+y),x,y) + e^{x+y} f(x,y,-(x+y)) + e^y f(y,-(x+y),x)\right) x$$
(59)
$$f_2^B(x,y) = \left(f(-(x+y),x,y) + e^{x+y} f(x,y,-(x+y)) + e^y f(y,-(x+y),x)\right) y .$$

Let us compute $f(x,y,z)$; one has

$$f(x,y,z) = \int_{0 \le u_1 \le u_2 \le u_3 \le 1} e^{(u_1 x + u_2 y + u_3 z)} \, du_1 \, du_2 \, du_3 ,$$

since

$$\int_{0 \le u_1 \le u_2 \le u_3 \le 1} u_1^{k_1} u_2^{k_2} u_3^{k_3} \, du_1 \, du_2 \, du_3 = \alpha_k .$$

We then obtain:

(60) $$f(x,y,z) = \frac{e^{x+y+z}-1}{x(x+y)(x+y+z)} - \frac{e^z-1}{x(x+y)z} - \frac{(e^{y+z}-1)}{xy(y+z)} + \frac{(e^z-1)}{xyz} .$$

Since only the restriction of f to the hyperplane $x+y+z = 0$ is involved, we can use the equality $\frac{e^a-1}{a} = 1$ for $a = 0$, to handle the first term. We get

(61) $$f(-(x+y),x,y) =$$
$$\frac{1}{-(x+y)(-y)} + \frac{(e^{x+y}-1)}{(x+y)x(x+y)} + (e^y-1)\left(\frac{1}{-(x+y)xy} - \frac{1}{(x+y)y^2}\right) ,$$

49

$$
(62) \qquad e^{x+y} \, f(x,y,-(x+y)) = \frac{e^{x+y}}{x(x+y)} + \frac{e^y - e^{x+y}}{x^2 y} + \frac{e^{x+y} - 1}{(x+y)^2 y} \; ,
$$

$$
(63) \qquad e^y \, f(y,-(x+y),x) = -\frac{e^y}{xy} + \frac{e^y - 1}{y^2(x+y)} + \frac{(e^x - 1)e^y}{x^2(x+y)} \; .
$$

Let us first compute the coefficient of e^{x+y} in the sum $(61) + (62) + (63)$. We have

$$
\frac{1}{x(x+y)^2} + \frac{1}{x(x+y)} - \frac{1}{x^2 y} + \frac{1}{(x+y)^2 y} + \frac{1}{x^2(x+y)}
$$
$$
= \frac{1}{x(x+y)} + \left(x^2 y (x+y)^2\right)^{-1} \left(xy - (x+y)^2 + x^2 + y(x+y)\right)
$$
$$
= \frac{1}{x(x+y)} \; .
$$

The coefficient of e^y is given by

$$
-\frac{1}{(x+y)y^2} - \frac{1}{xy(x+y)} + \frac{1}{x^2 y} - \frac{1}{xy} + \frac{1}{y^2(x+y)} - \frac{1}{x^2(x+y)}
$$
$$
= -\frac{1}{xy} + \left(x^2 y^2 (x+y)\right)^{-1} \left(-x^2 - xy + y(x+y) + x^2 - y^2\right)
$$
$$
= -\frac{1}{xy} \; .
$$

The rational term which remains is then

$$
\frac{1}{(x+y)y} - \frac{1}{x(x+y)^2} + \frac{1}{xy(x+y)} + \frac{1}{y^2(x+y)} - \frac{1}{y(x+y)^2} - \frac{1}{y^2(x+y)}
$$
$$
= \frac{1}{(x+y)y} + \left(xy^2 (x+y)^2\right)^{-1} \left(-y^2 + y(x+y) + x(x+y) - xy - x(x+y)\right)
$$
$$
= \frac{1}{(x+y)y} \; .
$$

Thus, the sum $(61) + (62) + (63)$ is equal to

$$
(64) \qquad \frac{e^{x+y}}{x(x+y)} - \frac{e^y}{xy} + \frac{1}{y(x+y)} \; .
$$

When we multiply (64) by $-x$ we get

$$
(65) \qquad \frac{1 - e^{x+y}}{x+y} - \left(\frac{1 - e^y}{y}\right) \; .
$$

When we multiply (64) by y we get:

$$
(66) \qquad -\frac{e^y}{x} + \frac{1}{x+y} + \frac{e^{x+y} \, y}{x(x+y)} \; .
$$

Comparing these expressions with (50) and (51) we then obtain

$$(67) \qquad f_1^b + f_1^B = 0 \quad , \quad f_2^b + f_2^B = 0 .$$

This shows that in the expression for $b\ \varphi_1 + B\ \varphi_3$ the coefficient of any of the $\tau(j; k_1, k_2)$ vanishes for $j = 1, 2$ and any k_1, k_2.

We have thus shown that $B\ \varphi_1 = 0$ and that $b\ \varphi_1 + B\ \varphi_3 = 0$. The proof for the general identity

$$b\ \varphi_n + B\ \varphi_{n+2} = 0$$

is based on a similar computation.

The above discussion only covers the case of simple poles. The general case follows in exactly the same way, by introducing the expression

$$D_{(z)} = \Gamma(z)|D|^{-2z},$$

for complex values of z, and performing the above manipulations with τ and $D_{(k)}$ replaced respectively by the usual trace *Trace* and by $D_{(k+\epsilon)}$, where ϵ is a small complex number. Taking the residue at $\epsilon = 0$ gives the desired identities.

b) With $a^0, \ldots, a^n \in \mathcal{A}$ fixed, we let

$$(68) \quad \zeta(z^0, \ldots, z^n) = \text{Trace}\left(a^0\ |D|^{-2z_0}\ da^1\ |D|^{-2z_1}\ da^2 \ldots |D|^{-2z_{n-1}}\ da^n\ |D|^{-2z_n}\right) .$$

This expression makes sense if $\sum \text{Re}\ z_i > \frac{p}{2}$ and we shall first express it in terms of the following functions of a single complex variable:

$$(69) \qquad h_k(z) = \text{Trace}\left(a^0(da^1)^{(k_1)}\ (da^2)^{(k_2)} \ldots (da^n)^{(k_n)}\ D^{-2\Sigma k_j - 2z}\right) ,$$

where $k = (k_1, \ldots, k_n)$ is a multi-index.

As in (35), one has

$$(70) \qquad |D|^{-2z}\ da = \sum \frac{(-1)^k}{k!}\ z^{(k)}\ (da)^{(k)}\ |D|^{-2z-2k} ,$$

where $z^{(k)} = z(z+1) \ldots (z+k-1)$.

We can then write the expansion

$$(71) \qquad \zeta(z_0, \ldots, z_n) = \sum P_k(z_0, \ldots, z_{n-1})\ h_k\left(\sum_0^n z_j\right) ,$$

51

where the polynomial P_k is given by

$$(72) \qquad P_k(z_0, \ldots, z_{n-1}) = \frac{(-1)^{k_1 + \cdots + k_n}}{k_1! \ldots k_n!} \, z_0^{(k_1)} \, (z_0 + k_1 + z_1)^{(k_2)}$$

$$(z_0 + k_1 + z_1 + k_2 + z_2)^{(k_3)} \ldots (z_0 + k_1 + z_1 + k_2 + z_2 + k_3 + \cdots + z_{n-1})^{(k_n)} .$$

In the expansion (71), if we sum the terms for which $|k| > p$, they contribute by a function of $(z_0, \ldots, z_n)$ which is holomorphic and bounded for $\sum \operatorname{Re} z_i > 0$. This follows from the Hölder inequality and the control (Theorem B.1 of Appendix B) of the remainder in the Taylor expansion (70).

Let $\lambda \in \mathbf{R}_+$, $\lambda > \frac{p}{2(n+1)}$; then using the equality

$$(73) \qquad e^{-uD^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma(\lambda + is) \, |D|^{-2(\lambda + is)} \, u^{-(\lambda + is)} \, ds \, , \qquad \forall u > 0 \, ,$$

we obtain:

$$(74) \qquad \operatorname{Trace} \left(a^0 \, e^{-u_0 D^2} \, da^1 \, e^{-u_1 D^2} \, da^2 \ldots e^{-u_{n-1} D^2} \, da^n \, e^{-u_n D^2} \right)$$

$$= (2\pi i)^{-(n+1)} \int_{C_\lambda^{n+1}} \Gamma(z_0) \ldots \Gamma(z_n) \, u^{-z} \, \zeta(z_0, \ldots, z_n) \, dz_0 \ldots dz_n \, ,$$

where $u_j > 0$, $u^{-z} = u_0^{-z_0} \ldots u_n^{-z_n}$, and $C_\lambda = \{\lambda + is \; ; \; s \in \mathbf{R}\}$.

We let $\theta(u_0, u_1, \ldots, u_n)$ be the function defined by (74) and we want to compute the coefficient of $\epsilon^{-n/2}$ in the expansion of

$$(75) \qquad \theta(\epsilon \, v_0, \epsilon \, v_1, \ldots, \epsilon \, v_n) \, , \quad \sum v_i = 1 \, , \; \epsilon \to 0 \, .$$

From (71) and the hypothesis on the dimension spectrum we see, using the boundedness of $\Gamma(z_0) \ldots \Gamma(z_n) \, \Gamma(z_0 + \cdots + z_n)^{-1}$ on C_λ^{n+1}, that except for finitely many values of λ the function

$$\Gamma(z_0) \ldots \Gamma(z_n) \, \zeta(z_0, \ldots, z_n)$$

is integrable on C_λ^{n+1} (for $\lambda > 0$ say). It follows then that the right hand side of (74), when evaluated for such a λ, at $(\epsilon \, v_i)$ is a $O(\epsilon^{-(n+1)\lambda})$. It is not equal to (75) because of the contribution of the residues of the differential form

$$(76) \qquad \omega = \Gamma(z_0) \ldots \Gamma(z_n) \, u^{-z} \, \zeta(z_0, \ldots, z_n) \, dz_0 \ldots dz_n \, .$$

Using the expansion (71), we let

$$(77) \qquad c(k, q) = \text{coeff of } \epsilon^{-q} \text{ in } h_k \left(\frac{n}{2} + \epsilon \right) \text{ at } \epsilon = 0$$

52

and concentrate on the contribution of the residue of the differential form

$$(78) \qquad \omega = \Gamma(z_0)\dots\Gamma(z_n)\, P_k(z_0,\dots,z_{n-1})\, u^{-z} \left(z_0 + \dots + z_n - \frac{n}{2}\right)^{-q} \prod_0^n dz_i$$

(which then needs to be multiplied by $c(k,q)$).

If we denote by X the differential operator

$$(79) \qquad X = \frac{1}{n+1} \sum_0^n \frac{\partial}{\partial z_j} \, ,$$

the contribution of the residue is given by

$$(80) \qquad \int_{\Sigma z_i = \frac{n}{2}} \frac{X^{q-1}}{(q-1)!} \left(\Gamma(z_0)\dots\Gamma(z_n)\, P_k(z_0,\dots,z_{n-1})\, u^{-z}\right) \prod_1^n dz_i \, ,$$

where $\mathrm{Re}\, z_i = \lambda = \frac{n}{2(n+1)}$.

Thus, with $u_j = \epsilon\, v_j$, $\sum_0^n v_j = 1$, the coefficient of $\epsilon^{-n/2}$ is given by (80) with v instead of u, since the derivatives of $\epsilon^{-\Sigma z_i}$ contribute by terms involving $\epsilon^{-n/2}(\log \epsilon)^k$.

Introducing an additional variable $t \in \mathbb{R}$ we can rewrite (using the Fourier expansion of the δ function) the result as

$$(81) \qquad (2\pi)^{-1} \int \frac{t^{q-1}}{(q-1)!}\, \Gamma(z_0)\dots\Gamma(z_n)\, P_k(z_0,\dots,z_{n-1})\, v^{-z}\, e^{-t\left(\Sigma z_i - \frac{n}{2}\right)} \prod_0^n dz_i\, dt$$

where $\mathrm{Re}\, z_i = \lambda = \frac{n}{2(n+1)}$ and one integrates in the $n+2$ remaining variables.

Before taking care of the polynomial P_k, we can already compute, for fixed t,

$$(82) \qquad \int_{\mathrm{Re}\, z_i = \lambda} \Gamma(z_0)\dots\Gamma(z_n)\, v^{-z}\, e^{-t\Sigma z_i} \prod_0^n dz_i = (2\pi i)^{(n+1)} \prod_0^n e^{-v_j\, e^t}\, e^{t\frac{n}{2}} \, ,$$

which holds for any value of $v_j > 0$.

Next, we have

$$(83) \qquad \left(\frac{\partial}{\partial v_0}\right)^k v_0^{-z_0} = (-1)^k\, z_0^{(k)}\, v_0^{-(z_0+k)} \, ,$$

or better

$$(84) \qquad \left(\frac{\partial}{\partial u}\right)^k (uv_0)^{-z_0} = (-1)^k\, z_0^{(k)}\, u^{-(z_0+k)}\, v_0^{-z_0} \, .$$

53

This means that the effect of $P_k(z_0, \ldots, z_{n-1})$ in the above integral is obtained as follows, for the term

$$(-1)^{\Sigma k_j} z_0^{(k_1)} (z_0 + k_1 + z_1)^{(k_2)} \ldots (z_0 + k_1 + z_1 + \cdots + z_{n-1})^{(k_n)}.$$

One starts with the integral (82) written as $f(v_0, \ldots, v_n)$, then one applies

$$(85) \qquad \left(\frac{\partial}{\partial u} \right)^{k_1} f(uv_0, v_1, \ldots, v_n) = f_1(u, v_0, v_1, \ldots, v_n)$$

and one continues with

$$(86) \qquad \left(\frac{\partial}{\partial u} \right)^{k_2} f_1(u, v_0, uv_1, v_2, \ldots, v_n) = f_2(u, v_0, \ldots, v_n) \, ,$$

$$\left(\frac{\partial}{\partial u} \right)^{k_3} f_2(u, v_0, v_1, uv_2, \ldots, v_n) = f_3(u, v_0, \ldots, v_n) \, ,$$

$$\cdots$$

$$\left(\frac{\partial}{\partial u} \right)^{k_n} f_{n-1}(u, v_0, \ldots, uv_{n-1}, v_n) = f_n(u, v_0, \ldots, v_n) \, ,$$

which is finally evaluated at $u = 1$.

Using (82), we are just applying this rule to

$$f(v_0, \ldots, v_n) = e^{-(\Sigma v_j)e^t} \, .$$

We get

$$f_1(u, v_0, \ldots, v_n) = (-v_0 \, e^t)^{k_1} \, f(uv_0, v_1, \ldots, v_n) \, ,$$

$$f_2(u, v_0, \ldots, v_n) = (-v_0 \, e^t)^{k_1} \left(-(v_0 + v_1)e^t \right)^{k_2} \, f(uv_0, uv_1, v_2, \ldots, v_n) \, , \, ,$$

$$f_3(u, v_0, \ldots, v_n) = (-v_0 \, e^t)^{k_1} \left(-(v_0 + v_1)e^t \right)^{k_2} \left(-(v_0 + v_1 + v_2)e^t \right)^{k_3}$$
$$f(uv_0, uv_1, uv_2, v_3, \ldots)$$

$$f_n(1, v_0, \ldots, v_n) = (-1)^{\Sigma k_j} \, e^{t \Sigma k_j} \, v_0^{k_1}(v_0 + v_1)^{k_2} \ldots (v_0 + \cdots + v_{n-1})^{k_n}$$
$$f(v_0, \ldots, v_n) \, .$$

We can thus write (81) as

$$(87) \qquad (2\pi)^{-1} \, (2\pi i)^{n+1} \, \frac{(-1)^{\Sigma k_j}}{k_1! \ldots k_n!} \int \frac{t^{q-1}}{(q-1)!} \, e^{t\left(\Sigma k_j + \frac{n}{2} \right)}$$
$$e^{-(\Sigma v_j)e^t} \, dt \, v_0^{k_1}(v_0 + v_1)^{k_2} \ldots (v_0 + \cdots + v_{n-1})^{k_n} \, .$$

54

We then have to integrate the result on the simplex $\sum v_j = 1$. The first task is thus to compute

(88)
$$\int_{-\infty}^{\infty} \frac{t^{q-1}}{(q-1)!} \, e^{t\alpha} \, e^{-e^t} \, dt \quad , \quad \alpha = \sum k_j + \frac{n}{2} \; .$$

It is obtained from the Taylor expansion at α of

$$\int_{-\infty}^{\infty} e^{t\alpha} \, e^{-e^t} \, dt = \Gamma(\alpha)$$

and is given by

(89)
$$\frac{\Gamma^{(q-1)}(\alpha)}{(q-1)!} \; .$$

The remaining part of the integral is

$$\int_{\substack{n \\ \sum_0 v_j = 1}} v_0^{k_1} (v_0 + v_1)^{k_2} \dots (v_0 + \dots + v_{n-1})^{k_n} \, \Pi \, dv_i$$
$$= (k_1 + 1)^{-1} \, (k_1 + k_2 + 2)^{-1} \dots (k_1 + \dots + k_n + n)^{-1} \; .$$

We can now complete the proof of 3 b).

We use the Chern character of $(\mathcal{A}, \mathcal{H}, D)$ in entire cyclic cohomology (cf. [Co]) given in the most efficient manner by the JLO formula, which defines the components of an entire cocycle in the (b, B) bicomplex:

(90)
$$\psi_n(a^0, \dots, a^n) = \sqrt{2i} \int_{\substack{n \\ \sum_0 v_i = 1, v_i \geq 0}}$$

$$\mathrm{Trace} \left(a^0 \, e^{-v_0 D^2} \, [D, a^1] \, e^{-v_1 D^2} \dots e^{-v_{n-1} D^2} \, [D, a^n] \, e^{-v_n D^2} \right) \; , \qquad \forall a^j \in \mathcal{A}$$

where n is odd.

We introduce a parameter ϵ by replacing D^2 by ϵD^2, which yields a cocycle ψ_n^ϵ which is cohomologous to ψ_n. One has moreover

(91)
$$\psi_n^\epsilon(a^0, \dots, a^n) = \sqrt{2i} \left(\int_{\substack{n \\ \sum_0 v_i = 1}} \theta(\epsilon \, v_0, \dots, \epsilon \, v_n) \, \pi \, dv_i \right) \epsilon^{n/2} \; ,$$

where θ was defined by (74). The coefficient $\epsilon^{n/2}$ comes from the n terms $[D, a^j]$ in the formula (90). Since n is odd, $n/2 \neq 0$. The above computation of the behavior of

55

$\epsilon^{n/2} \theta(\epsilon\, v_0, \ldots, \epsilon\, v_n)$ from the residues of the differential form ω (76) gives an expansion as a finite sum of terms:

$$(92) \qquad \theta(\epsilon\, v_0, \ldots, \epsilon\, v_n) = \sum \alpha_{m,\ell}\, \epsilon^{-p_m}\, (\log \epsilon)^\ell + O(\epsilon^{-n/2}) \,,$$

where the p_m correspond to the poles of h_k whose real part is larger than $\frac{n}{2}$. Moreover we have computed above the coefficient of $\epsilon^{-n/2}$ in this expansion, after integration of the result in v_i we get:

$$(93) \qquad \psi_n^\epsilon(a^0, \ldots, a^n) = \sum \beta_{m,\ell}\, \epsilon^{n/2-p_m}\, (\log \epsilon)^\ell + O(1) \,,$$

$$\beta_{0,0} = \sqrt{2i} \sum_{k_1, \ldots, k_n, q} (-1)^{\Sigma k_j}\ \frac{1}{k_1! \ldots k_n!}\ (k_1+1)^{-1}\, (k_1+k_2+2)^{-1} \ldots (k_1 + \cdots + k_n + n)^{-1}$$

$$(94) \qquad \frac{\Gamma^{(q)}\left(k_1 + \cdots + k_n + \frac{n}{2}\right)}{q!}\ c(k,q) \,.$$

When we pair the (b,B) cocycle ψ^ϵ with a cyclic cycle $c = (c_n)$ in entire cyclic homology (cf. [Co]), the pairing gives a scalar independent of ϵ and written as a sum of terms of the form (93). The total contribution of the terms ψ_n^ϵ, $n > p$ converges to 0 by the argument of [C-M].

Thus, we can assert that

$$(95) \qquad \sum_{n \le p} \langle \psi_n^\epsilon, c_n \rangle \underset{\epsilon \to 0}{\to} \langle ch_*(\mathcal{H}, D), c \rangle \,.$$

Using (93) this is possible only if the asymptotic expansion of the left hand side in terms of $\epsilon^{-p_m} (\log \epsilon)^\ell$, $\mathrm{Re}\, p_m \le 0$, only contains a constant term, and by (94) we know the value of this constant term, it is given by the pairing $\langle \varphi, c \rangle$ of the cyclic cocycle of Theorem II.1 a) with c.

To prove that the class of (φ_n) actually coincides with $ch_*(\mathcal{A}, \mathcal{H}, D)$, we need to recall that there is a canonical transgressed cochain $(\widetilde{\psi}_k)$ such that

$$(96) \qquad \frac{d}{dt}\, \psi_n^\varepsilon = b\widetilde{\psi}_{n-1}^\varepsilon + B\widetilde{\psi}_{n+1}^\varepsilon \,,$$

and to note that $(\widetilde{\psi}_k)$ also has an asymptotic expansion of the form (93). It then follows from [CM2, §4] that (ψ_n) is cohomologous with the *finite part* of (ψ_n^ϵ), which in turn, from the above discussion, can be seen to give precisely the cocycle (φ_n). ∎

56

3. Renormalization

There is one unpleasant feature of the formula II.1. a) for the cyclic cocycle φ, namely the occurence of the transcendental numbers which enter in the Taylor expansion of the Γ function at the points $\Gamma\left(\frac{1}{2}+q\right)$, $q \in \mathbb{N}$. Also the sum

$$(97) \qquad \sum \frac{\Gamma\left(|k|+\frac{n}{2}\right)^{(q)}}{q!} \operatorname*{Res}_{s=0} \left(s^q \, \zeta(s)\right)$$

is an infinite sum when ζ is not meromorphic at $s = 0$. We can of course rewrite it as

$$(98) \qquad \operatorname*{Res}_{s=0} \; \Gamma\left(|k|+\frac{n}{2}+s\right) \zeta(s) \, .$$

We shall however proceed to show how to obtain a modified cyclic cocycle φ', giving the same result Thm. II.1. b) as φ, but involving a *finite* linear combination *with rational coefficients* of the terms

$$(99) \qquad \sqrt{2i} \; \Gamma\left(\frac{1}{2}\right) \; \tau_q \left(a^0(da^1)^{(k_1)} \ldots (da^n)^{(k_n)} \, |D|^{-2\Sigma k_j - n}\right) \, .$$

To achieve this, we shall exploit the freedom of replacing the operator D by $\mu^{-1}D$, $\mu \in \mathbb{R}_+^*$ without affecting Thm. II.1. b). The effect of this transformation on the functionals τ_q is as follows:

$$(100) \qquad \tau_q^\mu = \sum \frac{(\log \mu)^m}{m!} \, \tau_{q+m} \, .$$

This implies that for any integer $m \geq 1$ the following formula defines the components of a cyclic cocycle which pairs trivially with cyclic homology:

$$(101) \qquad \varphi_n^{(m)}(a^0, \ldots, a^n) = \sum c_{n,k,q} \, \tau_{q+m} \left(a^0(da^1)^{(k_1)} \ldots (da^n)^{(k_n)} \, |D|^{-(n+2|k|)}\right) \, .$$

What we shall do now is add a suitable linear combination of counterterms $\varphi^{(m)}$ in order to cancel all the transcendental coefficients occuring in the Taylor expansion of $\Gamma(1/2)^{-1}\Gamma(s)$ at half integers. Even though we could right away write down the list of the coefficients β_m needed in front of $\varphi^{(m)}$, we shall rather explain carefully how they are obtained.

To begin with, there is no problem at all if Sd is *simple*, i.e. if one has at most a simple pole. In that case one simply writes

$$(102) \qquad \Gamma\left(\frac{1}{2}+q\right) = \frac{1}{2} \, \frac{3}{2} \cdots \left(\frac{1}{2}+q-1\right) \Gamma\left(\frac{1}{2}\right)$$

and since all τ_q's with $q \geq 1$ vanish one gets the desired answer.

Let us see what happens when Sd has multiplicity two, i.e. when we have at most a double pole. In that case by Proposition II.1 we know that τ_1 is a trace, while $\tau_q = 0$ for $q \geq 2$. This means that the formula for φ_n involves the combination

$$(103) \qquad \Gamma\left(|k| + \frac{n}{2}\right) \tau_0(A) + \Gamma'\left(|k| + \frac{n}{2}\right) \tau_1(A) ,$$

where A is some operator. Now since the Hochschild coboundary $b \tau_0$ is given rationally in terms of τ_1 (cf. Proposition II.1) we do not expect to need the transcendental coefficient

$$(104) \qquad \frac{\Gamma'\left(\frac{1}{2} + m\right)}{\Gamma\left(\frac{1}{2} + m\right)}$$

in order to compensate for the lack of trace property of τ_0. If we replace the term $\Gamma'\left(|k| + \frac{n}{2}\right) \tau_1(A)$ by $\Gamma\left(|k| + \frac{n}{2}\right) \tau_1(A)$, then we get exactly the components of $\varphi_n^{(1)}$ which we can subtract from φ without affecting Thm. II.1. b). Thus, we shall look for a coefficient λ such that

$$(105) \qquad \Gamma'\left(\frac{1}{2} + m\right) = \lambda \, \Gamma\left(\frac{1}{2} + m\right) + c_m \, \Gamma\left(\frac{1}{2} + m\right) , \quad m \in \mathbf{N} ,$$

where the c_m are rational numbers.

To obtain (105) one just uses the equality

$$(106) \qquad \frac{\Gamma'\left(\frac{1}{2} + m + \epsilon\right)}{\Gamma\left(\frac{1}{2} + m + \epsilon\right)} = \sum_{a=0}^{m-1} \frac{1}{\frac{1}{2} + a + \epsilon} + \frac{\Gamma'\left(\frac{1}{2} + \epsilon\right)}{\Gamma\left(\frac{1}{2} + \epsilon\right)} ,$$

which we write with ϵ for later use.

Thus, the constant λ is

$$\frac{\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = -(\gamma_E + 2\log 2),$$

where γ_E is Euler's constant.

If we replace φ by $\varphi - \lambda\varphi^{(1)}$ then using (105) we find that in the formula giving φ the terms $\Gamma'\left(|k| + \frac{n}{2}\right) \tau_1(A)$ should be replaced simply by $c_{|k| + \frac{n-1}{2}} \Gamma\left(|k| + \frac{n}{2}\right) \tau_1(A)$, where :

$$(107) \qquad c_\ell = \sum_{a=0}^{\ell-1} \frac{1}{\frac{1}{2} + a} .$$

Let us consider the next case, when Sd has multiplicity 3, i.e. when we have triple poles. This time we shall get the combination:

$$(108) \qquad \Gamma\left(|k| + \frac{n}{2}\right) \tau_0(A) + \Gamma'\left(|k| + \frac{n}{2}\right) \tau_1(A) + \frac{\Gamma''}{2!}\left(|k| + \frac{n}{2}\right) \tau_2(A) \,.$$

We want to use a further subtraction, say of $\lambda_2\, \varphi^{(2)}$, to get coefficients for $\tau_1(A)$ and $\tau_2(A)$ of the form $\Gamma\left(|k| + \frac{n}{2}\right) \times \mathbb{Q}$. From (106) we get the formula

$$(109) \qquad \Gamma'\left(\frac{1}{2} + m + \epsilon\right) = R_m(\epsilon)\, \Gamma\left(\frac{1}{2} + m + \epsilon\right) + f(\epsilon)\, \Gamma\left(\frac{1}{2} + m + \epsilon\right) \,,$$

where R_m is the rational fraction

$$(110) \qquad R_m(\epsilon) = \sum_{a=0}^{m-1} \frac{1}{\frac{1}{2} + a + \epsilon}$$

and where the function f is given by

$$(111) \qquad f(\epsilon) = \frac{\Gamma'\left(\frac{1}{2} + \epsilon\right)}{\Gamma\left(\frac{1}{2} + \epsilon\right)} \,.$$

If we differentiate (109) we get:

$$(112) \qquad \Gamma''\left(\frac{1}{2} + m + \epsilon\right) = R'_m(\epsilon)\, \Gamma\left(\frac{1}{2} + m + \epsilon\right) + R_m(\epsilon)\, \Gamma'\left(\frac{1}{2} + m + \epsilon\right)$$
$$+ f'(\epsilon)\, \Gamma\left(\frac{1}{2} + m + \epsilon\right) + f(\epsilon)\, \Gamma'\left(\frac{1}{2} + m + \epsilon\right) \,.$$

We have to transform the term $R_m(\epsilon)\, \Gamma'\left(\frac{1}{2} + m + \epsilon\right)$ because it involves at the same time a function of m, $R_m(\epsilon)$ and a derivative of Γ. To do this we replace $\Gamma'\left(\frac{1}{2} + m + \epsilon\right)$ by its value (109) which yields:

$$(113) \qquad R_m(\epsilon)^2\, \Gamma\left(\frac{1}{2} + m + \epsilon\right) + R_m(\epsilon)\, f(\epsilon)\, \Gamma\left(\frac{1}{2} + m + \epsilon\right)$$

and we use again (109) to replace the second term of the formula by

$$(114) \qquad f(\epsilon)\, \Gamma'\left(\frac{1}{2} + m + \epsilon\right) - f(\epsilon)^2\, \Gamma\left(\frac{1}{2} + m + \epsilon\right) \,.$$

Coming back to all the terms of (112) we thus proved:

$$(115) \qquad \Gamma''\left(\frac{1}{2} + m + \epsilon\right) = \left(R'_m(\epsilon) + R_m(\epsilon)^2\right) \Gamma\left(\frac{1}{2} + m + \epsilon\right)$$
$$+ \left(f'(\epsilon) - f(\epsilon)^2\right) \Gamma\left(\frac{1}{2} + m + \epsilon\right) + 2\, f(\epsilon)\, \Gamma'\left(\frac{1}{2} + m + \epsilon\right) \,.$$

59

This shows that, if we replace φ by $\varphi - \lambda_1 \, \varphi^{(1)} - \lambda_2 \, \varphi^{(2)}$, where $\lambda_1 = \lambda = f(0)$ as above and

$$(116) \qquad \lambda_2 = \frac{1}{2!} \, f_2(0) \quad , \quad f_2(\epsilon) = f'(\epsilon) - f(\epsilon)^2 \ ,$$

then the combination (108) gets replaced by

$$(117) \quad \Gamma\left(|k| + \frac{n}{2}\right) \, \tau_0(A) + c_{|k| + \frac{n-1}{2}} \, \Gamma\left(|k| + \frac{n}{2}\right) \, \tau_1(A) + c'_{|k| + \frac{n-1}{2}} \, \Gamma\left(|k| + \frac{n}{2}\right) \, \tau_2(A) \ ,$$

where the rational number c'_m is $\frac{1}{2!} \, R_m^{(1)'}(0)$, with

$$(118) \qquad R_m^{(1)}(\epsilon) = R'_m(\epsilon) + R_m(\epsilon)^2 \ .$$

We can now proceed by induction to the general case. One proves by induction on ℓ the following formula on the ℓth derivative $\Gamma^{(\ell)}$ of the Γ function:

$$(119) \quad \Gamma^{(\ell)}\left(\frac{1}{2} + m + \epsilon\right) = R_m^{(\ell-1)}(\epsilon) \, \Gamma\left(\frac{1}{2} + m + \epsilon\right) + \sum_{j=1}^{\ell} C_\ell^j \, f_j(\epsilon) \, \Gamma^{(\ell-j)}\left(\frac{1}{2} + m + \epsilon\right)$$

where $R_m^{(\ell)}$ and f_j are defined inductively by

$$(120) \qquad R_m^{(\ell+1)}(\epsilon) = R_m^{(\ell)}(\epsilon)' + R_m(\epsilon) \, R_m^{(\ell)}(\epsilon) \ ,$$

$$(121) \qquad f_{j+1}(\epsilon) = f'_j(\epsilon) - f(\epsilon) \, f_j(\epsilon) \ .$$

The proof uses the same patern as above and is straightforward. It shows that if we replace φ by

$$(122) \qquad \varphi' = \varphi - \lambda_1 \, \varphi^{(1)} - \lambda_2 \, \varphi^{(2)} - \cdots - \lambda_\ell \, \varphi^{(\ell)} \cdots$$

where $\lambda_\ell = \frac{1}{\ell!} \, f_\ell(0)$, then the combination of terms:

$$(123) \qquad \sum \frac{1}{q!} \, \Gamma\left(|k| + \frac{n}{2}\right)^{(q)} \, \tau_q(A)$$

in the expression of the cocycle φ, gets replaced by

$$(124) \qquad \Gamma\left(|k| + \frac{n}{2}\right) \sum \frac{1}{q!} \, R_m^{(q-1)}(0) \, \tau_q(A) \qquad m = |k| + \frac{n-1}{2} \ .$$

60

Now the functions $R_m^{(\ell)}(\epsilon)$ are easy to compute, since

$$R_m(\epsilon) = \sum_{a=0}^{m-1} T_a(\epsilon) \quad \text{with} \quad T_a'(\epsilon) = -T_a(\epsilon)^2 \ , \ T_a(\epsilon) = \frac{1}{\frac{1}{2} + a + \epsilon} \ .$$

One gets that $R_m^{(\ell)}(\epsilon)$ is the $(\ell + 1)$th symmetric function of the m terms $\frac{1}{\frac{1}{2} + a + \epsilon}$:

$$(125) \qquad \qquad \prod_{a=0}^{m-1} \left(1 + \frac{z}{\frac{1}{2} + a + \epsilon} \right) = \sum R_m^\ell(\epsilon) \, z^{\ell+1} \ .$$

We can then easily compute the product $\Gamma\left(\frac{1}{2} + m\right) \, R_m^{(q-1)}(0)$ which appears in (124) and get

$$(126) \qquad \qquad \Gamma\left(\frac{1}{2} + m\right) \, R_m^{(q-1)}(0) = \Gamma\left(\frac{1}{2}\right) \, \sigma_{m-q}(m) \ ,$$

where $\sigma_j(m)$ is the jth symmetric function of the first m odd $1/2$ integers:

$$(127) \qquad \qquad \prod_{\ell=0}^{m-1} \left(z + \frac{(2\ell + 1)}{2} \right) = \sum z^j \, \sigma_{(m-j)}(m) \ .$$

What is remarkable now is that these coefficients vanish if q is larger than m so that not only we transformed them to elements of $\Gamma\left(\frac{1}{2}\right)\mathbb{Q}$ but we have also eliminated all but a finite number.

We note that the function $f_j(\epsilon)$ is not difficult to compute, and by induction we get the formula

$$(128) \qquad \qquad f_j(\epsilon) = -\Gamma\left(\frac{1}{2} + \epsilon\right) \left(\frac{1}{\Gamma\left(\frac{1}{2} + \epsilon\right)} \right)^{(j)}$$

as can be seen by interpreting the transformation

$$(129) \qquad \qquad T(h) = h' - f h$$

as $T(h) = \Gamma\left(\frac{h}{\Gamma}\right)'$, and using (121).

It follows then that

$$(130) \qquad \qquad \lambda_\ell = -\Gamma\left(\frac{1}{2}\right) \frac{1}{\ell!} \left(\frac{1}{\Gamma\left(\frac{1}{2} + \epsilon\right)} \right)^{(\ell)_{\epsilon=0}}$$

61

and that the above operation of subtraction has a very simple interpretation, namely the following. In the proof of Theorem II.1 a), we were applying the linear form $\mathrm{Res}_{s=0}$ to meromorphic functions of the form

$$(131) \qquad \Gamma\left(|k| + \frac{n}{2} + s\right) \, \mathrm{Trace}\left(A\, |D|^{-2s}\right) = \zeta(s)\,,$$

where A is an operator. But any other linear form such as

$$(132) \qquad \zeta \to \mathrm{Res}_{s=0}\, g(s)\, \zeta(s)\,,$$

with g holomorphic at 0, would have worked equally well. What the subtraction of $\sum \lambda_\ell\, \varphi^{(\ell)}$ is doing is exactly to take

$$g(s) = \frac{\Gamma(1/2)}{\Gamma(1/2 + s)}.$$

If we combine this with

$$(133) \qquad \frac{\Gamma\left(\frac{1}{2} + m + s\right)}{\Gamma\left(\frac{1}{2} + s\right)} = \prod_0^{m-1} \left(\frac{1}{2} + a + s\right)\,,$$

we can summarize the above discussion by the following variant of Theorem II.1.

Theorem II.2. *The statements of Theorem II.1 are true for the cocycle φ'_n given, for n-odd, $n \leq p$, by the formula*

$$\varphi'_n(a^0, \ldots, a^n) =$$

$$\sqrt{2\pi i}\, \sum_{k,q} \frac{(-1)^{|k|}}{k_1! \ldots k_n!}\, \alpha_k\, \frac{1}{q!}\, \sigma_{m-q}(m)\, \tau_q\left(a^0 (da^1)^{(k_1)} \ldots (da^n)^{(k_n)}\, |D|^{-(2|k|+n)}\right)\,,$$

with $m = |k| + \frac{n-1}{2}$, $\alpha_k^{-1} = (k_1 + 1)(k_1 + k_2 + 2) \ldots (k_1 + \cdots + k_n + n)$ and σ defined in (127).

Not only the coefficients are all rational multiples of the overall factor $\sqrt{2\pi i}$, but also the total number of terms in the formula is now finite and bounded in terms of p alone and not the order of the poles. Indeed,

$$q \leq |k| + \frac{n-1}{2} \quad \text{and} \quad |k| + n \leq p,$$

so that the formula does not involve more than p terms in the Laurent expansion.

Let us see what this formula looks like for small values of p.

$p = 1$. Then only φ_1' is non zero; we have $k = 0$ and $q = 0$, also

(134) $$\varphi_1'(a^0, a^1) = \sqrt{2\pi i} \; \tau_0 \left(a^0 \; da^1 \; |D|^{-1}\right) .$$

This shows that, even if we had poles of arbitrary order for the function $\zeta(s) = \text{Trace} \left(a^0 \; da^1 \; |D|^{-1-2s}\right)$ at $s = 0$, they do not contribute to φ_1' except for the residue of ζ at $s = 0$.

If we had used the formula of Theorem II.1 we would be taking the residue of $\Gamma\left(\frac{1}{2} + s\right) \zeta(s)$ at $s = 0$ which involves infinitely many of the functionals τ_k. Note also that here τ_0 is a trace.

$p = 2$. Again, only φ_1' is non zero, but now we can have $k_1 = 1$ and also $q = 1$ if $k_1 = 1$. Thus, we get three terms:

(135) $$\varphi_1'(a_0, a_1) = \sqrt{2\pi i} \; \left(\tau_0 \left(a^0 \; da^1 \; |D|^{-1}\right) - \frac{1}{2} \; \tau_0 \left(a^0 (da^1)^{(1)} \; |D|^{-3}\right)\right.$$
$$\left. - \frac{1}{2} \; \tau_1 \left(a^0 (da^1)^{(1)} \; |D|^{-3}\right)\right) .$$

This time τ_0 is no longer a trace, as one can see using Proposition II.1, and the formula involves τ_1, i.e. the coefficient of s^{-2} in some ζ-function. However, no higher order coefficient is involved, unlike the formula for φ in Thm. II.1. a).

$p = 3$. Let us look at φ_3' in this case. Here, we must have $k = 0$ but since $q \leq |k| + \frac{n-1}{2}$, we can have $q = 1$. Thus, we get two terms for φ_3':

(136) .
$$\varphi_3'(a_0, a_1, a_2, a_3) = \sqrt{2\pi i} \left(\tau_0 \left(a^0 \; da^1 \; da^2 \; da^3 \; |D|^{-3}\right) + \tau_1 \left(a^0 \; da^1 \; da^2 \; da^3 |D|^{-3}\right)\right) .$$

This shows that, even for φ_3', the coefficient of s^{-2} in the expansion of the ζ-function is playing a role, i.e. that τ_1 enters into play.

4. Local index formula

To get more insight into the content of Theorems II.1 and II.2, we shall now write down a corollary whose statement does not involve cyclic cohomology or noncommutative geometry but computes a Fredholm index (called *spectral flow*) as a sum of residues of ζ-functions attached to the problem.

To formulate the problem we just need a pair (D, U) of operators in Hilbert space, where D is selfadjoint with discrete spectrum, while U is unitary. The main assumption that we

need is that $[D,U]$ is bounded, which implies immediately that the compression PUP of U of the *positive* part of D, $\left(P = \frac{1+F}{2}\ ,\ F = \text{Sign } D\right)$ is a Fredholm operator. The index

(137) $$\text{Index } PUP = \dim \text{Ker } PUP - \dim \text{Ker } PU^*P$$

can be interpreted as spectral flow, i.e. as the number of eigenvalues which cross the origin in the natural homotopy between D and $UDU^* = D + U[D, U^*]$. In any case, it is an integer, and we shall compute it as a sum of residues.

We make the following hypotheses:

(138)

a) If S is the spectrum of D (with multiplicity), then

$$\sum_{\lambda \in S} |\lambda|^{-s} < \infty \text{ for some finite } s \ .$$

(We call p the lower bound of such s).

b) The operators U and $[D, U]$ are in the domain of δ^k, $\delta = [|D|, \cdot]$ for $1 \leq k \leq N$, $N >> 0$.

c) The following functions, holomorphic for $\text{Re} s >> 0$, are meromorphic, with finitely many poles for $\text{Re} s > -\epsilon$,

$$\zeta_{(k,n)}(s) = \text{Trace }\left(U^{-1}\,[D,U]^{(k_1)}\,[D,U^{-1}]^{(k_2)} \ldots [D,U]^{(k_n)}\,|D|^{-2|k|-n-s}\right)\ ,$$

where we use the notation $X^{(k)} = \nabla^k(X)$, $\nabla(X) = [D^2, X]$.

In c) only *finitely many* functions are involved because of the inequality $|k| + n \leq p$. At the technical level, we need to assume that $\Gamma(s)\,\zeta(s)$ restricted to vertical lines is of rapid decay.

Corollary II.1. *Let D and U be as above. Then*

$$\text{Index } PUP = \sum_{n \leq p}(-1)^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)! \sum_{k,q}\frac{(-1)^{|k|}}{k_1! \ldots k_n!}\,\alpha_k\,\frac{1}{q!}\,\sigma_{m-q}(m)\,\text{Res}_{s=0}\,s^q\,\zeta_{(k,n)}(s)\ ,$$

with $m = |k| + \frac{n-1}{2}$.

Proof. We just apply Theorem II.2 to the special case when $\mathcal{A} = C^\infty(S^1)$, acting on $\mathcal{H}$ by the unique representation which sends the function $f(e^{i\theta})$ to $f(U)$. We use the formula

64

for the pairing between K^1-theory and odd cyclic cohomology, together with the index formula (cf. [Co]),

$$(139) \qquad \langle \varphi', U \rangle = \frac{1}{\sqrt{2\pi i}} \sum_{n \text{ odd}} (-1)^{\frac{n-1}{2}} \left(\frac{n-1}{2} \right)! \; \varphi'_n(U^{-1}, U, \ldots, U^{-1}, U) \; .$$

The proof of Theorem II.1 b) shows that the hypothesis (138) is sufficient to conclude. ∎

At this point we should stress the considerable freedom that one has in applying Corollary II.1. The data is a discrete subset (perhaps with multiplicity) of $\mathbb{R}$,

$$(140) \qquad\qquad\qquad S = \text{Spectrum } D \; ,$$

together with a unitary matrix, $u(\lambda, \lambda')_{\lambda, \lambda' \in S}$ which signifies a "unitary correspondence" on the list S. The main hypothesis is that when D is shifted by this correspondence (i.e. UDU^* is considered), it stays at bounded distance from D. Then one writes down a finite number of ζ-functions, the $\zeta_{(k,n)}$ above, which can be expressed as Dirichlet series of the form

$$(141) \qquad\qquad\qquad \sum a_n^{\pm} \; |\lambda_n^{\pm}|^{-s}$$

when one computes the trace in the basis of eigenvectors

$$(142) \qquad\qquad\qquad D \; e_n^{\pm} = \lambda_n^{\pm} \; e_n^{\pm}$$

for the operator D.

The statement is that a certain rational combination of residues of these functions gives the index of PUP or spectral flow. In particular, one has:

Corollary II.2. *If* Index $PUP \neq 0$, *at least one of the functions* $\zeta_{(k,n)}(s)$ *has a non trivial pole at* $s = 0$.

5. Concluding remarks

We shall now briefly discuss the analogues of Theorems II.1 and II.2 in the even case, i.e. when we have a $\mathbb{Z}/2$-grading γ of the Hilbert space $\mathcal{H}$ such that:

$$(143) \qquad\qquad \gamma a = a\gamma \quad \forall a \in \mathcal{A} \; , \qquad D\gamma = -\gamma D \; .$$

65

Let us come back to Theorem II.1 and consider for n *even*, $n > 0$, the cochain given by the similar formula:

$$(144) \qquad \varphi_n(a^0, \ldots, a^n) = \sum_{q \geq 0, k_j \geq 0} c_{n,k,q} \, \tau_q \left(\gamma a^0 (da^1)^{(k_1)} \ldots (da^n)^{(k_n)} \, |D|^{-(n+2\Sigma k_j)} \right) ,$$

$\forall a^j \in \mathcal{A}$, where the $c_{n,k,q}$ are given by:

$$(145) \qquad\qquad\qquad\qquad\qquad c_{n,k,q} =$$

$$(-1)^{k_1 + \cdots + k_n} \, (k_1! \ldots k_n!)^{-1} \, \Gamma^{(q)} \left(k_1 + \cdots + k_n + \frac{n}{2} \right) \frac{1}{q!} ((k_1 + 1)(k_1 + k_2 + 2) \ldots)^{-1} .$$

One can compute $B\varphi_2$ exactly as we did in the proof of Thm. II.1. a). One obtains:

$$(146) \qquad B\varphi_2(a^0, a^1) = -\sum_{\substack{k, q \\ k \geq 1}} (-1)^k \, \frac{\Gamma(k)^{(q)}}{k! q!} \, \tau_q \left(\gamma a^0 (da^1)^{(k)} \, |D|^{-1-2k} \right) .$$

One checks as in the proof of a) that $b\varphi_n + B\varphi_{n+2} = 0$ for all $n > 0$ (n even), but to get the correct expression for φ_0 we need to go back to the proof of b), in the case $n = 0$.

In this case we get:

$$(147) \qquad\qquad\qquad \psi_0^\epsilon(a) = \text{Trace}(\gamma a \, e^{-\epsilon D^2}) \qquad \forall a \in \mathcal{A}$$

and this time we look for the *constant* terms in the asymptotic expansion at 0 of this function of $\theta(\epsilon)$. It is of course related to the analogue of (68) for $n = 0$,

$$(148) \qquad\qquad\qquad \zeta(z) = \text{Trace}(\gamma a \, |D|^{-2z}) \, ;$$

however, this time it is no longer the residue of (148) which matters, but rather its value at 0, which governs the residue of the product $\Gamma(z)\zeta(z)$

$$(149) \qquad\qquad\qquad \varphi_0(a) = \underset{s=0}{\text{Res}} \, (\Gamma(s) \, \text{Trace}(\gamma a \, |D|^{-2s})) .$$

The right hand side is equal to

$$\sum_{q \geq 0} \frac{\Gamma(1)^{(q)}}{q!} \, \tau_{q-1}(\gamma a) ,$$

where we need to define τ_{-1} following Proposition II.1, by

$$(150) \qquad\qquad\qquad \tau_{-1}(b) = \underset{s=0}{\text{Res}} \, s^{-1} \, \text{Trace}(b \, |D|^{-2s}) .$$

66

We can now simplify the cocycle φ as we did in Theorem II.2. The expression (144) for φ_n involves

$$(151) \qquad \operatorname*{Res}_{s=0} \, \Gamma\left(|k| + \frac{n}{2} + s\right) \, \mathrm{Trace}(A|D|^{-2s}) \, ,$$

for some operator A. As in the proof of Theorem II.2 we replace this by

$$(152) \qquad \operatorname*{Res}_{s=0} \, \Gamma(1+s)^{-1} \, \Gamma\left(|k| + \frac{n}{2} + s\right) \, \mathrm{Trace}(A|D|^{-2s})$$

and we use the equality

$$(153) \qquad \Gamma(m+s) \, \Gamma(1+s)^{-1} = \sum_{0}^{m-1} \sigma_j(m) \, s^j \, ,$$

where the σ_j are the elementary symmetric functions of the numbers $1, 2, \ldots, m-1$.

We thus obtain the following analogue of Theorem II.2 :

Theorem II.3. a) *The following formula defines a cocycle in the (b, B) bicomplex of $\mathcal{A}$:*

$$\varphi_n(a^0, \ldots, a^n) =$$

$$\sum_{k,q} \frac{(-1)^{|k|}}{k_1! \ldots k_n!} \, \alpha_k \, \sigma_q\left(|k| + \frac{n}{2}\right) \, \tau_q\left(\gamma a^0 (da^1)^{(k_1)} \ldots (da^n)^{(k_n)} \, |D|^{-(2|k|+n)}\right)$$

for n even, $\neq 0$, while

$$\varphi_0(a^0) = \tau_{-1}(\gamma a^0) \, .$$

b) *The cohomology class of the cocycle (φ_n), n even, in $HC^{\mathrm{even}}(\mathcal{A})$ coincides with the cyclic cohomology Chern character $ch_*(\mathcal{A}, \mathcal{H}, D)$.*

Remark II.1. To see where the classical index theorem for manifolds fits in this picture, let us consider the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of the Dirac operator D acting on the Hilbert space $\mathcal{H}$ of L^2-spinors over a closed Riemannian Spin manifold M, of dimension p, and with $\mathcal{A} = C^\infty(M)$. Then:

a) *the dimension spectrum of the triple $(\mathcal{A}, \mathcal{H}, D)$ is simple and contained in the set*

$$\{n \in \mathbb{N} \; ; \; n \leq p\};$$

b) *with $\tau = \tau_0$ and otherwise using the above notation, one has*

$$\tau\left(a^0 (da^1)^{(k_1)} \ldots (da^n)^{(k_n)} \, |D|^{-(n+2|k|)}\right) = 0 \, , \quad \text{for} \quad |k| \neq 0 \; ;$$

67

c) *for* $k_1 = k_2 = \ldots = k_n = 0$, *one has*

$$\tau\left(a^0 [D, a^1] \ldots [D, a^n]\, |D|^{-n}\right) = \nu_n \int_M \hat{A}(R) \wedge a^0\, da^1 \wedge \ldots \wedge da^n \,,$$

where ν_n *is a numerical factor and* $\hat{A}(R)$ *stands for the* $\hat{A}$-*form of the Riemannian curvature of* M.

Assertion a) follows from the standard pseudodifferential calculus, while b) and c) can be easily checked, e.g. by means of the symbolic calculus for asymptotic operators, as in [C-M, §3].

A much more interesting example for the general local index formalism developed in this section is provided by the spectral triples of Section 1, associated to triangular structures on manifolds. In that case, the foliation-preserving diffeomorphisms contribute (via Gelfand-Fuchs classes) to the "higher residues" corresponding to $|k| \neq 0$. This application will be discussed in a forthcoming paper.

Appendix A. Inequalities for eigenvalues and the Dixmier trace

Let $\mathcal{H}$ be a (separable) Hilbert space. We let $\mathcal{K} \subset \mathcal{L}(\mathcal{H})$ be the ideal of compact operators and $\mathcal{L}^1 \subset \mathcal{K}$ the ideal of trace class operators:

$$(1) \qquad T \in \mathcal{L}^1 \Leftrightarrow \sum_0^\infty \mu_n(T) < \infty$$

where $\mu_n(T) = \mathrm{Inf}\left\{ \|T/E^\perp\| \; ; \; \dim E = n \right\}$ is the nth characteristic value of T. Note that:

$$(2) \qquad \mu_n(T) = \mathrm{dist}(T, \mathcal{R}_n) \; , \; \mathcal{R}_n = \{\text{operators of rank } \leq n\}$$

$$(3) \qquad \mu_n(T) = n + 1\text{'th eigenvalue of } |T|.$$

(This last equality is the minimax principle.)

One defines $\mathrm{Trace}(T)$ for $T \in \mathcal{L}^1$ as:

$$(4) \qquad \mathrm{Trace}(T) = \sum \langle T\xi_n, \xi_n \rangle \qquad (\xi_n) \text{ orthonormal basis}.$$

This converges because T is an ℓ^1 sum of rank one operators $|\eta\rangle\langle\eta|$ and it is independent of the basis as can be seen for such an operator.

Lemma A.1.
$$\sum_0^\infty \mu_n(T) = \mathrm{Sup} \, |\mathrm{Trace}(TX)| \; ; \; \|X\| \leq 1 \, .$$

Proof. First note that $|\mathrm{Trace}(T)| \leq \sum_0^\infty \mu_n(T)$ using the polar decomposition $T = U|T|$, $|T| = \sum \mu_n |\eta_n\rangle\langle\eta_n|$ and $|\mathrm{Trace} \, |U\eta_n\rangle\langle\eta_n|| \leq 1$.

Then $|\mathrm{Trace}(TX)| \leq \sum_0^\infty \mu_n(TX) \leq \sum_0^\infty \mu_n(T)$, if $\|X\| \leq 1$. For the converse use $U^* = X$. ∎

One lets $\|T\|_1 = \mathrm{Trace}(|T|)$. It is a norm, the trace norm.

Definition A.1. *For each integer $N \geq 1$, let, for $T \in \mathcal{K}$*

$$\sigma_N(T) = \mathrm{Sup}\left\{ \|TE\|_1 \; ; \; E \text{ subspace} , \dim E = N \right\} \, .$$

By construction, σ_N is a norm,

(5)
$$\sigma_N(T_1 + T_2) \leq \sigma_N(T_1) + \sigma_N(T_2) \qquad \forall T_j \in \mathcal{K} \; .$$

Proposition A.1.
$$\sigma_N(T) = \sum_0^{N-1} \mu_n(T) \; .$$

Proof. We can assume $T \geq 0$. Then taking $E = E_N(T)$ the spectral projection on the spectral subspace corresponding to the first N eigenvalues one gets the inequality $\geq$. Conversely if $\dim E = N$ then $\|TE\|_1 = \sum_0^\infty \mu_n(TE) = \sum_0^{N-1} \mu_n(TE) \leq \sum_0^{N-1} \mu_n(T)$ (since $\mathrm{dist}(TE, \mathcal{R}_N) = 0$). ∎

We shall now extend the definition of σ_N to a function σ_λ of the positive "scale" parameter λ as follows:

Lemma A.2. *Let $T \in \mathcal{K}$. The following function of $\lambda \in \mathbb{R}_+^*$ agrees with $\sigma_N(T)$ at integer values:*

$$\sigma_\lambda(T) = \mathrm{Inf} \, \{ \|x\|_1 + \lambda \|y\|_\infty \; ; \; x + y = T \} \; .$$

Proof. One can assume $T \geq 0$. Let $N \in \mathbb{N}^*$, we compare $\sigma_N(T)$ of Definition A.1 with the r.h.s. of 4.

Assume $T = x + y$, with $\|x\|_1 + N\|y\|_\infty \leq 1$. Then

$$\sigma_N(T) \leq \sigma_N(x) + N\|T - x\|_\infty \leq \|x\|_1 + N\|y\|_\infty \leq 1 \; .$$

Conversely write $T = (T - \mu_N(T)1) \, E_N + (\mu_N(T) \, E_N + T(1 - E_N)) = x + y$. One has $\|x\|_1 = \sigma_N(T) - N\mu_N(T)$ and $\|y\|_\infty = \mu_N(T)$. Thus $\|x\|_1 + N\|y\|_\infty = \sigma_N(T)$. ∎

By construction, the function $\sigma_\lambda(T)$ is increasing as a function of λ. Also the unit ball for the norm σ_λ is the convex hull of the unit ball of $\mathcal{L}_1$ and λ^{-1} times the unit ball of $\mathcal{K}$.

The slope of σ_λ, such as $\sigma_{N+1} - \sigma_N = \mu_N$, is decreasing with λ, thus σ_λ is a concave function of λ.

Between 0 and 1 one has $\sigma_\lambda(T) = \lambda\|T\|$ as follows from:

$$\|x\|_\infty \leq \|x\|_1 \; .$$

70

One can check as in the proof of Lemma A.2 that σ_λ is affine between N and $N+1$ for any N (see Fig.). In particular (5) holds for all real values $\lambda > 0$.

For a positive operator T we view $\sigma_\lambda(T)$ as the trace of T *cutoff* at the (inverse) scale λ. We shall now investigate the additivity of σ_λ for $T \geq 0$.

Lemma A.3. *For $T_1, T_2 \geq 0$ and $\lambda_1, \lambda_2 \in \mathbb{R}_+^*$*

$$\sigma_{\lambda_1 + \lambda_2}(T_1 + T_2) \geq \sigma_{\lambda_1}(T_1) + \sigma_{\lambda_2}(T_2) .$$

Proof. Let us assume that N_1, N_2 are integers. First for $T \geq 0$ one has:

(6) $$\sigma_N(T) = \text{Sup} \{ \text{Trace}(TE) \ ; \ \dim E = N \} .$$

(The r.h.s. is smaller than $\|TE\|_1$ and hence than $\sigma_N(T)$, the other inequality is clear.) Then $\text{Trace}(T_1 E_1) + \text{Trace}(T_2 E_2) \leq \text{Trace}(T_1 E) + \text{Trace}(T_2 E)$ where $E = E_1 \vee E_2$ has dimension $\leq N_1 + N_2$. One can then deduce the result for arbitrary λ_1, λ_2 by piecewise linearity. ∎

We shall now concentrate on the $\log \lambda$ divergence of $\sigma_\lambda(T)$ and average the coefficient of $\log \lambda$ over various scales by considering the following function:

(7) $$\tau_\lambda(T) = \frac{1}{\log \lambda} \int_a^\lambda \frac{\sigma_u(T)}{\log(u)} \frac{du}{u} \qquad \text{(we fix } a > e) .$$

By construction τ_λ is subadditive (using 5). Let us now evaluate a lower bound for $\tau_\lambda(T_1 + T_2)$, $T_j \geq 0$.

Lemma A.4. *Let T_1, T_2 be such that $\sigma_\lambda(T_j) \leq C_j \log \lambda \quad \forall \lambda \geq a$. If $T_j \geq 0$, then:*

$$|\tau_\lambda(T_1 + T_2) - \tau_\lambda(T_1) - \tau_\lambda(T_2)| \leq (C_1 + C_2) \, ((\log \log \lambda + 2) \log 2)/ \log \lambda .$$

Proof. One has $\sigma_{2u}(T_1 + T_2) \geq \sigma_u(T_1) + \sigma_u(T_2)$. Thus:

$$\tau_\lambda(T_1) + \tau_\lambda(T_2) \leq \frac{1}{\log \lambda} \int_a^\lambda \frac{\sigma_{2u}(T_1 + T_2)}{\log u} \frac{du}{u} = \frac{1}{\log \lambda} \int_{2a}^{2\lambda} \frac{\sigma_u(T_1 + T_2)}{\log(u/2)} \frac{du}{u} .$$

On $[a, +\infty[$ one has $\sigma_u(T_1 + T_2) \leq (C_1 + C_2) \, \log u$, thus:

$$\left| \frac{1}{\log \lambda} \int_a^\lambda \frac{\sigma_u(T_1 + T_2)}{\log u} \frac{du}{u} - \frac{1}{\log \lambda} \int_{2a}^{2\lambda} \frac{\sigma_u(T_1 + T_2)}{\log u} \frac{du}{u} \right| \leq (C_1 + C_2) \frac{2 \log 2}{\log \lambda} .$$

Next,

$$\frac{1}{\log \lambda} \int_{2a}^{2\lambda} \left| \frac{\sigma_u(T_1 + T_2)}{\log(u/2)} - \frac{\sigma_u(T_1 + T_2)}{\log u} \right| \frac{du}{u} \le (C_1 + C_2) \frac{1}{\log \lambda} \int_{2a}^{2\lambda} \frac{\log u - \log u/2}{\log u/2} \frac{du}{u}$$

$$\int_{2a}^{2\lambda} \frac{\log 2}{\log(u/2)} \frac{du}{u} = \log 2 \int_a^\lambda \frac{du}{u \log u} = \log 2 (\log \log \lambda - \log \log a) \ . \qquad \blacksquare$$

Let $\mathcal{A}$ be the space of functions $h(\lambda)$; $\lambda \in [a, \infty[$, which are bounded and are taken modulo those of order $0(\log \log \lambda / \log \lambda)$. The latter form an ideal for the obvious pointwise product and thus $\mathcal{A}$ is an algebra.

By Lemma A.4 we have a well defined additive map:

$$(8) \qquad\qquad \tau : \mathcal{L}^{(1,\infty)} \to \mathcal{A}$$

defined by $\tau(T) = $ class of $(\tau_\lambda(T))_{\lambda \ge a}$ for any $T \ge 0$. Here we used:

Definition A.2. $\mathcal{L}^{(1,\infty)}$ *is the normed ideal with norm:*

$$\operatorname*{Sup}_{u \ge a} \frac{\sigma_u(T)}{\log u} = \|T\|_{1,\infty} \ .$$

Note that the image by τ of the ideal $\mathcal{L}^1$ of trace class operators is $\{0\}$.

The above definition of $\tau(T)$ for $T \ge 0$ has been extended to any T using linearity. For instance if we write $T = T^*$ in two ways as $T_1 - T_2 = T_1' - T_2'$ (all $T_i, T_j' \ge 0$) we have:

$$(9) \qquad\qquad \tau(T_1) - \tau(T_2) = \tau(T_1') - \tau(T_2')$$

since the equality $T_1 + T_2' = T_2 + T_1'$ can be combined with Lemma A.4.

Proposition A.2. τ *is a linear positive map from* $\mathcal{L}^{(1,\infty)}$ *to* $\mathcal{A}$ *such that for any bounded operator S in* $\mathcal{H}$:

$$\tau(ST) = \tau(TS) \qquad \forall T \in \mathcal{L}^{(1,\infty)} \ .$$

Proof. One has for any unitary U and $T \ge 0$ in $\mathcal{L}^{(1,\infty)}$ the equality:

$$\tau(UTU^*) = \tau(T) \ .$$

This extends by linearity to arbitrary T's and then using TU instead of T one gets $\tau(UT) = \tau(TU)$ and the conclusion by linearity. $\blacksquare$

Corollary A.1. *For $T \in \mathcal{L}^{(1,\infty)}$ the class $\tau(T)$ only depends upon the locally convex topology of $\mathcal{H}$, not the inner product.*

Proof. $\tau(STS^{-1}) = \tau(T)$ for any invertible S. $\blacksquare$

Let us now consider states ω on the C^* algebra:

$$(10) \qquad\qquad A = C_b\left([a,\infty[\right)/C_0\left([a,\infty[\right) .$$

Lemma A.5. *Let $f \in C_b\left([a,\infty[\right)$ then $f(\lambda)$ has a limit for $\lambda \to \infty$ iff $\omega(f)$ is independent of ω.*

Proof. If $f \to L$ then $f - L \in C_0$ on which any ω vanishes. Conversely if f has two distinct limit points, one gets two states ω_1, ω_2 whose values on f are different. $\blacksquare$

Remark A.1. For any separable C^*-subalgebra of $C_b\left([a,\infty[\right)/C_0$ the construction of states can be effectively performed without using the axiom of uncountable choice. So whenever we apply Tr_ω (cf. below) to any separable subspace of $\mathcal{L}^{(1,\infty)}$ we may assume that it is effectively constructed.

Definition A.3. *For any ω we let:*

$$\mathrm{Tr}_\omega(T) = \omega(\tau(T)) \qquad \forall T \in \mathcal{L}^{(1,\infty)} .$$

One has, using Proposition A.2:

Proposition A.3. a) Tr_ω *is a positive linear form on $\mathcal{L}^{(1,\infty)}$ and $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$ $\forall S \in \mathcal{L}(\mathcal{H})$.*

b) $\mathrm{Tr}_\omega(T)$ *is independent of ω iff $\tau(T)$ converges for $\lambda \to \infty$ (and the limit is then equal to $\mathrm{Tr}_\omega(T)$).*

By construction Tr_ω is continuous for the norm $\| \ \|_{1,\infty}$ and vanishes on the closure $\mathcal{L}_0^{(1,\infty)}$ of finite rank operators in this norm.

One has:

(11) $$T \in \mathcal{L}_0^{(1,\infty)} \Leftrightarrow \sigma_\lambda(T) = o(\log \lambda) \ .$$

In particular one has:

(12) $$\mu_n(T) = o\left(\frac{1}{n}\right) \Rightarrow T \in \mathcal{L}_0^{(1,\infty)} \ .$$

There is an easy case where Tr_ω is independent of ω and can be computed:

Proposition A.4. *Let $T \geq 0$, $\mu_n(T) = 0(1/n)$ and $\zeta(s) = \mathrm{Trace}(T^s)$. Then the following are equivalent:*

a) $(s-1)\,\zeta(s) \to L \quad as \quad s \to 1+ \quad,\quad$ b) $(\log N)^{-1} \displaystyle\sum_0^{N-1} \mu_n(T) \to L \quad as \quad N \to \infty \ .$

When this holds one has $\mathrm{Tr}_\omega(T) = L$ independently of ω.

In particular take $T = \Delta^{-p}$, $p > 0$, where one knows the asymptotic behavior of $\mathrm{Trace}(e^{-t\Delta}) \sim L\ t^{-p}$ for $t \to 0$.

Assume $\Delta \geq c > 0$ and write:

(13) $$\Delta^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\Delta}\ t^s\ \frac{dt}{t} \ .$$

With $s = p(1+\varepsilon)$ and applying the trace on both sides we get:

$$\zeta(1+\varepsilon) = \frac{1}{\Gamma(p+\varepsilon p)} \int_0^\infty \mathrm{Trace}(e^{-t\Delta})\ t^{p+\varepsilon p}\ \frac{dt}{t}$$

and for $\varepsilon \to 0$ it is equivalent to:

$$\frac{1}{\Gamma(p)}\ L \int_0^1 t^{\varepsilon p}\ \frac{dt}{t} = \frac{1}{\varepsilon}\ \frac{1}{\Gamma(p+1)}\ L \ .$$

Thus we get in that case:

(14) $$\mathrm{Tr}_\omega(\Delta^{-p}) = \frac{1}{\Gamma(p+1)}\ \lim_{t\to 0}\ t^p\ \mathrm{Trace}(e^{-t\Delta}) \ .$$

If we let Δ be the Laplacian on the n torus $\mathbb{T}^n$ where the length of $\mathbb{T}$ is 2π, one can use

$$\sum_{\mathbb{Z}} e^{-tk^2} \sim \frac{\pi^{1/2}}{\sqrt{t}} \qquad t \to 0$$

and get $\mathrm{Tr}_\omega(\Delta^{-n/2}) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$.

More generally for ordinary pseudodifferential operators (ΨDO) on a manifold the Dixmier trace is given by the Wodzicki residue,

Proposition A.5. *Let V be an n-dimensional manifold and $P \in OP^{-n}(V)$ a ΨDO of order $-n$ then*

$$1) \quad P \in \mathcal{L}^{(1,\infty)} \qquad 2) \quad \mathrm{Tr}_\omega(P) = \frac{1}{n} \, (2\pi)^{-n} \int_{S^*} a_{-n}(x,\xi) \, d^n x \, d^{n-1}\xi \ .$$

Recall that in local coordinates x a ΨDO is written as

$$(P\eta)\,(x) = (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} \, \alpha(x,\xi) \, \eta(y) \, dy \, d\xi$$

where $\alpha \sim a_q + a_{q-1} + \cdots$ and $a_\ell(x,\xi)$ is homogeneous of degree ℓ in ξ.

The principal symbol $a_q(x,\xi)$ is invariantly defined as a function on the cotangent bundle T^*V by:

$$\left(P\left(e^{i\tau\varphi}\,\eta\right)\right)(x) \sim \tau^q \, a_q \, (x,d\varphi) \, \eta(x) \qquad \forall \eta \ .$$

Next, let us consider on the complement of the 0-section V in T^*V the measure $dx \, d\xi$ associated to the symplectic structure. For any homogeneous function of degree $-n$, $a_{-n}(x,\xi) \, dx \, d\xi$ is now invariant under $\xi \to \lambda\xi$.

If we let $E = r \, \frac{\partial}{\partial r}$ be the vector field generating this one parameter group we have $\partial_E \, \mu = 0$, $d \, i_E \, \mu = 0$, where μ is viewed as a form of top degree. Thus $i_E \, \mu$ is a form of degree $2n - 1$ and its integral on any two homologous cycles of dimension $2n - 1$ gives the same result.

In particular we can choose a Riemannian metric on V and take as cycle its unit sphere bundle S^*. Then we get

$$\int_{S^*} a_{-n}(x,\xi) \, d^n x \, d^{n-1}\xi$$

where $d^{n-1}\xi$ is the volume form on the $n-1$ sphere with its induced metric. If we compute it for the constant 1 we get $|S^{n-1}| \int_V d^n x$, where

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \ .$$

Let us check the equality 2) for the above torus $\mathbb{T}^n$ and the Laplacian. The l.h.s. is $\frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$ the r.h.s. is

$$\frac{1}{n} \, (2\pi)^{-n} \, (2\pi)^n \, |S^{n-1}| = \frac{2}{n} \, \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \ .$$

The general conclusion follows by positivity and invariance of Tr_ω. ∎

Appendix B. Spectral triple and pseudodifferential calculus

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. For each $s \in \mathbb{R}$ we let $\mathcal{H}^s = \text{Domain } (|D|^s)$ and

$$\mathcal{H}^\infty = \bigcap_{s \geq 0} \mathcal{H}^s \quad , \quad \mathcal{H}^{-\infty} = \text{dual of } \mathcal{H}^\infty .$$

We obtain in this way a scale of Hilbert spaces, and for each r we define op^r to be the linear space of operators in $\mathcal{H}^\infty$ which are continuous for every s:

$$op^r : \mathcal{H}^s \to \mathcal{H}^{s-r} .$$

We shall use the following smoothness condition on $\mathcal{A}$: $\forall a \in \mathcal{A}$, both a and $[D, a]$ are in the domain of all powers of the derivation $\delta = [|D|, \cdot]$.

Lemma B.1. *Then a, $[D, a]$ are in op^0 and*

$$b - |D| \, b \, |D|^{-1} \in op^{-1} \qquad (b = a \text{ or } [D, a]) .$$

Proof. Let us first check that $|D|^n \, b \, |D|^{-n}$ is bounded for $n \geq 0$. With $\sigma(\cdot) = |D| \cdot |D|^{-1}$, one has:

$$\sigma = id + \varepsilon \quad , \quad \varepsilon(b) = \delta(b) \, |D|^{-1} .$$

Since $\varepsilon^k(b)$ is bounded, equal to $\delta^k(b) \, |D|^{-k}$, we get the result using $\sigma^n = \sum C_n^k \, \varepsilon^k$.

Moreover $\sigma^{-1}(b) = |D|^{-1} \, b \, |D| = -|D|^{-1} \, \delta(b)$ and the same argument shows that $\sigma^n(b)$ is bounded for $n < 0$. Then one uses interpolation.

For the second part one applies the above argument to $\delta(b)$; thus,

$$\delta(b) \in op^0 \quad , \quad \delta(b) \, |D|^{-1} \in op^{-1} .$$

∎

It is important to note that the above smoothness hypothesis can be replaced by:

$$a \quad \text{and} \quad [D, a] \in \cap \text{ Dom } L^k R^q \quad , \quad L(b) = |D|^{-1} \, [D^2, b] \quad , \quad R(b) = [D^2, b] \, |D|^{-1} .$$

Indeed, assuming the above, one has

$$L(b) = |D|^{-1} \, (|D| \, \delta(b) + \delta(b) \, |D|) \in op^0 \quad , \quad R(b) \in op^0$$

and the same applies to $L^k R^q(b)$.

Corollary B.1. *Under the above hypothesis one has*

$$\left[D^2, \left[D^2, \ldots \underbrace{[D^2, b]\right] \cdots}_{n}\right] \in op^n \qquad \forall b \in \mathcal{A} , \ or \ [D, \mathcal{A}] .$$

Let us now show that if $b \in \cap \ \mathrm{Dom} \ L^k R^q$ then $b \in \mathrm{Dom} \ \delta$. The proof is more subtle than one would expect, because the obvious argument, using

$$|D| = \pi^{-1} \int_0^\infty \frac{D^2}{D^2 + \mu} \ \mu^{-1/2} \ d\mu ,$$

requires some care. Indeed, one gets from the above

$$[|D|, b] = \pi^{-1} \int_0^\infty (D^2 + \mu)^{-1} \ [D^2, b] \ (D^2 + \mu)^{-1} \ \mu^{1/2} \ d\mu .$$

We can replace $[D^2, b]$ by $|D|$, which has the same size, and get

$$\int_0^\infty (D^2 + \mu)^{-2} \ |D| \ \mu^{1/2} \ d\mu = \int_0^\infty (1 + t)^{-2} \ t^{1/2} \ dt .$$

For this to work, we need to move $[D^2, b]$ in front of the above integral, i.e. use the finiteness of the norm of

$$\int_0^\infty \underbrace{\left[(D^2 + \mu)^{-1}, [D^2, b]\right]}_{-(D^2+\mu)^{-1}[D^2,[D^2,b]](D^2+\mu)^{-1}} \ (D^2 + \mu)^{-1} \ \mu^{1/2} \ d\mu .$$

This finiteness follows from:

1) $(D^2 + \mu)^{-1} \ [D^2, [D^2, b]]$ bounded since $b \in \mathrm{Dom} \ L^2$

2) $\int_0^\infty \|(D^2 + \mu)^{-2}\| \ \mu^{1/2} \ d\mu \leq C \int_0^1 \mu^{1/2} \ d\mu + \int_1^\infty \mu^{-3/2} \ d\mu < \infty.$

Once $[D^2, b]$ is moved in front the above calculation applies.

It follows that $b \in \mathrm{Dom} \ \delta$ and applying the same proof to $\delta(b), \ldots$ we get $b \in \cap \ \mathrm{Dom} \ \delta^k$. We thus obtain:

Lemma B.2. $\underset{k,q}{\cap} \ \mathrm{Dom} \ L^k R^q = \underset{n}{\cap} \ \mathrm{Dom} \ \delta^n.$

We shall define the order of operators by the following filtration:

$$P \in OP^\alpha \quad \text{iff} \quad |D|^{-\alpha} \ P \in \cap \ \mathrm{Dom} \ \delta^n .$$

77

Thus $OP^0 = \cap \text{Dom } \delta^n$ and we have:

$$OP^\alpha \subset op^\alpha \qquad \forall \alpha .$$

Let us now describe the general pseudodifferential calculus.

We let ∇ be the derivation: $\nabla(T) = [D^2, T]$, and consider the algebra generated by the $\nabla^n(T)$, $T \in \mathcal{A}$ or $[D, \mathcal{A}]$.

We view this algebra $\mathcal{D}$ as an analogue of the algebra of differential operators. In fact by Corollary B.1 we have a natural filtration of $\mathcal{D}$ by the total power of ∇ applied, and moreover:

$$(1) \qquad\qquad \mathcal{D}^n \subset OP^n .$$

We want to develop a calculus for operators of the form:

$$(2) \qquad\qquad A \, |D|^z \qquad z \in \mathbb{C} , \; A \in \mathcal{D} .$$

We shall use the notation $\Delta = D^2$ and begin by understanding the action of $\mathbb{C}$ given by:

$$(3) \qquad\qquad \sigma^{2z} = \Delta^z \cdot \Delta^{-z}.$$

By construction $\mathcal{D}$ is stable under the derivation ∇ and:

$$(4) \qquad\qquad \nabla(\mathcal{D}^n) \subset \mathcal{D}^{n+1} .$$

Also for $A \in \mathcal{D}^n$ and $z \in \mathbb{C}$ one has

$$(5) \qquad\qquad A \, |D|^z \in OP^{n+\text{Re}(z)} .$$

We shall use the group σ^{2z} to understand how to multiply operators of complex order modulo OP^{-k} for any k. One has: $\sigma^2 = 1 + \mathcal{E}$,

$$(6) \qquad\qquad \mathcal{E}(T) = \nabla(T) \, \Delta^{-1} .$$

Lemma B.3. *Let* $T \in \mathcal{D}^q$ *then* $\mathcal{E}^k(T) \in OP^{q-k} \quad \forall k \geq 0.$

Proof.

$$\mathcal{E}^k(T) = \nabla^k(T) \, \Delta^{-k} \in OP^{q+k} \, \Delta^{-k} \subset OP^{q-k} . \qquad\qquad \blacksquare$$

78

We just wish to justify the formal expansion:

$$\sigma^{2z}(T) = \left(1 + z\,\mathcal{E} + \frac{z(z-1)}{2!}\,\mathcal{E}^2 + \cdots\right)(T)\,.$$

It should give a control of $\sigma^{2z}(T)$ modulo OP^{q-k-1} if we stop at $\mathcal{E}^k(T)$.

To do this we need to control the remainder in the Taylor formula:

$$(7) \qquad (1+\mathcal{E})^{n+1-\alpha} = 1 + (n+1-\alpha)\mathcal{E} + \frac{(n+1-\alpha)(n-\alpha)}{2!}\,\mathcal{E}^2 + \cdots +$$

$$(n+1-\alpha)\cdots(n+1-k-\alpha)\,\frac{\mathcal{E}^{k+1}}{(k+1)!} + \cdots + (n+1-\alpha)\cdots(2-\alpha)\,\frac{\mathcal{E}^n}{n!} +$$

$$\mathcal{E}^{n+1}\int_0^1 (n+1-\alpha)\cdots(1-\alpha)(1+t\mathcal{E})^{-\alpha}\,\frac{(1-t)^n}{n!}\,dt\,.$$

The main lemma is the following:

Lemma B.4. *Let $\alpha \in \mathbb{C}$, $0 < \mathrm{Re}\,\alpha < 1$ and $\beta > 0$, $\beta < a = \mathrm{Re}\,\alpha$. Then the following operator preserves the space OP^α for any α:*

$$\Psi = \sigma^{2\beta}\int_0^1 (1+t\mathcal{E})^{-\alpha}\,(1-t)^n\,dt\,.$$

Proof. This will be done by expressing Ψ as an integral of the form

$$(8) \qquad \Psi = \int \sigma^{2is}\,d\mu(s) \qquad \|\mu\| < \infty\,.$$

One writes

$$(9) \qquad (1+t\mathcal{E})^{-\alpha} = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{1}{1+t\mathcal{E}+\mu}\,\mu^{-\alpha}\,d\mu$$

using the standard formula:

$$(10) \qquad x^{-\alpha} = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \frac{1}{x+\mu}\,\mu^{-\alpha}\,d\mu\,.$$

Let us then consider the resolvent of $-\sigma^2$, namely

$$R(\lambda) = (\lambda + \sigma^2)^{-1}\,.$$

One has, with $\beta \in \,]0,1[$ as above,

$$(11) \qquad R(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \sigma^{-2(\beta+is)} \, \lambda^{\beta+is-1} \, \frac{ds}{\sin \pi(\beta + is)}$$

which follows from

$$(12) \qquad \frac{1}{1+y} = \frac{1}{2} \int_{-\infty}^{\infty} y^{-(\beta+is)} \, \frac{ds}{\sin \pi(\beta + is)} \ .$$

(With $y = e^u$, this means that $\frac{e^{\beta u}}{1+e^u}$ is the Fourier transform of $\frac{1}{\sin \pi(\beta+is)}$, which also follows from (10) written as $\frac{\pi}{\sin(\pi(1-\beta-is))} = \int_{-\infty}^{\infty} \frac{e^{(1-\alpha)u}}{1+e^u} \, du$, $\alpha = 1 - \beta - is$.)

Thus, from (11) we get,

$$(13) \qquad \sigma^{2\beta} \, R(\lambda) = \frac{1}{2} \int_{-\infty}^{\infty} \sigma^{-2is} \, \lambda^{\beta-1} \, \frac{\lambda^{is} \, ds}{\sin \pi(\beta + is)}$$

where the measure $\frac{\lambda^{is} \, ds}{\sin \pi(\beta+is)}$ is well controled by $e^{-|s|} \, ds$. By (9) we have:

$$(1 + t\mathcal{E})^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \int_0^{\infty} \frac{1}{t} \, R\left(\frac{\mu+1}{t} - 1\right) \, \mu^{-\alpha} \, d\mu \ ,$$

$$\sigma^{2\beta}(1 + t\mathcal{E})^{-\alpha} = \frac{\sin \pi\alpha}{\pi} \, \frac{1}{2} \int_0^{\infty} \frac{1}{t} \int_{-\infty}^{\infty} \sigma^{-2is} \left(\frac{\mu+1}{t} - 1\right)^{\beta-1} \mu^{-\alpha} \, \frac{\lambda^{is} \, ds}{\sin \pi(\beta + is)} \, d\mu \ ,$$

with $\lambda = \left(\frac{\mu+1}{t} - 1\right)$.

For fixed s we are thus dealing with the size:

$$\frac{1}{t} \int_0^{\infty} \left(\frac{\mu+1}{t} - 1\right)^{\beta-1} \mu^{-\alpha} \, d\mu = I \ .$$

One has $0 < t < 1$ so that the behavior at $\mu = 0$ is fine, also the integral converges for $\mu \to \infty$ as $\mu^{(\beta-\alpha)-1}$, since $\beta < \alpha$.

We get

$$\begin{aligned}
I &= \frac{1}{t} \int_0^{\infty} \left(u + \frac{1}{t} - 1\right)^{\beta-1} \, t^{-\alpha} \, u^{-\alpha} \, t \, du \\
&= \frac{1}{t} \int_0^{\infty} \left(\frac{1}{t} - 1\right)^{\beta-1} (v+1)^{\beta-1} \, t^{-\alpha} \left(\frac{1}{t} - 1\right)^{-\alpha} v^{-\alpha} \, t \left(\frac{1}{t} - 1\right) \, dv \\
&= t^{-\alpha} \left(\frac{1}{t} - 1\right)^{\beta-\alpha} \int_0^{\infty} (v+1)^{\beta-1} \, v^{-\alpha} \, dv \\
&= (1-t)^{\beta-\alpha} \, t^{\beta} \, c(\alpha,\beta) \ .
\end{aligned}$$

Finally, we get an equality:

$$\int_0^1 \sigma^{2\beta}(1+t\mathcal{E})^{-\alpha} \frac{(1-t)^n}{n!} \, dt = \int_{-\infty}^{\infty} \sigma^{2is} \, d\nu(s)$$

where the total mass of the measure ν is finite.

Let us check this in another way, by looking directly at the L^1-norm of the Fourier transform of the function

$$u \rightarrow \int_0^1 e^{\beta u} \left(1 + t(e^u - 1)\right)^{-\alpha} \frac{(1-t)^n}{n!} \, dt \ .$$

Thus, it is enough to check that the following function of u is in the Schwartz space $\mathcal{S}(\mathbf{R})$:

$$\varphi_n(u) = (e^u - 1)^{-(n+1)} e^{\beta u} \Big(e^{(n+1-\alpha)u} - 1 - (n+1-\alpha)(e^u - 1)$$

$$-\frac{(n+1-\alpha)(n-\alpha)}{2!} (e^u - 1)^2 - \cdots - \frac{(n+1-\alpha)(n-\alpha)\cdots(2-\alpha)}{n!} (e^u - 1)^n \Big) \ .$$

First, for $u \to \infty$ the size is $\sim e^{-(n+1)u} \, e^{\beta u} \, e^{(n+1-\alpha)u} = e^{(\beta-\alpha)u} \to 0$. For $u \to -\infty$ it behaves like $e^{\beta u} \to 0$. We need to know that it is smooth at $u = 0$ but this follows from the Taylor expansion. The same argument applies to all derivatives. Thus this gives another proof of the lemma. ∎

We are now ready to prove:

Theorem B.1. *Let $T \in \mathcal{D}^q$ and $n \in \mathbf{N}$. Then for any $z \in \mathbf{C}$*

$$\sigma^{2z}(T) - \left(T + z\,\mathcal{E}(T) + \frac{z(z-1)}{2!}\,\mathcal{E}^2(T) + \cdots + \frac{z(z-1)\cdots(z-n+1)}{n!}\,\mathcal{E}^n(T) \right)$$

$$\in OP^{q-(n+1)} \ .$$

Proof. First for any $z \in \mathbf{C}$ and $k \in \mathbf{N}$ one has:

(14) $$\mathcal{E}^k \left(\sigma^{2z}(T) \right) \in OP^{q-k} \ .$$

Indeed, by (8) we know that σ^{2z} leaves any OP^n invariant; as $\mathcal{E}^k \circ \sigma^{2z} = \sigma^{2z} \circ \mathcal{E}^k$, we just use Lemma B.3.

One has, for $0 < \operatorname{Re} \alpha < 1$, β as above and $z = (n+1) - \alpha$:

$$\sigma^{2(\beta+(n+1)-\alpha)}(T) - \left(\sigma^{2\beta}(T) + z\,\sigma^{2\beta}\,\mathcal{E}(T) + \cdots + \frac{z(z-1)\cdots(z-n+1)}{n!}\,\sigma^{2\beta}\,\mathcal{E}^n(T) \right)$$

(15) $$= \lambda\,\Psi\left(\mathcal{E}^{n+1}(T)\right) \ .$$

81

If we apply this equality to $\sigma^{2s}(T)$ and use (14) and Lemma B.4 we see that for any s there exists a polynomial $P_s(\alpha)$ of degree n in α such that:

$$\sigma^{2(s-\alpha)}(T) - P_s(\alpha) \in OP^{q-(n+1)} \qquad \beta < \mathrm{Re}\ \alpha < 1 \ .$$

The polynomials $P_s(\alpha + s)$ have to agree modulo $OP^{q-(n+1)}$ on the overlap of the bands $\beta < \mathrm{Re}\ \alpha < 1$ and thus the differences between two will belong to $OP^{q-(n+1)}$ for all z. It follows then that there is $P(z)$ which works for all z. To obtain its coefficients one takes the integral values $z = 0, 1, \ldots, n$ which yields the formula of Theorem B.1. ∎

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