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The Sharp Bound of the Hankel Determinant of the Third Kind for Starlike Functions of Order 1/2

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Abstract

In the present paper, we proved the sharp inequality $|H_{3,1}(f)| \leq 1/9$ for analytic functions f with $a_n := f^{(n)}(0)/n!,\ n \in \mathbb{N},\ a_1 := 1$, such that

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > \frac{1}{2}, \quad z \in \mathbb{D} := \{z \in \mathbb{C} : |z| < 1\},$$

where

$$H_{3,1}(f) := \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}$$

is the third Hankel determinant.

Keywords Starlike functions of order 1/2 · Carathéodory functions · Hankel determinant · Coefficients

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1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{A}$ be its subclass of functions f normalized by $f(0) := 0$, $f'(0) := 1$, i.e., of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 := 1,\ z \in \mathbb{D}. \tag{1.1}$$

Given $\alpha \in [0, 1)$, let $\mathcal{S}^*(\alpha)$ denote the subclass of $\mathcal{A}$ of functions f such that

$$\operatorname{Re} \frac{zf'(z}{f(z)} > \alpha, \quad z \in \mathbb{D}, \tag{1.2}$$

called starlike of order α. In particular, $\mathcal{S}^*(0) =: \mathcal{S}^*$ is the class of starlike functions, i.e., the family of all univalent functions in $\mathcal{A}$ which map $\mathbb{D}$ onto starlike domains (with respect to the origin). Starlike functions of order α were introduced by Robertson [19] (see also [7, Vol. I, p. 138]). An important role is played by the class $\mathcal{S}^*(1/2)$. One of the significant results belongs to Marx [15] and Strohhäcker [23]. They proved that

$$\mathcal{S}^c \subset \mathcal{S}^*(1/2) \tag{1.3}$$

(see also [16, Theorem 2.6a, p. 57]), where $\mathcal{S}^c$ means the class of convex functions, i.e., the family of all univalent functions in $\mathcal{A}$ which map $\mathbb{D}$ onto convex domains. By the well known result due to Study ([24], see also [6, p. 42]) a function f is in $\mathcal{S}^c$ if and only if

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0, \quad z \in \mathbb{D}.$$

What is interesting, a function

$$f(z) := \frac{z}{1-z}, \quad z \in \mathbb{D}, \tag{1.4}$$

is extremal for many computational problems in both these two classes, i.e., in $\mathcal{S}^c$ and $\mathcal{S}^*(1/2)$.

For $q, n \in \mathbb{N}$, the Hankel determinant $H_{q,n}(f)$ of function $f \in \mathcal{A}$ of the form (1.1) is defined as

$$H_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

Given a subfamily $\mathcal{F}$ of $\mathcal{A}$, q and n, computing the upper bound of $H_{q,n}$ is an interesting problem to study. Recently many authors examined the Hankel determinant $H_{2,2}(f) =$

$a_2a_4 - a_3^2$ of order 2 (see e.g., [4,5,8,9,12,17]). Note also that $H_{2,1}(f) = a_3 - a_2^2$ is the well known coefficient functional which for $\mathcal{S}$ was estimated in 1916 by Bieberbach (see e.g., [7, Vol. I, p. 35]). To find the upper bound of the Hankel determinant

$$H_{3,1}(f) = \begin{vmatrix} a_1 & a_1 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) \tag{1.5}$$

of the third kind, is more difficult if we expect to get sharp estimate. Results in this direction however not sharp were obtained by various authors, e.g., [1,2,4,5,20–22,25].

In this paper, we found the sharp bound of the Hankel determinant $H_{3,1}$ over the class $\mathcal{S}^*(1/2)$, namely, we proved that $|H_{3,1}(f)| \leq 1/9$ for $f \in \mathcal{S}^*(1/2)$ and that the inequality is sharp. Since the class $\mathcal{S}^*(1/2)$ has a representation with using the Carathéodory class $\mathcal{P}$, i.e., the class of functions $p \in \mathcal{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{1.6}$$

having a positive real part in $\mathbb{D}$, the coefficients of functions in $\mathcal{S}^*(1/2)$ have a suitable representation expressed by coefficients of functions in $\mathcal{P}$. Therefore to get the upper bound of $H_{3,1}$, we based our computing on the well known formulas on coefficient c_2 (e.g., [18, p. 166]), the formula c_3 due to Libera and Zlotkiewicz [13,14] and the formula for c_4 recently found in [11].

At the end let us mention that in [10] the authors proved that $|H_{3,1}(f)| \leq 4/135 = 0.0296\ldots$ for $f \in \mathcal{S}^{\rfloor}$ and that the result is sharp. Looking at the inclusion (1.3) we can state that the the corresponding bounds of $H_{3,1}$ carry some information about the richness of classes. Classical estimates of coefficients does not necessarily include such a distinction, e.g., both in the class $\mathcal{S}^c$ and in the class $\mathcal{S}^*(1/2)$ modules of all coefficients are bounded by 1 (see [7, Theorem 7, p. 117; Theorem 2, p. 140]) with the extremal function given by (1.4).

2 Main Result

The basis for proof of the main result is the following lemma which contains the well known formula for c_2 (e.g., [18, p. 166]), the formula for c_3 due to Libera and Zlotkiewicz [13,14] and the formula for c_4 found in [11].

Lemma 2.1 *If $p \in \mathcal{P}$ is of the form* (1.6) *with $c_1 \geq 0$, then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta, \tag{2.1}$$

$$4c_3 = c_1^3 + (4 - c_1^2)c_1\zeta(2 - \zeta) + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{2.2}$$

and

$$8c_4 = c_1^4 + (4-c_1^2)\zeta\left[c_1^2(\zeta^2-3\zeta+3)+4\zeta\right] \\ -4(4-c_1^2)(1-|\zeta|^2)\left[c_1(\zeta-1)\eta+\overline{\zeta}\eta^2-\left(1-|\eta|^2\right)\xi\right] \tag{2.3}$$

for some $\zeta, \eta, \xi \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

We will now estimate the third order Hankel determinant $H_{3,1}(f)$ for $f \in \mathcal{S}^*(1/2)$.

Theorem 2.2

$$\max\left\{|H_{3,1}(f)| : f \in \mathcal{S}^*(1/2)\right\} = \frac{1}{9} \tag{2.4}$$

with the extremal function

$$f(z) := \frac{z}{\sqrt[3]{1-z^3}}, \quad z \in \mathbb{D}, \ \sqrt[3]{1} := 1. \tag{2.5}$$

Proof Let $f \in \mathcal{S}^*(1/2)$ be of the form (1.1). Then by (1.2) we have

$$zf'(z) = \frac{1}{2}(p(z)+1)f(z), \quad z \in \mathbb{D}, \tag{2.6}$$

for some function $p \in \mathcal{P}$ of the form (1.6). Since the classes $\mathcal{P}$ and $\mathcal{S}^*(1/2)$ are invariant under the rotations, by Carathéodory Theorem we may assume that $c := c_1 \in [0,2]$ ([3], see also [7, Vol. I, p. 80, Theorem 3]). Putting the series (1.1) and (1.6) into (2.6) and equating coefficients we get

$$a_2 = \frac{1}{2}c, \quad a_3 = \frac{1}{8}\left(2c_2 + c^2\right), \quad a_4 = \frac{1}{48}\left(8c_3 + 6cc_2 + c^3\right), \\ a_5 = \frac{1}{384}\left(48c_4 + 32cc_3 + 12c_2^2 + 12c^2c_2 + c^4\right).$$

Hence and by (1.5) we have

$$H_{3,1}(f) = \frac{1}{9216}\left(-c^6 + 6c^4c_2 - 72c_2^3 + 32c^3c_3 \right. \\ \left. + 192cc_2c_3 - 256c_3^2 - 36c^2c_2^2 + 288c_2c_4 - 144c^2c_4\right). \tag{2.7}$$

To simplify computation, let $t := 4 - c^2$. Thus formulas (2.1)-(2.3) we can rewrite as

$$c_2 = \frac{1}{2}(c^2 + t\zeta), \quad c_3 = \frac{1}{4}\left(c^3 + 2ct\zeta - ct\zeta^2 + 2t(1-|\zeta|^2)\eta\right), \\ c_4 = \frac{1}{8}\left[c^4 + 3c^2t\zeta + (4-3c^2)t\zeta^2 + c^2t\zeta^3 + 4t(1-|\zeta|^2)\left(c\eta - c\zeta\eta - \overline{\zeta}\eta^2\right) \right. \\ \left. + 4t(1-|\zeta|^2)(1-|\eta|^2)\xi\right].$$

Hence by straightforward algebraic computation we have

$$
\begin{aligned}
6c^4c_2 &= 3(c^6 + c^4 t\zeta),\\
72c_2^3 &= 9\left[c^6 + 3c^4 t\zeta + 3c^2t^2\zeta^2 + t^3\zeta^3\right],\\
32c^3c_3 &= 8\left[c^6 + 2c^4 t\zeta - c^4 t\zeta^2 + 2c^3 t(1-|\zeta|^2)\eta\right],\\
192cc_2c_3 &= 24\left[c^6 + 3c^4 t\zeta + 2c^2t^2\zeta^2 - c^4t\zeta^2 - c^2t^2\zeta^3\right.\\
&\quad \left. + 2t(c^3 + ct\zeta)(1-|\zeta|^2)\eta\right],\\
256c_3^2 &= 16\left[c^6 + 4c^4 t\zeta + 4c^4t^2\zeta^2 - 2c^4t\zeta^2 - 4c^2t^2\zeta^3 + c^2t^2\zeta^4\right.\\
&\quad \left. + 4t(c^3 + 2ct\zeta - ct\zeta^2)(1-|\zeta|^2)\eta + 4t^2(1-|\zeta|^2)^2\eta^2\right],\\
36c^2c_2^2 &= 9\left[c^6 + 2c^4 t\zeta + c^2t^2\zeta^2\right],\\
144(2c_2c_4 - c^2c_4) &= 18\left[c^4 t\zeta + 3c^2t^2\zeta^2 + (4-3c^2)t^2\zeta^3 + c^2t^2\zeta^4\right.\\
&\quad + 4t^2c\zeta(1-\zeta)(1-|\zeta|^2)\eta\\
&\quad \left. - 4t^2(1-|\zeta|^2)|\zeta|^2\eta^2 + 4t^2(1-|\zeta|^2)(1-|\eta|^2)\zeta\xi\right].
\end{aligned}
$$

Setting the above expression to (2.7) we get

$$
\begin{aligned}
&H_{3,1}(f)\\
&= \frac{1}{9216}(4-c^2)^2\left[\gamma_1(c,\zeta) + \gamma_2(c,\zeta)\eta + \gamma_3(c,\zeta)\eta^2 + \gamma_4(c,\zeta,\eta)\xi\right],
\end{aligned}
\tag{2.8}
$$

where for $\zeta,\ \eta,\ \xi \in \overline{\mathbb{D}}$,

$$
\begin{aligned}
\gamma_1(c,\zeta) &:= \zeta^2\left[2c^2 + (36-5c^2)\zeta + 2c^2\zeta^2\right],\\
\gamma_2(c,\zeta) &:= -8c\zeta(1+\zeta)(1-|\zeta|^2),\\
\gamma_3(c,\zeta) &:= -8(8+|\zeta|^2)(1-|\zeta|^2),
\end{aligned}
$$

and

$$
\gamma_4(c,\zeta,\eta) := 72(1-|\zeta|^2)(1-|\eta|^2)\zeta.
$$

Let $x := |\zeta| \in [0,1]$ and $y := |\eta| \in [0,1]$. Since $|\xi| \le 1$, from (2.8) we obtain

$$
\begin{aligned}
|H_{3,1}(f)| &\le \frac{1}{9216}(4-c^2)^2\Big[|\gamma_1(c,\zeta)|\\
&\quad + |\gamma_2(c,\zeta)||\eta| + |\gamma_3(c,\zeta)||\eta|^2 + |\gamma_4(c,\zeta,\eta)|\Big]\\
&\le \frac{1}{9216}(4-c^2)^2 F(c,x,y),
\end{aligned}
\tag{2.9}
$$

where

$$F(c,x,y) := f_1(c,x) + f_4(c,x) + f_2(c,x)y + (f_3(c,x) - f_4(c,x))y^2,$$

with

$$\begin{aligned} f_1(c,x) &:= x^2\left[2c^2 + (36-5c^2)x + 2c^2x^2\right], \\ f_2(c,x) &:= 8cx(1+x)(1-|x|^2), \\ f_3(c,x) &:= 8(8+x^2)(1-x^2) \end{aligned}$$

and

$$f_4(c,x) := 72(1-x^2)x.$$

Now, we will show that

$$F(c,x,y) \le 64, \quad c \in [0,2],\ x \in [0,1],\ y \in [0,1]. \tag{2.10}$$

Since $f_2(c,x) > 0$ and

$$f_3(c,x) - f_4(c,x) = 8(1-x)(8-x)(1-x^2) > 0$$

for $c \in (0,2)$ and $x \in (0,1)$, so for $c \in (0,2)$ and $x \in (0,1)$,

$$\begin{aligned} F(c,x,y) &\le F(c,x,1) \\ &= f_1(c,x) + f_2(c,x) + f_3(c,x) \\ &= x^2(x-2)(2x-1)c^2 + 8x(x+1)(1-x^2)c \\ &\quad - 4(2x^4 - 9x^3 + 14x^2 - 16) =: G(c,x). \end{aligned} \tag{2.11}$$

For $x = 1/2$ the function $(0,2) \ni c \mapsto G(c,1/2)$ has no critical point, obviously. When $x \ne 1/2$, then $\partial G/\partial c = 0$ iff

$$c = \frac{4x(x+1)(1-x^2)}{x^2(2-x)(2x-1)} =: c_0 \in (0,2),$$

which holds only for $x \in ((2+3\sqrt{2})/7, 1)$. Thus

$$\frac{\partial G}{\partial x}(c_0, x) = 0$$

iff

$$\begin{aligned} &4(8x^2 - 15x + 4)(x+1)^2(1-x^2)^2 \\ &\quad + 8(4x^3 + 3x^2 - 2x - 1)(x+1)(1-x^2)(x-2)(2x-1) \\ &\quad - x^2(8x^2 - 27x + 28)(x-2)^2(2x-1)^2 = 0 \end{aligned}$$

which after simplifying reduces to

$$-64x^7 + 320x^6 - 788x^5 + 1503x^4 - 1624x^3 + 760x^2 - 80x - 36 = 0$$

for $x \in ((2 + 3\sqrt{2})/7, 1)$. As we can check the above equation has no solution in $((2 + 3\sqrt{2})/7, 1)$ (real solutions are $x_1 \approx -0.1513,\ x_2 \approx 1.0622,\ x_3 \approx 2.4952$). Thus the function G has no critical point in $(0, 2) \times (0, 1)$.

For $c = 0$ and $c = 2$ both functions

$$g_1(x) := F(0, x, 1) = 4(-2x^4 + 9x^3 - 14x^2 + 16), \quad x \in [0, 1],$$

and

$$g_2(x) := F(2, x, 1) = 16(-x^4 - 2x^2 + 4), \quad x \in [0, 1],$$

are decreasing, so

$$g_i(x) \le g_i(0) = 64, \quad i = 1, 2,\ x \in [0, 1]. \tag{2.12}$$

For $x = 0$ and $x = 1$ we have respectively,

$$F(c, 0, 1) = 64, \quad c \in [0, 2],$$

and

$$F(c, 1, 1) = -c^2 + 36 \le 36, \quad c \in [0, 2].$$

Hence, by (2.12) and (2.11) it follows that the (2.9) holds. This together with (2.9) shows that $|H_{3,1}(f)| \le 1/9$.

For the function (2.5) which is in $\mathcal{S}^*(1/2)$, we have $a_2 = a_3 = a_5 = 0$ and $a_4 = 1/3$. Thus $H_{3,1}(f) = -1/9$, which makes equality in (2.4). □

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