THRUST NETWORK ANALYSIS: A NEW METHODOLOGY FOR THREE-DIMENSIONAL EQUILIBRIUM

Philippe Block [1)] **and John Ochsendorf** [2)]

1) Research Assistant, Building Technology Program, MIT
5-418, 77 Massachusetts Avenue, Cambridge, MA 02139 USA
tel. +1 (617) 452-2286, fax. +1 (617) 253-6152
e-mail: ph_block@mit.edu
2) Associate Professor of Building Technology, MIT

ABSTRACT

This paper presents a new methodology for generating compression-only vaulted surfaces and networks. The method finds possible funicular solutions under gravitational loading within a defined envelope. Using projective geometry, duality theory and linear optimisation, it provides a graphical and intuitive method, adopting the same advantages of techniques like graphic statics, but offering a viable extension to fully three-dimensional problems. The proposed method is applicable for the analysis of vaulted historical structures, specifically in unreinforced masonry, as well as the design of new vaulted structures. This paper introduces the method and shows examples of applications in both fields.

1. INTRODUCTION

Medieval vault builders created complex forms carefully balanced in compression. The structural properties of these sophisticated forms are still poorly understood because of a lack of appropriate analysis methods, i.e. methods relating stability and form. Understanding the mechanics of these vaulted structures leads to new insights for both analysis and design.

Thrust Line Analysis is a powerful graphical method for calculating the range of lower-bound equilibrium solutions of compression-only systems, such as unreinforced masonry structures. It visualises the stability of these structures and suggests possible collapse mechanisms [1]. Unfortunately, thrust line analysis is primarily suitable for 2-D cases and this limitation has prevented it from being used for the assessment of complex 3-D structures. While numerical methods based on elastic solutions give one possible answer, they no longer suggest better form as was inherent to the more holistic graphical methods.

There is a real need for tools to better understand and visualise the stability of compression-only structures, such as historic unreinforced masonry structures, as well as design tools that suggest better form. Both problems are related to finding axial force structures in equilibrium acting only in compression or tension. Currently, graphic statics provides a holistic analysis and design tool for two-dimensional structures. With today's availability of powerful virtual 3-D and parametric environments, the following question arises: can a fully three-dimensional version of thrust-line analysis provide the same freedom to explore the infinite equilibrium solutions for a certain loading condition?

2. METHODOLOGY

The *Thrust-Network Method* presented in this paper is inspired by O'Dwyer's work on funicular analysis of vaulted masonry structures [2]. It is extended by adding the concept of duality between geometry and the in-plane internal forces of networks [3].

2.1. Reciprocal figures

The duality between the geometry of a network and its internal forces is an old concept, first explained by Maxwell [4]. He called this relationship *reciprocal* and defined it as follows: "*Two plane figures are reciprocal when they consist of an equal number of lines, so that corresponding lines in the two figures are parallel, and corresponding lines which converge to a point in one figure form a closed polygon in the other.*" This means that the equilibrium of a node in the first diagram is represented by a closed polygon in the second diagram and vice versa (Fig. 1). Graphic statics is based on this principle [5].

2.2. Assumptions

The proposed method produces funicular (compression-only) solutions for loading conditions where all loads are applied in the same direction, as is the case for gravitational loading. Since the solutions are compression-only this also means that the vaults can never curl back onto themselves, which would demand some elements to go into tension. The resulting three-dimensional networks can represent load paths throughout a structure. There is no constraint on the length of the branches or the planarity of the facets of the solution.

2.3. Thrust Network Method

Key elements in the proposed process are (1) force networks, representing possible forces in equilibrium in the structure; (2) interactive reciprocal diagrams, visualising the proportional relationship of the horizontal forces in the network and providing a high level of control for the user to manipulate the force distributions in the system; (3) the use of envelopes defining the solution space; and (4) linear optimisation, resulting in fast computation of results.

2.3.1. Overview of main steps

Thrust Network Analysis has been implemented using Matlab [6] and RhinoScripting in Rhinoceros [7]. The set-up of the program is explained in more detail below:

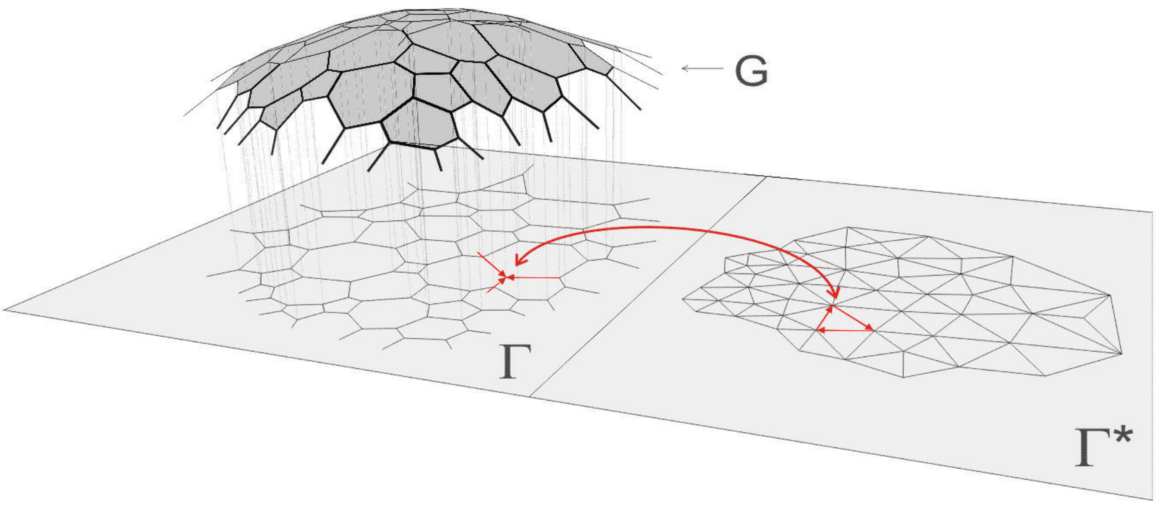

Fig. 1. Relationship between compression shell (G), its planar projection (primal grid Γ) and the reciprocal diagram (dual grid Γ^*) to determine equilibrium.

(a) Defining a solution envelope:

The solutions must lie within given boundaries defined by an intrados and an extrados (Fig. 3b). These put height constraints on the nodes of the solution. These limits can be the design envelope or the actual vault geometry for the analysis of existing masonry vaults.

(b) Constructing the primal grid Γ:

In plan, a possible force pattern topology is constructed. This is the primal grid Γ in Fig. 1. The branches represent possible load paths throughout the structure. These force patterns can be drawn by the user or generated automatically. The primal grid Γ is the horizontal projection of the final solution G.

(c) Attributing weights:

The weights attributed to each node come from lumping the dead load of the 3-D tributary area around that node. In addition to self weight, loads such as asymmetric live loads can be applied.

(d) Generating the dual grid Γ^*:

The dual grid Γ^* is produced from the primal grid Γ according to Maxwell's definition of reciprocal figures: corresponding branches stay parallel and nodal equilibrium in the primal grid is guaranteed by closed polygons in the dual grid. The applied loads do not appear in the dual grid because they disappear in the horizontal projection. Therefore, the dual grid has an unknown scale ζ since the relation between the primal and dual grid is true regardless of their relative scales.

(e) Updating the dual grid:

In the case of an indeterminate primal grid, i.e. a grid with nodes with a higher valency than 3, the user can manually change the force distribution by manipulating the dual grid (Fig. 2).

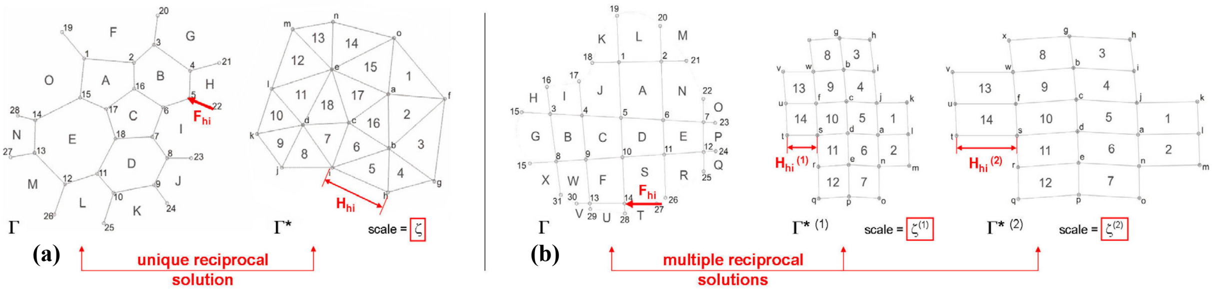

Fig. 2. For a determinate (i.e. 3-valent) primal grid, there is a unique relationship between primal and dual (a). For an indeterminate primal grid, multiple dual grids, which all satisfy Maxwell's definition, are possible (b).

(f) Solving for the result G:

Using the geometry of both primal (Γ) and dual (Γ^*) grid, the weights applied at the nodes and the boundary conditions, this problem can be solved using a one-step linear optimisation. We solve simultaneously for the nodal heights of G and the scale of the dual grid ζ^*. The horizontal components of the forces in the solution G can easily be found by measuring the lengths of the branches in the dual grid and multiplying them by the actual scale ζ^*.

2.3.2. Linear optimisation formulation

The first set of constraints comes from enforcing static equilibrium at all nodes (Fig. 3a). The vertical equilibrium of a typical internal node *i* gives

$$F_{ji}^{V} + F_{ki}^{V} + F_{li}^{V} = P_i \tag{1}$$

We describe (1) as a function of the horizontal components of the forces

$$F_{ji}^{H} \cdot \frac{(z_i - z_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} + F_{ki}^{H} \cdot \frac{(z_i - z_k)}{\sqrt{(x_i - x_k)^2 + (y_i - y_k)^2}} + F_{li}^{H} \cdot \frac{(z_i - z_l)}{\sqrt{(x_i - x_l)^2 + (y_i - y_l)^2}} = P_i \tag{2}$$

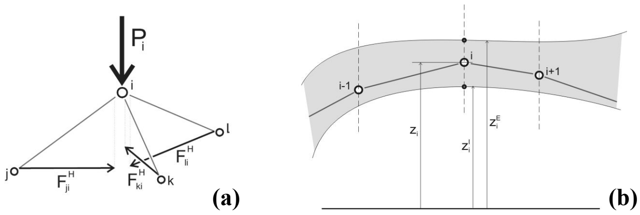

Fig. 3. The constraints come from (a) static equilibrium in every node under the applied loading and (b) the given boundaries, resulting in nodal height constraints.

The lengths of branch ij in the primal and dual grids are defined respectively as $H_{i,j}$ and $H_{i,j}^{\ *}$. The horizontal components of the forces in the branches, F_{ji}^{H}, can be expressed as a function of the dual branch lengths $H_{i,j}^{\ *}$, measured from the dual grid Γ^* and multiplied by the as-yet unknown scale factor ζ.

$$F_{ji}^{H} = \zeta \cdot H_{i,j}^{*}, \quad F_{ki}^{H} = \zeta \cdot H_{i,k}^{*}, \quad F_{li}^{H} = \zeta \cdot H_{i,l}^{*} \tag{3}$$

Rearranging equation (2) and writing it as a function of the branch lengths in both grids using equations (3) gives

$$\left(\frac{H_{i,j}^{*}}{H_{i,j}} + \frac{H_{i,k}^{*}}{H_{i,k}} + \frac{H_{i,l}^{*}}{H_{i,l}} \right) \cdot z_i - \frac{H_{i,j}^{*}}{H_{i,j}} \cdot z_j - \frac{H_{i,k}^{*}}{H_{i,k}} \cdot z_k - \frac{H_{i,l}^{*}}{H_{i,l}} \cdot z_l - P_i \cdot r = 0 \tag{4}$$

where r is the inverse of the unknown scale of the dual grid, ζ. The equilibrium constraints of the nodes can be written as a linear combination of z_i, the unknown nodal heights, and r. This emphasises the importance of using the information provided by the dual grid (3). Thanks to this insight, the nonlinear constraints (2) can be made linear by treating r as a variable. The constants of the linear function (4) are in function of the primal and dual branch lengths. Note that, because lengths (absolute values) are used, this formulation guarantees that all solutions G will be compression-only.

A second set of constraints comes from the limits put on the nodal heights (Fig.3.b). We want the solutions to lie within the given boundaries defined by an intrados and an extrados.

$$z_i^I \leq z_i \leq z_i^E \tag{5}$$

Since we are interested in the range of possible solutions that fit within the given envelope, we want to minimise or maximise r (= $1/\zeta$), resulting in respectively the shallowest or deepest solution still contained within the limits, for a chosen combination of primal and dual grid. This then becomes the objective function of the linear optimisation problem.

3. APPLICATIONS FOR THE ANALYSIS OF VAULTED MASONRY STRUCTURES

Using the proposed methodology for the assessment of unreinforced masonry structures fits within the realm of lower-bound analysis. Put simply, if a compression-only network can be found that fits within the boundaries of a vault, then the vault will stand in compression. This is a powerful concept for understanding the stability and proximity to collapse of such structures. Additional reading on this topic can be found in Heyman [8], O'Dwyer [2], Boothby [9] and Block et al. [1].

The method is particularly appropriate for historic masonry structures because their self-weight is the dominant load. The range of possible equilibrium states, bounded by a minimum and maximum thrust, can be produced (Fig. 4a). Figure 4b shows a solution with three-dimensional web action. The distribution of the horizontal components in the network is represented in its dual grid (Fig. 4c). Such a 3-D equilibrium analysis is valuable because it informs us very clearly about the stability of a vault. A forthcoming publication will show more detailed applications of this methodology to masonry vaults [10].

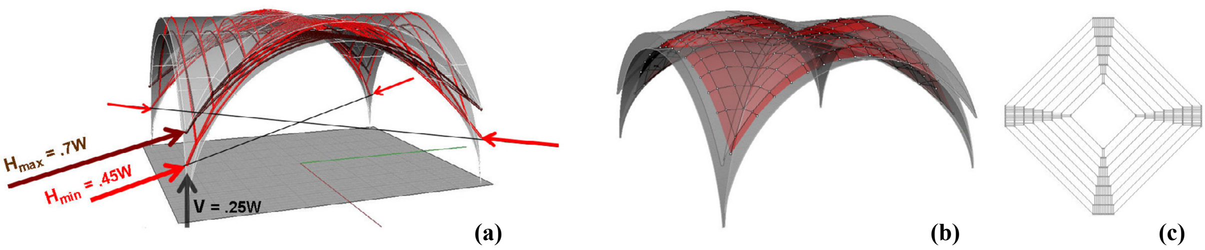

Fig. 4. (a) Possible thrust values for this groin vault range from 45% to 70% of its total weight. (b) 3-D web and rib action with the forces mainly spanning between the ribs as represented in the dual grid (c).

4. APPLICATIONS FOR THE DESIGN OF COMPRESSION-ONLY STRUCTURES

Figure 5 gives a series of compression-only solutions for a uniformly applied loading, starting from a regular rectangular grid. It shows the relationship between the dual grid and the corresponding solution. From the dual grids, the internal distribution of all horizontal forces in the networks can be understood in a glimpse, and since all dual grids are drawn at the same scale, the overall magnitude of the forces in the different solutions can immediately be compared. Some special features are that force lines do not have to go through loaded nodes (e-i) or that the edges can be freed, arching in space (h-i).

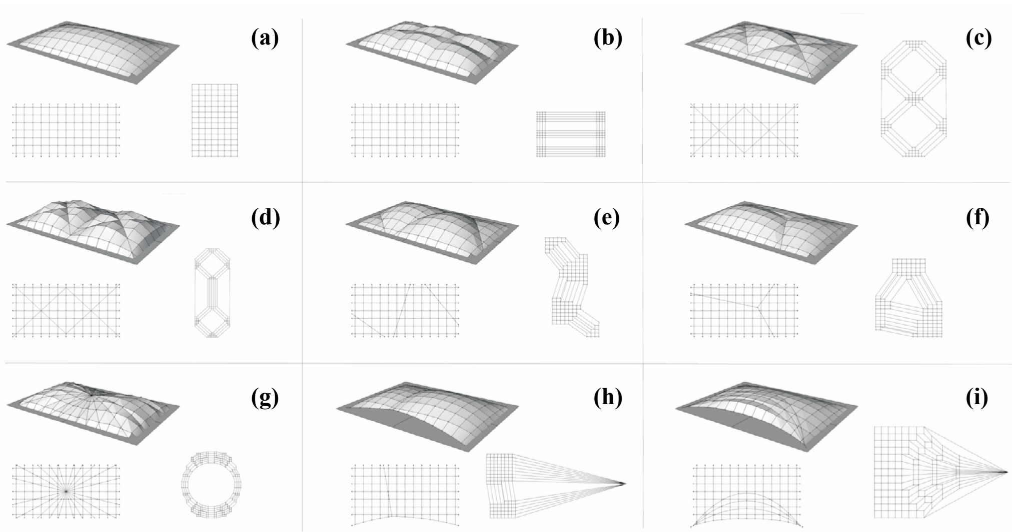

Fig. 5. A series of examples, starting from a regular rectangular grid, showing the relationship between the primal and dual grid and the corresponding solution.

The examples in figure 5 show the potential of the method for design through a series of examples, which were possible because of the flexibility and intuitiveness of the method: forces can be redistributed internally within the network by 'tweaking' the dual grids (e.g. 5a versus 5b); more force can be attracted in primary force lines (e.g. 5e or 5f); and different boundary conditions can be explored (e.g. 5h).

5. DISCUSSION AND CONCLUSIONS

This paper has proposed the Thrust Network Analysis method. It provides

- a viable three-dimensional extension for thrust-line analysis;
- a flexible, intuitive and interactive design tool for finding three-dimensional equilibrium of compression-only surfaces and systems; and
- an improved lower-bound method for the assessment of the stability of masonry vaults with complex geometries.

Key features are

- clear graphical representation of forces in the system (through the use of force diagrams, i.e. the dual grids);
- a high level of control, allowing the exploration of different possible equilibrium solutions; and
- fast solving times because of the formulation as a simple linear optimisation problem.

Currently, the number of elements in the network is limited by the implementation in Matlab and the user has to switch between programs. Future work includes going towards true interactivity and bi-directionality between both grids, implemented in a fully parametric environment, and the automatic generation of possible network topologies according to e.g. curvature, openings, support conditions or architectural preferences. We believe that this methodology has great potential for the use in both design and analysis of compression-only vaulted structures.

ACKNOWLEDGMENTS

The authors would like to thank Professor Chris Williams of Bath University, for his comments on Bow's notation for 3-D networks that sparked the ideas underlying this work. The first author is also grateful to Xuan Vinh Doan at MIT for his help in setting up the linear optimisation problems.

REFERENCES

1. Block, P., Ciblac, T., Ochsendorf, J., Real-Time Limit Analysis of Vaulted Masonry Buildings, *Computers and Structures*, 2006, **Vol. 84**, No. 29-30, 1841-1852.
2. O'Dwyer, D., Funicular analysis of masonry vaults, *Computers and Structures*, 1999, **Vol. 73**, 187-197.
3. Williams, C.J.K. (1986) "Defining and designing curved flexible tensile surface structures," *The mathematics of surfaces*, J.A. Gregory (Ed.), Oxford, Clarendon Press, 143-177.
4. Maxwell, J.C., On reciprocal figures and diagrams of forces, *Phil. Mag. Series*, 1864, **Vol. 4**, 27, 250-261.
5. Cremona, L. (1879) *Le Figure Reciproche nella Statica Grafica*. Milan, Ulrico Hoepli.
6. Matlab: http://www.mathworks.com/products/matlab/
7. Rhinoceros: http://www.rhino3d.com/
8. Heyman, J. (1995) *The Stone Skeleton: Structural engineering of masonry architecture*. Cambridge, Cambridge University Press.
9. Boothby, T.E., Analysis of masonry arches and vaults, *Progress in Structural Engineering and Materials*, 2001. **Vol. 3**, 246-256.
10. P. Block and J. Ochsendorf (2008) "Lower-bound analysis of unreinforced masonry vaults," Submitted for the *6th International Conference on Structural Analysis of Historical Construction*, Bath, UK.